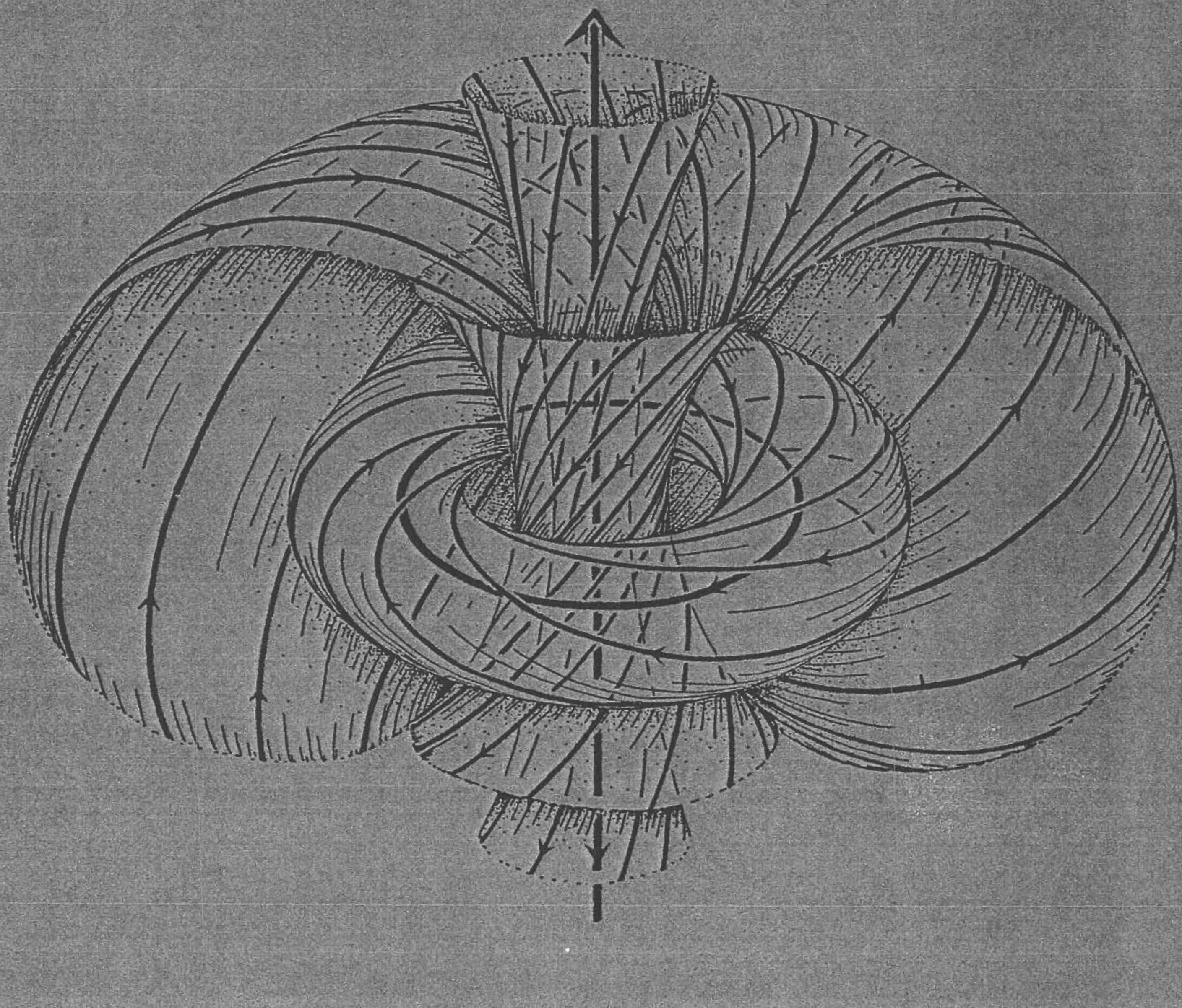
Twistor Newsletter (no 34: 11 May, 1992)



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The most general {22} self-dual vacuum: a googly approach

