Cohomology of the scalar product diagram in higher dimensions

As an example for the cohomological treatment of twistor diagrams in higher dimensions (cf. [1]) we discuss the scalar product of helicity $-(1+\frac{r}{2})$ massless fields "based on a line $\mathcal{L}$". Essentially we can just adapt [2] to the case of arbitrary dimensions. We also refer to [2] for notation.
Choosing homogeneous coordinates for $\mathbf{C}P^n, n > 3$, we let π_n be the forgetful map $\pi_n : \mathbf{C}P^n - \mathbf{C}P^{n-4} \to \mathbf{C}P^3$. We write $\mathcal{L}^{n-2} := \pi_n^{-1}(\mathcal{L}^1) \cup \mathbf{C}P^{n-4}$ where $\mathcal{L}^1$ is a line in $\mathbf{C}P^3$. The fibration

$$\begin{array}{ccccc} \mathbf{C}^{n-3} & \to & \mathbf{C}P^n - \mathcal{L}^{n-2} & \ni & [(Z^0,\ldots,Z^3,\ldots,Z^n)] \\ & & \pi_n \downarrow & & \downarrow \\ & & \mathbf{C}P^3 - \mathcal{L}^1 & \ni & [(Z^0,\ldots,Z^3)] \end{array} \tag{1}$$

induces an injection

$$\pi_n^* : H^1(\mathbf{C}P^3 - \mathcal{L}^1; \mathcal{O}(r)) \hookrightarrow H^1(\mathbf{C}P^n - \mathcal{L}^{n-2}; \mathcal{O}(r)), \quad r \in \mathbf{Z}, \tag{2}$$

in the following way: $(\pi_n^{-1}U_1, \pi_n^{-1}U_2)$ is a Stein cover for $\mathbf{C}P^n - \mathcal{L}^{n-2}$ if (U_1, U_2) is a Stein cover for $\mathbf{C}P^3 - \mathcal{L}^1$. If f_{12} is a Čech representative for $f \in H^1(\mathbf{C}P^3 - \mathcal{L}^1; \mathcal{O}(r))$ then $f_{12} \circ \pi_n$ is a representative for $\pi_n^* f$.
As in the case $n = 3$ we have

$$\begin{array}{ll} H^k(\mathbf{C}P^n; \mathcal{O}(r)) = 0, & \text{if } 0 < k < n, \\ H^0(\mathbf{C}P^n - \mathcal{L}^{n-2}; \mathcal{O}(r)) = H^0(\mathbf{C}P^n; \mathcal{O}(r)), & \\ H^k(\mathbf{C}P^n - \mathcal{L}^{n-2}; \mathcal{O}(r)) = 0 & \text{for } k \geq 2. \end{array} \tag{3}$$

Let $X := \mathbf{C}P^n, U := \mathbf{C}P^n - \mathcal{L}^{n-2}$. The relative cohomology exact sequence

$$\begin{array}{ccccccc} & 0 & \to & H^0(X; \mathcal{O}(r)) & \xrightarrow{\cong} & H^0(U; \mathcal{O}(r)) & \to \\ \to & H^1(X, U; \mathcal{O}(r)) & \to & \underbrace{H^1(X; \mathcal{O}(r))}_{=0} & \to & H^1(U; \mathcal{O}(r)) & \to \\ \to & H^2(X, U; \mathcal{O}(r)) & \to & \underbrace{H^2(X; \mathcal{O}(r))}_{=0} & \to & H^2(U; \mathcal{O}(r)) & \to \end{array} \tag{4}$$

then gives us

$$\begin{array}{l} H^1(U; \mathcal{O}(r)) \cong H^2(X, U; \mathcal{O}(r)), \\ H^0(X, U; \mathcal{O}(r)) = H^1(X, U; \mathcal{O}(r)) = 0. \end{array} \tag{5}$$

