The Bogomolny Hierarchy and Higher Order Spectral Problems.

I.A.B.Strachan

The starting point for the construction and solution of a wide range of integrable models is to write the equation as the integrability condition for the otherwise overdetermined linear system (where $\lambda \in \mathbb{CP}^1$ is the spectral parameter):

$$\begin{aligned} \partial_x s &= -U(\lambda).s\,, \\ \partial_t s &= -V(\lambda).s\,. \end{aligned} \tag{1}$$

The integrability conditions for (1) is

$$\partial_x V - \partial_t U + [U, V] = 0\,, \tag{2}$$

and equating powers of λ (if U and V are polynomial in λ) yields the equation in question. Many of those systems which are known to have a twistorial description (such as the KdV, mKdV, NLS, SG and N-wave equations) arise from a so-called first order spectral problem, with

$$\begin{aligned} U &= \lambda A + Q(x,t)\,, \\ V &= \sum_i \lambda^i A_i(x,t)\,. \end{aligned}$$

In this article the matrices will be taken to be $sl(2,\mathbb{C})$-valued, with* $A = \begin{pmatrix} \kappa & 0 \\ 0 & -\kappa \end{pmatrix}$ and $Q = \begin{pmatrix} 0 & p \\ q & 0 \end{pmatrix}$, i.e. $A \in \boldsymbol{h}$ and $Q \in \boldsymbol{k}$, where $\boldsymbol{h}$ is the Cartan subalgebra and $\boldsymbol{k}$ is the complement. A higher order spectral problem is one for which U and V are general polynomial functions, namely:

$$\begin{aligned} U &= \lambda^p A + \lambda^{p-1} Q_1 + \ldots + Q_p\,, \\ V &= \lambda^n V_0 + \lambda^{n-1} V_1 + \ldots + V_n\,. \end{aligned}$$

The simplest example ($p = 2, n = 4$ and $Q_2 = 0$) results in the derivative Non-Linear Schrödinger (or DNLS) equation. The purpose of this article is two-fold: firstly to show how such systems are nothing more than a reduction of the Bogomolny hierarchy introduced in [1], and secondly to generalise these systems to (2+1)-dimensions while retaining their integrability.

$\star$ In the terminology of [1], the fields are of type β; type α fields will not be considered here.

The following method to generate the matrices $Q_1, \ldots, Q_p, V_0, \ldots, V_n$ for these higher order problems is due to Crumey [2]. Let

$$u = \lambda^p . A \,,$$
$$v = \lambda^n . A \,.$$

These trivially satisfy (2). However, this equation is gauge invariant, so if $\omega(x,t)$ is a λ-dependent gauge transformation (often called a 'dressing transformation'), defined by $\omega = \exp \sum_{i=1}^{\infty} \omega_i \lambda^{-i}$ with $\omega_i \in sl(2,\mathbb{C})$, then U and V, defined by

$$U = \omega u \omega^{-1} - \omega_x \omega^{-1} \,,$$
$$V = \omega v \omega^{-1} - \omega_t \omega^{-1} \,,$$

will also satisfy (2). Assuming that ω is chosen so that U and V involve only non-negative powers of λ yields, on projecting onto positive (including the λ^0 term) and negative powers of λ, the equations

$$U = (\lambda^p \omega A \omega^{-1})_+ \,, \qquad V = (\lambda^n \omega A \omega^{-1})_+ \,,$$
$$\omega_x \omega^{-1} = (\lambda^p \omega A \omega^{-1})_- \,, \qquad \omega_t \omega^{-1} = (\lambda^p \omega A \omega^{-1})_- \,.$$

These simplify further by decomposing ω as $\omega = h.k$, where $h = \sum_{i=1}^{\infty} h_i(x,t)\lambda^{-i}$, $h_i(x,t) \in \boldsymbol{h}$ and $k = \sum_{i=1}^{\infty} k_i(x,t)\lambda^{-i}$, $k_i(x,t) \in \boldsymbol{k}$. One then has

$$U = (\lambda^p k A k^{-1})_+ \,, \qquad V = (\lambda^n k A k^{-1})_+ \,.$$

Let A_{n-i} denote the coefficient of λ^{-i} in the expansion of kAk^{-1} (the reason for this skew choice will become apparent later), i.e.

$$A_{n-i} = \sum_{r=1}^{i} \frac{1}{r!} \sum_{(\{s_j\}:\sum s_j = i)} [k_{s_1}, [k_{s_2}, \ldots, [k_{s_r}, A] \ldots]] \,.$$

From this proceadure one obtains the general form of the functions U and V. The matrices $k_1, \ldots, k_p$ are matrix valued fields. The integrable equation itself (which connects the time evolution of these fields with their spacial derivatives), together with the remaining matrices, may be found using the above equations, or equivalently, equation (2).