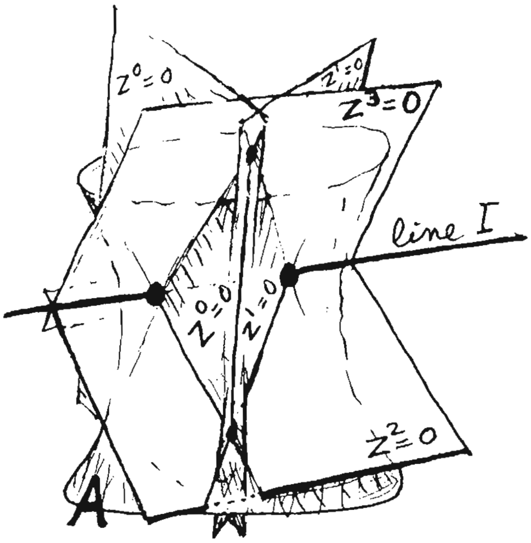
Twistor functions f of homogeneity degree -6 can be used to generate curved dual twistor spaces and, thence, self-dual (complex) vacuum "space-times" $\mathcal{M}$. There are some plausible advantages in proceeding in this "googly" fashion (cf. R.P. in TN 16,17,23). Any symmetry that the twistor function f possesses will be carried over to a symmetry of $\mathcal{M}$, and there is a natural specialization to the linearized limit. Moreover, since in linear theory a pole-multiplicity m in the -6 homogeneity function f is reflected in the (linearized) Weyl curvature possessing a $(5-m)$-fold principal null direction (cf. R.P., J. Math. Phys. 10 (1969) 38-9), we may anticipate something similar occurring in the fully non-linear self-dual case. We have been able to carry through this programme explicitly in the case when f is the inverse cube of a generic quadratic form, and find that, indeed, the resulting self-dual $\mathcal{M}$ has {2,2} Weyl tensor, being the self-dual specialization of the Plebański-Demiański (Annals of Physics 98 (1976) 98-127) solution, which is itself a NUT-type generalization of Levi-Civita's C-metric. (Our solution belongs to the class described by Tod and Ward, Proc. Roy. Soc. Lond. A 368 (1979) 411-427, and also one by Fette, Janis and Newman, J. Math. Phys 17 (1976) 660.)

The general quadratic form $A_{\alpha\beta}Z^\alpha Z^\beta$, in the twistor variable Z^α can be put in the form

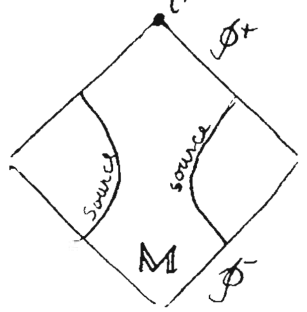
$$A_{\alpha\beta}Z^\alpha Z^\beta = Z^0Z^1 + a^2 Z^2 Z^3$$
$$= \omega^0\omega^1 + a^2\pi_{0'}\pi_{1'} \quad = \frac{1}{2}(Z^0 \dots Z^3)\begin{pmatrix} & 1 & & \\ 1 & & & \\ & & & a \\ & & a & \end{pmatrix}\begin{pmatrix} Z^0 \\ \vdots \\ Z^3 \end{pmatrix}$$

by a suitable choice of twistor coordinates ($Z^0=\omega^0$, $Z^1=\omega^1$, $Z^2=\pi_{0'}$, $Z^3=\pi_{1'}$) and our twistor function can be taken as

$$f = \left(\frac{k}{A_{\alpha\beta}Z^\alpha Z^\beta}\right)^3 = \frac{k^3}{(Z^0Z^1 + a^2Z^2Z^3)^3}.$$



This function has just 3-fold poles lying along the quadric **A** given by $A_{\alpha\beta}Z^\alpha Z^\beta = 0$. The line **I** is given by $Z^2 = Z^3 = 0$, and we have taken the generic case whereby **I** meets **A** in two distinct points. If we put f into a twistor integral, we derive the linearized Weyl spinor of a linearized self-dual C-metric (uniformly accelerating NUT-like source). We have included



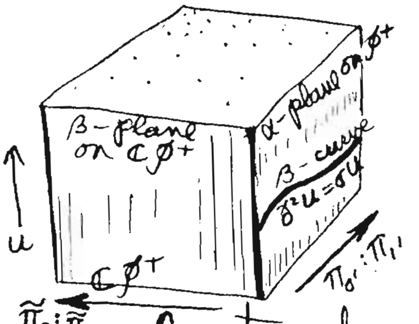
"a" as an explicit parameter so as to allow for the limit $a \to 0$, which gives an elementary state in linear theory and the Eguchi-Hanson-Sparling-Tod solution in the full theory (K.P.T., G.B.-S., R.P. in TN 9).