Thus, by the Künneth formula,

$$\begin{array}{rl} & H^1(\mathbf{C}P^n - \mathcal{L}_1^{n-2}; \mathcal{O}(r)) \otimes H^1(\mathbf{C}P^{n*} - \mathcal{L}_2^{n-2*}; \mathcal{O}(r)) \\ \cong & H^4(\mathbf{C}P^n \times \mathbf{C}P^{n*}, \mathcal{L}_1^{n-2} \times \mathcal{L}_2^{n-2*}; \mathcal{O}(r,r)), \end{array} \tag{6}$$

as a straightforward extension of proposition 3.1. in [2].
The "higher dimensional propagator"

$$h^n_{-r} = \frac{(n+r)!\,\mathcal{D}^n Z \mathcal{D}^n W}{(2\pi i)^{n-3}(Z^i W_i)^{n+1+r}} \in H^0(\mathbf{C}P^n \times \mathbf{C}P^{n*} - \Sigma; \Omega^{2n}(-r,-r)) \tag{7}$$

(where $n+1+r>0$ and Σ is the singularity set of h^n_{-r}) together with a contour in

$$\begin{array}{cl} & H_{2n+4}(\mathbf{C}P^n \times \mathbf{C}P^{n*} - \Sigma, \mathcal{L}_1^{n-2} \times \mathcal{L}_2^{n-2*} - \Sigma; \mathbf{C}) \\ \overset{\text{Thom}}{\cong} & H_{2(n-2)}(\mathcal{L}_1^{n-2} \times \mathcal{L}_2^{n-2*} - \Sigma; \mathbf{C}) \end{array} \tag{8}$$

induces a continuous functional on (6) (see §3.2. in [2]) and hence via π_n^* (and its analogue for $\mathbf{C}P^{n*}$ which we again denote by π_n^*) a continuous functional

$$F_n : H^1(\mathbf{C}P^3 - \mathcal{L}_1^1; \mathcal{O}(r)) \otimes H^1(\mathbf{C}P^{3*} - \mathcal{L}_2^{1*}; \mathcal{O}(r)) \to \mathbf{C}. \tag{9}$$

If $\mathcal{L}_1^{n-2}$ and $\mathcal{L}_2^{n-2*}$ are in general position, i.e.

$$(\mathcal{L}_1^{n-2})^\perp \cap \mathcal{L}_2^{n-2*} = \emptyset, \tag{10}$$

then $V := \mathcal{L}_1^{n-2} \times \mathcal{L}_2^{n-2*} - \Sigma$ has the topology of

$$\{([Z],[W]) \in \mathbf{C}P^{n-2} \times \mathbf{C}P^{n-2*} \mid Z^i W_i \neq 0\}$$

which fibres over $\mathbf{C}P^{n-2}$ with contractible fibre $\mathbf{C}^{n-2}$:

$$\begin{array}{ccccc} \mathbf{C}^{n-2} & \to & V & \ni & ([Z],[W]) \\ & & \downarrow & & \downarrow \\ & & \mathbf{C}P^{n-2} & \ni & [Z] \end{array} \tag{11}$$

so that $H_*(V;\mathbf{C}) = H_*(\mathbf{C}P^{n-2};\mathbf{C})$. Therefore there is a unique contour $C \sim \mathbf{C}P^{n-2}$, $[C] \in H_{2(n-2)}(V;\mathbf{C})$, which associates the functional F_n of (9) to the kernel (7).

It remains to be seen that this functional does indeed coincide with the scalar product: We assume that the field

$$f_r^1 = \partial^* f_r^0 \in H^1(\mathbf{C}P^3 - \mathcal{L}_1^1; \mathcal{O}(r)) \tag{12}$$

is given as image under the Mayer-Vietoris map ∂^* of

$$f_r^0 \in H^0(\mathbf{C}P^3 - H_1^2 - H_2^2; \mathcal{O}(r)), \tag{13}$$

where the hyperplanes H_1^2, H_2^2 define $\mathcal{L}_1^1 = H_1^2 \cap H_2^2$, and similarly for g_r^1 based on $\mathcal{L}_2^{1*} = H_3^{2*} \cap H_4^{2*}$. Then $\pi_n^* f_r^1 = \partial_n^* \pi_n^* f_r^0$ (with ∂_n^* defined in the obvious way) and by theorem 1 of [3], carried over to higher dimensions, $F(f_r^1, g_r^1)$ can be evaluated as an integral