Having found the general form of U and V it remains to show how these are contained within the Bogonolny hierarchy. Assuming $m \equiv n - p \geq 0$, the matrix V may be written in the form

$$V = \lambda^m . U + \sum_{i=0}^{m-1} \lambda^i A_i \, ,$$

and hence the original system (1) may be rewritten as

$$\begin{aligned} \partial_x s &= -\Big\{ \sum_{i=0}^{p} \lambda^i A_{m+i} \Big\} s \, , \\ \partial_t s - \lambda^m \partial_x s &= -\Big\{ \sum_{i=0}^{m-1} \lambda^i A_i \Big\} s \, . \end{aligned} \tag{3}$$

Recall [1] that given the minitwistor space $\mathcal{O}(n)$, the line bundle over the Riemann sphere of Chern class $n \geq 1$, the Ward construction gives rise to the linear system

$$\big\{ [\partial_{z_i} + A_i] - \lambda [\partial_{z_{i+1}} + B_{i+1}] \big\} s = 0 \, , \qquad i = 0 \, , \ldots , n-1 \, ,$$

where A_i and B_{i+1} are $sl(2, \mathbb{C})$-valued gauge potentials. With the symmetry generated by ∂_{z_n}, together with $B_i = 0 \, , i = 1, \ldots, , n-1 \, , B_n \equiv A_{n+1} \, , A_0 = A$, relabelling $z_0 = t \, , z_m = x$, and eliminating the other variables results in (3) :

$$\left. \begin{aligned} [\partial_t - \lambda \partial_{z_1}] s &= -A . s \\ &\vdots \\ [\partial_{z_{m-1}} - \lambda \partial_x] s &= -A_{m-1} . s \end{aligned} \right\} \Rightarrow [\partial_t - \lambda^m \partial_x] s = -\Big\{ \sum_{i=0}^{m-1} \lambda^i A_i \Big\} . s \, ,$$

$$\left. \begin{aligned} [\partial_x - \lambda \partial_{z_{m+1}}] s &= -A_m . s \\ &\vdots \\ [\partial_{z_{n-1}} - \lambda \partial_{z_n}] s &= -[A_{n-1} + \lambda A_n] . s \end{aligned} \right\} \Rightarrow \partial_x s = -\Big\{ \sum_{i=0}^{p} \lambda^i A_{m+i} \Big\} . s \, .$$

Thus these higher order spectral problems may all be embedded within the Bogomolny hierarchy. Solutions of the simplest example, that of the DNLS equation, correspond to bundles over the space $\mathcal{O}(4)$ with certain symmetries.

These systems have an elegant generalisation to $(2+1)$-dimensions [3]. By replacing the term $\lambda^m \partial_x$ in (3) by $\lambda^m \partial_y$ one naturally obtains examples of $(2+1)$-dimensional integrable systems. Thus the DNLS equation has the following generalisation:

$$\begin{aligned} i\partial_t\psi &= \partial_{xy}\psi + 2i\partial_x[V.\psi]\,, \\ \partial_x V &= \partial_y|\psi|^2\,. \end{aligned}$$

These may be given a twistorial description by introducing a weighted twistor space defined by $\mathbb{T}_{m,p} = \{(Z_0, Z_1, Z_2, Z_3)\}/\sim$, where Z_0, Z_1 are coördinates on the Riemann sphere, Z_2, $Z_3 \in \mathbb{C}$, and $\sim$ is the equivalence relation

$$(Z_0, Z_1, Z_2, Z_3) \sim (\mu Z_0, \mu Z_1, \mu^m Z_2, \mu^p Z_3)\,, \quad \forall \mu \in \mathbb{CP}^1\,.$$

Reimposing the symmetry $\partial_x = \partial_y$ corresponds to factoring out by a non-vanishing holomorphic vector field on $\mathbb{T}_{m,p}$ to recover $\mathcal{O}(m+p)$, exactly analogous to the construction of the minitwistor space $\mathcal{O}(2)$ from standard twistor space.

Ian Strachan

References

[1] L.J.Mason and G.A.J.Sparling: ‘*Twistor Correspondences for the Soliton Hierarchies* in *Journal of Geometry and Physics*, April 1992.

[2] A.Crumey: ‘*Integrable Hierarchies, Homogeneous Spaces and Kac-Moody Algebras*’ Leeds preprint, Jan.1992.

[3] I.A.B.Strachan: ‘*The Twistor Description of some Integrable models in* $(2+1)$*-dimensions*’ Durham Preprint, Sept.1991.