We do not recover the linearized or full Schwarzschild-NUT specialization with our particular coordinate choice, according to which the line I would lie <u>on</u> A and the twistor function could be written in the form $k(Z^0Z^2 + Z^1Z^3)^{-3}$. (This solution might encounter difficulties with a googly-type approach in any case since its $\mathbb{C}\mathscr{I}^+$ "looks" flat.) We do recover <u>flat</u> space, however, taking $k \to 0$.

The procedure is first to derive a "Bondi" $\sigma\ (=\sigma^\circ)$ on $\mathbb{C}\mathscr{I}^+$ (identified with the standard $\mathbb{C}\mathscr{I}^+$ for $\mathbb{CM}$ by matching the two at i^+) according to the formula

$$\frac{\partial^2\sigma}{\partial\omega^A\partial\omega^B} = \oint \tilde{\pi}_A\tilde{\pi}_B\, f(\omega^A + \lambda\tilde{\pi}^A, \pi_{A'})\, d\lambda .$$

We then solve the Newman equation "$\eth^2 u = \sigma u$" to derive the dual twistor lines (β-curves) on $\mathbb{C}\mathscr{I}^+$. The space of these lines provides us with a curved **dual** projective



twistor space $\mathbb{P}\mathcal{T}^*$, and we find patching relations for $\mathcal{T}^*$ explicitly. Holomorphic cross-sections of $\mathcal{T}^*$, i.e. "good cuts" of $\mathbb{C}\mathscr{I}^+$, give us the points of $\mathcal{M}$, and these can also be found explicitly.

A standard u-coordinate for $\mathbb{C}\mathscr{I}^+$ for $\mathbb{M}$ can be given as

$$u = i\tilde{\omega}^{A'}\pi_{A'} = -i\tilde{\pi}_A\omega^A$$

(where the flat space twistor $(\omega^A, \pi_{A'})$ and dual twistor $(\tilde{\pi}_A, \tilde{\omega}^{A'})$ are incident and define a null ray in $\mathbb{C}\mathbb{M}$ meeting $\mathbb{C}\mathscr{I}^+$ in a point with coordinate u). We can specialize either to $\pi_{0'} = 1$ or $\pi_{1'} = 1$ and either to $\tilde{\pi}_0 = 1$ or $\tilde{\pi}_1 = 1$, if desired, but we prefer to consider u as a homogeneity $(1,1)$ weighted expression and define a $(0,0)$ quantity ρ by

$$\frac{u^2}{\pi_{0'}\pi_{1'}\tilde{\pi}_0\tilde{\pi}_1} = \frac{1}{\rho}(\rho + a^2)^2 .$$

Note that u is invariant under $\rho \mapsto a^4/\rho$.

It turns out that with our choice of f, the solution of Newman's equation for the β-curves on $\mathbb{C}\mathscr{I}^+$ is given by

$$F\frac{\pi_{1'}}{\pi_{0'}} = (w + P)\left(\frac{w+Q}{w-Q}\right)^{a^2/Q}$$

— or the same with $\pi_{0'}$ and $\pi_{1'}$ interchanged — where w is related to ρ by

$$\rho = \frac{w^2 - Q^2}{2(w+P)} ,$$

Q being a constant

$$Q^2 = a^4 - 8k^3 ,$$

and F and P being functions of $\tilde{\pi}_0, \tilde{\pi}_1$ (weights $(0,0)$ each).

