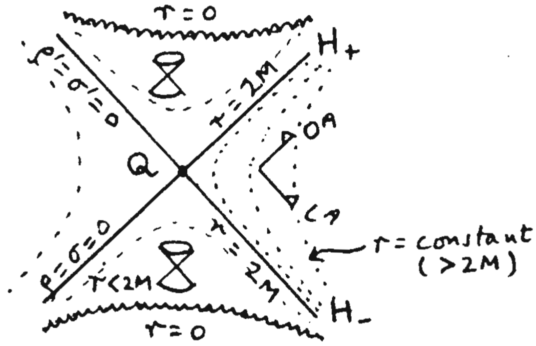
GROWING THE KERR GEOMETRY FROM SEED
PART ONE: FIRST CHOOSE A SUITABLE SEED

We take as our starting point Kruskal's map of the (analytically extended) Schwarzschild geometry.



In the diagram, θ and ϕ are suppressed. The future and past horizons (H_+ and H_-) appear in the diagram as null lines, but they are actually null hypersurfaces with "cylindrical" topology $S^2 \times R$. In each case the null generators are all parallel to each other (ρ, σ = 0 on H_+, ρ', σ' = 0 on H_-). The intersection Q of the two hypersurfaces is a spacelike 2-surface, which (in this case) is a sphere of radius 2M.

The geometry of the spacetime may be regarded as being determined by <u>initial data</u> on Q and on H_+ and H_-, the null hypersurfaces emanating orthogonally from Q. The data which is required consists of Ψ_0 on H_+, Ψ_4 on H_-, and $\rho, \sigma, \rho', \sigma'$ on Q, together with the intrinsic geometry of Q and its "complex curvature" K (the real part of which is half the Gaussian curvature of Q). If the spacetime is to correspond to a stationary black hole (with H_+ and H_- of constant area) all of the data must be zero except for K, which in this case is the same as $-\Psi_2$. Thus any stationary black hole corresponds to an especially simple set of initial data. Hawking [1] used the fact that the only non-zero datum is Ψ_2 to show that a rotating, stationary black hole must be axisymmetric (there has to be a second Killing vector, distinct from the time-translation Killing vector, which at H_+ and H_- points along the generators of the horizons).

For the Schwarzschild geometry, the initial data, the "seed" of the geometry, is simply the above-mentioned sphere, with $\Psi_2 = -1/8M^2$ (no imaginary part).

We note that at Q the curvature spinor is algebraically special (of type D):

$$\Psi_{ABCD} = 6\Psi_2 \, o_{(A} o_B \iota_C \iota_{D)}$$

(Ψ_1 and Ψ_3 vanish as well as Ψ_0 and Ψ_4 because $\rho, \sigma, \rho', \sigma'$ are zero). In the case of Schwarzschild the type D property extends from the "seed" throughout the whole spacetime. We are going to investigate the question: What other "seeds" grow into spacetimes that are type D?

We obtain a necessary condition for this by looking at the GHP equations for a type D spacetime and picking out the ones that can be applied "intrinsically" to Q. These are

$$ð\Psi_2 = 3\tau\Psi_2 \qquad \text{and} \qquad ð\tau = \tau^2$$

together with their primed versions. (Note that Ψ_2 has spin-weight 0 since $\Psi_2' = \Psi_2$). Letting $X = \Psi_2^{-1/3}$, we have

$$ðX = -X\tau \qquad \text{and} \qquad ð^2 X = 0 .$$

Thus we are seeking a surface Q whose complex curvature to the power of −1/3 satisfies $ð^2 X = ð'^2 X = 0$. (This problem has already been considered in a similar context by Ludvigsen [2]. However, he needed to impose an extra, arbitrary condition, namely $\oint \mathrm{Im}[X]dS = 0$, in order to arrive at the Kerr horizon. This does not seem to be necessary: see paragraph (5) below.)

Note that $\mu = \mathrm{Re}[X]$ is a real solution of $ð^2\mu = 0$. Now $ð^2\mu = 0$ can be solved on any surface, but a real solution (not a constant) gives rise to an isometry of Q. Using μ as one of the coordinates it turns out that the metric must have the form

$$ds^2 = d\mu^2/F(\mu) + F(\mu)\, d\phi^2 .$$

The Gaussian curvature of such a metric is

$$G = -\tfrac{1}{2}\, d^2F/d\mu^2 .$$

If the real part of X is constant the argument fails but we can use the imaginary part for μ, instead. If both real and imaginary parts are non-constant, they both give rise to isometries. Assuming that Q is not a sphere, both isometries must be the same, and in all cases we have that X must be of the form

$$X = A + B\mu,$$

with A and B complex constants (there is no implication, at this stage, that B must be pure imaginary: c.f.[2]). Next, we equate the above expression for G with the real part of $-2\Psi_2$:

$$-\tfrac{1}{2}d^2F/d\mu^2 = -2\,\mathrm{Re}[1/X^3] = -2\,\mathrm{Re}[1/(A + B\mu)^3] .$$

Integrating (and disallowing B = 0, the case of the sphere), we get

$$F = 2\,\mathrm{Re}[1/(B^2(A + B\mu))] + C\mu + D$$

with C and D real constants. This may be described as the local solution of the problem. At this point there are 6 real degrees of freedom. All but 2 of these degrees of freedom can be removed by applying appropriate global conditions, as follows.

1) We want $F(\mu)$ to have two zeroes, corresponding to the North and South poles of the surface Q.

2) We want ϕ to range from 0 to 2π. This can be achieved by replacing μ by a real constant times μ.

3) A coordinate transformation $\mu \to \mu$ + constant gives rise to a new F with different A and D, but to the same geometry. So without loss of generality we may take the zeroes of F to be at $\mu = \pm\mu_0$.

4) The Gauss-Bonnet theorem, generalized to the complex curvature, tells us

$$\oint \Psi_2 \, dS = -2\pi.$$

This amounts to two real conditions on A, B, C and D.

5) The condition $\oint G dS = 4\pi$ fails to exclude the possibility of equal and opposite conical singularities at the poles of our axisymmetric surface. This possibility must be excluded by one extra condition, namely, $dF/d\mu = -2$ at $\mu = \mu_0$.

When these constraints are worked through in detail we are left with an F of the form

$$F = \mu_0 (1 + u^2) (\mu_0^2 - \mu^2)/(\mu_0^2 + u^2\mu^2)$$

with u and μ_0 real parameters. Making the substitutions

$$\mu = \mu_0 \cos\theta; \quad u = a/r_+; \quad \mu_0 = r_+^2 + a^2,$$

we can bring the metric to the form

$$ds^2 = (r_+^2+a^2\cos^2\theta)\, d\theta^2 + (r_+^2+a^2)^2 \sin^2\theta \, d\phi^2/(r_+^2+a^2\cos^2\theta)$$

which is indeed the metric of the horizon of a Kerr black hole with mass $M = (r_+^2+a^2)/2r_+$ and angular momentum $J = aM$. Note that any spatial cross-section of H_+ or H_- has the same shape as Q.

We are now in a position to "grow" the geometry out from Q using the radial Newman-Penrose (or GHP) equations.

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References

[1] S.W.Hawking (1972) Commun.Math.Phys 25, 152-166.
[2] M.Ludvigsen (1987) Class.Qu.Grav. 4, 619-623.

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