$$\int_{(S^1)^4 \times C} (\pi_n^* f_r^0)(Z)\,(\pi_n^* g_r^0)(W)\, h^n_{-r}(Z,W) \tag{14}$$

over an $(S^1)^4$-bundle. Let for example $H_{1,2}^2, H_{3,4}^{2*}$ be given by

$$\begin{array}{l} H_{i+1}^2 = \{[Z] \in \mathbf{C}P^3 \mid Z^i = 0\}, \\ H_{i+3}^{2*} = \{[W] \in CP^{3*} \mid W_i = 0\}, \end{array} \quad i = 0, 1; \tag{15}$$

and define $(S^1)^4 \times C$ in these coordinates as

$$\left\{\begin{array}{l} ([Z],[W]) \in \mathbf{C}P^n \times \mathbf{C}P^{n*} \mid \\ [Z] = [(\epsilon e^{i\phi_0}, \epsilon e^{i\phi_1}, ra_2, ra_3, \sqrt{1-r^2}a_4, \ldots, \sqrt{1-r^2}a_n)], \\ [W] = [(\epsilon e^{i\psi_0}, \epsilon e^{i\psi_1}, r\bar{a}_2, r\bar{a}_3, \sqrt{1-r^2}\bar{a}_4, \ldots, \sqrt{1-r^2}\bar{a}_n)]. \\ a_i = r_i e^{i\phi_i}; a_2\bar{a}_2 + a_3\bar{a}_3 = a_4\bar{a}_4 + \ldots + a_n\bar{a}_n = 1; \\ r, r_i \in [0,1]; \phi_i, \psi_i \in [0, 2\pi]. \end{array}\right\} \qquad (16)$$

We could now just insert elementary states, based on $\mathcal{L}_1^1, \mathcal{L}_2^{1*}$, for $f_r^0(Z), g_r^0(W)$ to verify agreement with the scalar product on a dense subset (cf.[4]). Alternatively, using the fact that $\pi_n^* f_r^0, \pi_n^* g_r^0$ are constant along the fibres of π_n (i.e. they do not depend on the var iables $Z^4, \ldots, Z^n; W_4, \ldots, W_n$), one can try to reduce (14) to the familiar $\mathbf{C}P^n \times \mathbf{C}P^{n*}$-integral which is known to represent the scalar product in the case $r > -4$. Integrating out the $2n-7$ variables $r_5^2, \ldots, r_n^2; \phi_4, \ldots, \phi_n$ we are left with

$$\frac{(n+r)!}{(n-4)!}\int_{(S^1)^4 \times CP^1 \times [0,\infty]} \frac{f_r^0(Z) g_r^0(W)(Z^2W_2 + Z^3W_3)^{n-3} D^3 Z D^3 W\, t^{n-4} dt}{(Z^0W_0 + Z^1W_1 + (t+1)(Z^2W_2 + Z^3W_3))^{n+1+r}}$$

where $t = \frac{r^2}{1-r^2}$ and $(S^1)^4 \times \mathbf{C}P^1$ is the standard contour for the scalar product-twistor diagram in $\mathbf{C}P^3 \times \mathbf{C}P^{3*}$. $n-3$ partial integrations w.r.t. t give the standard integral for this diagram if $r > -4$. Of course it would be desirable to establish the n-independence of the functionals F_n (9) and other relations between representatives of functionals in different dimensions (see [1]) in more abstract terms, probably starting off from [5].

We note however that an increase in dimensions genuinely widens the possibility of representing such functionals by twistor diagrams without boundaries. Moreover, one could try to accommodate some additional space-time structure by a more essential use of the extra dimensions.

References

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