We have ρ unchanged under

$$w \mapsto \hat{w} = -\frac{Pw + Q^2}{w + P} .$$

This leads to the patching relation

$$\boxed{\hat{W}_0 = W_0 ,\quad \hat{W}_1 = W_1 ,\quad \hat{W}_2 = \left(1 + Q\frac{W_0W_1}{W_2W_3}\right)^{\frac{1}{2}\left(1-\frac{a^2}{Q}\right)} W_2 ,\quad \hat{W}_3 = \left(1 + Q\frac{W_0W_1}{W_2W_3}\right)^{\frac{1}{2}\left(\frac{a^2}{Q}-1\right)} W_3}$$

where

$$W_0 = \tilde{\pi}_0, \; W_1 = \tilde{\pi}_1, \quad P - Q = \frac{2W_2W_3}{W_0W_1}, \quad \frac{W_2}{W_3} = \frac{Q-P}{F}, \quad \frac{\hat{W}_2}{\hat{W}_3} = \frac{\hat{F}}{Q-P}$$

and

$$\hat{P} = P, \quad \hat{F} = (P^2 - Q^2)\left(\frac{P+Q}{P+Q}\right)^{a^2/Q} \cdot \frac{1}{F}$$

(opposite choice of $\pi_{0'}, \pi_{1'}$ in β-curve equation for $\hat{F}$). The patching relation between $\hat{W}_\alpha$ and W_α defines our required dual twistor space $\mathcal{T}^*$. We have $d\hat{W}_2 \wedge d\hat{W}_3 = dW_2 \wedge dW_2$ on constant W_0, W_1, as required.

Note that $\hat{W}_2\hat{W}_3 = W_2W_3$, and that the two commuting symmetries

$$W_0 \mapsto \alpha W_0, \; W_1 \mapsto \alpha^{-1}W_1 \quad \text{and} \quad W_2 \mapsto \beta W_2, \; W_3 \mapsto \beta^{-1} W_3$$

hold for $\mathcal{T}^*$ and therefore give two commuting Killing vector symmetries for $\mathcal{M}$. The holomorphic cross-sections for $\mathcal{T}^*$ can be found explicitly. The Weyl curvature for $\mathcal{M}$ is indeed self-dual $\{22\}$, the solution being of Plebanski-Demianski type.

It is of interest to examine the weak field limit, where $Q = a^2 + \varepsilon$ (ε small). We find

$$\hat{W}_2 = W_2\left(1 + \frac{\varepsilon}{2a^2}\log\left(1 + a^2\frac{W_0W_1}{W_2W_3}\right)\right)$$

$$\hat{W}_3 = W_3\left(1 - \frac{\varepsilon}{2a^2}\log\left(1 + a^2\frac{W_0W_1}{W_2W_3}\right)\right)$$

with $\hat{W}_0 = W_0$, $\hat{W}_1 = W_1$. This comes from the standard formula

$$\hat{W}_\alpha = W_\alpha + \varepsilon I_{\alpha\beta}\frac{\partial}{\partial W_\beta}\tilde{f}(W)$$

where

$$2\tilde{f}(W) = B\log B - C\log C - D\log D$$

with

$$B = W_0W_1 + \frac{1}{a^2}W_2W_3, \quad C = \frac{W_2W_3}{a^2}, \quad D = W_0W_1,$$

so $B = C + D$, ensuring $+2$ homogeneity. The term $B\log B$ is the kind of thing one expects from twistor transform considerations in relation to our original $f(Z)$, since $B^{\alpha\beta}$ is the inverse of $A_{\alpha\beta}$, where $B = B^{\alpha\beta}W_\alpha W_\beta$.

Linda Hackebunt & Roger Penrose

A comment on the preceding article

In the preceding article, Linda Haslehurst and Roger Penrose derive the twistor space of a self-dual type D vacuum metric. As they remark, their solution is one of a class constructed by Tod and Ward from solutions of Laplace's equation in three variables, and by following through the analysis in [1], one can find an explicit form of the metric by solving a linear splitting problem.

Since the solution has two commuting Killing vectors, it also has a 'Yang-Mills' twistor description [2]: the solution is encoded in a holomorphic vector bundle over a one-dimensional non-Hausdorff Riemann surface and can be recovered from its patching matrix P, a 2×2 matrix-valued holomorphic function of a single complex variable w [3,4]. The purpose of this note is to describe the connection between the two twistor constructions. The example is of interest in the context of the Yang-Mills construction because it illustrates what seems to be a general property of type D solutions, that their patching matrices are rational functions of a particularly simple form. Thomas von Schroeter has verified this by direct calculation in the case of the Lorentzian type D metrics, which are known explicitly, but the geometric reason for the special rational form is not yet clear. This self-dual example is another instance of the phenomenon, and it is one in which the underlying geometry is rather easier to understand.

To fit with the conventions of [4], I shall consider the dual version of the solution, for which the twistor space is determined by the coordinate relations

$$\hat{Z}^0 = \left(1 + \frac{Q}{w}\right)^{p/Q} Z^0, \; \hat{Z}^1 = \left(1 + \frac{Q}{w}\right)^{-p/Q} Z^1, \; \hat{Z}^2 = Z^2, \; \hat{Z}^3 = Z^3,$$

where $w = Z^0Z^1/Z^2Z^3$ and $p = \frac{1}{2}(Q - a^2)$. The surfaces of constant w are the leaves of the foliation of $\mathbb{P}\mathcal{T}$ spanned by the Killing vectors. The connection between the two constructions is made by considering the holomorphic tangent bundle of $\mathcal{T}$. This is the pull-back to $\mathcal{T}$ of the rank-4 bundle $E \to \mathbb{P}\mathcal{T}$ that has local sections of the form

$$A^\alpha \frac{\partial}{\partial Z^\alpha},$$

where the As are holomorphic functions of the Zs, homogeneous of degree zero: E is the Ward transform of the anti-self-dual Yang-Mills connection that defines local twistor transport in space-time.

Let $L \to \mathbb{P}\mathcal{T}$ be the line bundle with sections represented by holomorphic functions of degree one in Z^α. Construct an open cover of $\mathbb{P}\mathcal{T}$ by taking V_0, V_1, V_2, V_3 to be suitable neighbourhoods of $Z^0 = 0$, $\hat{Z}^1 = 0$, $Z^2 = 0$, and $\hat{Z}^3 = 0$, and trivialize $E \otimes L$ by the four frame fields defined on the respective neighbourhoods by

$$\begin{array}{ll} (V_0) & X,\ \dfrac{Z^0}{w}\dfrac{\partial}{\partial Z^0},\ -Z^2\dfrac{\partial}{\partial Z^2},\ X - Z^3\dfrac{\partial}{\partial Z^3} \\ (V_1) & \dfrac{\hat{Z}^1}{w}\dfrac{\partial}{\partial \hat{Z}^1},\ X,\ X - \hat{Z}^2\dfrac{\partial}{\partial Z^2}, -\hat{Z}^3\dfrac{\partial}{\partial \hat{Z}^3} \\ (V_2) & X,\ Z^0\dfrac{\partial}{\partial Z^0},\ -wZ^2\dfrac{\partial}{\partial Z^2},\ X - Z^3\dfrac{\partial}{\partial Z^3} \\ (V_3) & \hat{Z}^1\dfrac{\partial}{\partial \hat{Z}^1},\ X,\ X - \hat{Z}^2\dfrac{\partial}{\partial Z^2}, -w\hat{Z}^3\dfrac{\partial}{\partial \hat{Z}^3}. \end{array}$$

where $X = -Z^0\partial/\partial Z^0 + Z^1\partial/\partial Z^1 = -\hat{Z}^0\partial/\partial\hat{Z}^0 + \hat{Z}^1\partial/\partial\hat{Z}^1$ (the generator one of the symmetries). The corresponding transition matrices are $P_{20} = \mathrm{diag}(1, w^{-1}, w^{-1}, 1)$, $P_{31} = \mathrm{diag}(w^{-1}, 1, 1, w^{-1})$, and

$$P_{01} = \begin{pmatrix} g(w) & 1 & wg(w) & wg(w) \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

where $g(w) = (q + w)/w(Q + w)$, with $q = \frac{1}{2}(Q + a^2)$. We interpret w as the coordinate on the non-Hausdorff reduced twistor space.

Note that the vectors parallel to $Z^\alpha\partial/\partial Z^\alpha$ span a line sub-bundle of E is isomorphic to L^{-1}, and that $F := E/L^{-1} = T\mathbb{P}\mathcal{T} \otimes L^{-1}$.

The general anti-self-dual vacuum metric with two commuting orthogonally transitive Killing vectors is

$$\mathrm{d}s^2 = f(\mathrm{d}t - \omega\mathrm{d}\theta)^2 + f^{-1}(\mathrm{d}r^2 + \mathrm{d}z^2 + r^2\mathrm{d}\theta^2)\ (*)$$

where ω and f are functions of r and z such that $f^2\omega_z = rf_r$ and $f^2\omega_r = -rf_z$. By considering local twistor transport in such a background, one can show that the bundle $E \otimes L$ always has transition matrices of this form, where in the general case $g(z) = f(0, z)^{-1}$. The top left-hand two-by-two block in P_{01}

is the patching matrix P, which characterizes the solution uniquely. We can immediately read off, therefore, that the type D solution in the preceding article is given by (*), with

$$\frac{1}{f(0,z)} = \frac{q}{Qz} + \frac{p}{Q(Q+z)}.$$

Since $f^{-1}(r,z)$ is an axisymmetric harmonic function in cylindrical polars, we conclude that it is the potential of a pair of point masses p/Q and q/Q separated by Q.

A vacuum solution of the form (*) is type D if it admits a non-null Killing spinor ω^{AB} (this is a nontrivial condition, although there are many Killing spinors with primed indices whatever the form of the curvature). A Killing spinor with unprimed indices determines a holomorphic section A of the symmetric tensor product $F \otimes_S F = L^{-2} \otimes (T\mathbb{P}T \otimes_S T\mathbb{P}T)$. By dropping the fourth element of the local frame for E in V_0 and V_2, and the third element in V_1 and V_3, we can represent $L \otimes F = T\mathbb{P}T$ by the transition matrices $M_{20} = \mathrm{diag}(1, w^{-1}, w^{-1})$, $M_{31} = \mathrm{diag}(w^{-1}, 1, w^{-1})$, and

$$M_{01} = \begin{pmatrix} g(w) & 1 & -wg(w) \\ 1 & 0 & -2w \\ 0 & 0 & 1 \end{pmatrix}$$

A section of $F \otimes_S F$ has local representatives $A_i(w)$, where $i = 0, 1, 2, 3$ and the As are symmetric 3×3 matrices. For a global section

$$A_2 = wM_{20}A_0M_{20}^t, \ A_3 = wM_{31}A_1M_{31}^t, \ A_0 = M_{01}A_1M_{01}^t,$$

with A_2 and A_3 well behaved near $w = \infty$, and A_0 and A_1 well behaved near $w = 0$. The first two transition relations imply that the dependence of A_0 and A_1 on w must be of the form

$$A_0 = \begin{pmatrix} 0 & C & C \\ C & L & L \\ C & L & L \end{pmatrix}, \ A_1 = \begin{pmatrix} L & C & L \\ C & 0 & C \\ L & C & L \end{pmatrix}$$

where C means 'constant' and L means 'linear'. After a little algebra, the third relation then implies that g must be of the form $g = g_1(w)/g_2(w)$, where g_1 is at most linear in w and g_2 is at most quadratic. For the solution in the previous article, $g_1 = q - pw$ and $g_2 = Qw + w^2$: this is essentially the generic case. For the 'Euclidean Taub-NUT' solution, $g_1 = w$ and $g_2 = w + 2m$.

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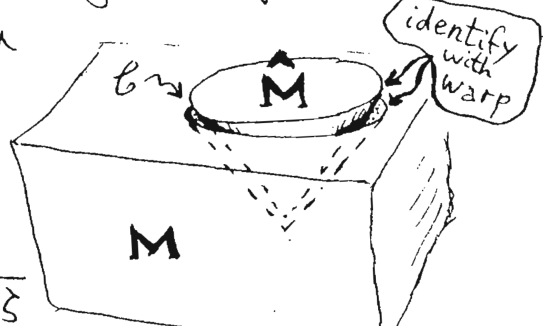
Nick Woodhouse

On Impulsive Gravitational Waves

The general pure δ-function solution of the Einstein vacuum equations was described in Penrose (1972), using a "scissors and paste" construction. Such waves can have plane or spherical wave fronts, the former being limiting cases of the latter. We discuss only the spherical case here. The scissors and paste description can be given by matching a region $\mathbf{M}$ of Minkowski space outside a (say future) light cone $\mathscr{C}$, with metric



$$ds^2 = 2\,du\,dv - 2u^2\,d\zeta\,d\bar\zeta$$

to another Minkowskian region $\hat{\mathbf{M}}$ inside $\mathscr{C}$, with metric

$$ds^2 = 2\,d\hat u\,d\hat v - 2\hat u^2\,d\hat\zeta\,d\bar{\hat\zeta},$$

where the metric identification ("warp") at $\mathscr{C}$ is given by

$$\hat v = 0 = v$$
$$\hat\zeta = f(\zeta) \qquad\qquad *$$
$$\hat u = u/|f'(\zeta)|,$$

f being holomorphic, apart from at its singular regions, which may be thought of as singular "wires" on $\mathscr{C}$. The Einstein vacuum equations are satisfied across $\mathscr{C}$, along which there is a δ-function in the Weyl curvature.

It is of interest to note that the (u, ζ) transformation can be described neatly as a non-linear holomorphic transformation of the spin-space, at the vertex $\mathbf{O}$ of $\mathscr{C}$, preserving

$$\boxed{\xi_A\,d\xi^A} \quad \text{i.e. both of } \boxed{d\xi_A \wedge d\xi^A} \text{ and } \boxed{\xi^A \frac{\partial}{\partial \xi^A}}.$$

Here, the position vector of a point on the cone is given by $\xi^A \bar\xi^{A'}$ $(= x^a)$, which in coordinates is $(\frac{1}{\sqrt2}\,x)$

$$\begin{bmatrix} u & \bar{\zeta}u \\ \zeta u & \zeta\bar{\zeta}u \end{bmatrix} = \begin{bmatrix} \xi^0\bar{\xi}^{0'} & \xi^0\bar{\xi}^{1'} \\ \xi^1\bar{\xi}^{0'} & \xi^1\bar{\xi}^{1'} \end{bmatrix},$$

so $*$ becomes $\xi^A \mapsto \hat{\xi}^A$ according to

$$\xi^0 \mapsto \hat{\xi}^0 = \xi^0 (f'(\zeta))^{-1/2}$$
$$\xi^1 \mapsto \hat{\xi}^1 = \xi^0 f(\zeta)(f'(\zeta))^{-1/2}$$

where
$$\zeta = \xi^1/\xi^0$$

i.e.
$$\zeta \mapsto \hat{\zeta} = f(\zeta)$$
$$\eta \mapsto \hat{\eta} = \eta / f'(\zeta)$$
where $\begin{pmatrix} \xi^0 \\ \xi^1 \end{pmatrix} = \begin{pmatrix} \eta^{1/2} \\ \zeta\eta^{1/2} \end{pmatrix}$.

This preserves

$$\xi_A d\xi^A = \xi^0 d\xi^1 - \xi^1 d\xi^0 = (\xi^0)^2 d\left(\frac{\xi^1}{\xi^0}\right) = \eta\, d\zeta .$$

This fact is closely related to the Hamiltonian nature of the twistor transformation between $\mathbf{M}$'s and $\hat{\mathbf{M}}$'s twistor spaces, that was noted in Penrose & MacCallum (1972).

The metric on the entire space $\mathcal{M} = \mathbf{M} \cup \mathcal{C} \cup \hat{\mathbf{M}}$ can be described as a $\mathbf{C}^0$ metric form

$$ds^2 = 2\,du\,dv - 2\left|u\,d\bar{\zeta} + v\{h;\zeta\}\,d\zeta\right|^2$$

where $\{\;;\;\}$ stands for the Schwarzian derivative

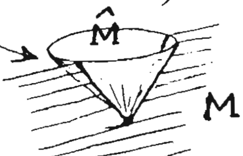
$$\{h;\zeta\} = -\frac{1}{2}\left(\frac{h_{\zeta\zeta\zeta}}{h_\zeta} - \frac{3}{2}\left(\frac{h_{\zeta\zeta}}{h_\zeta{}^2}\right)^2\right).$$

The curvature is defined by the only surviving Weyl component

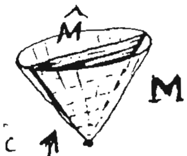
$$\Psi_4 = \frac{1}{u}\{h;\zeta\}_{\zeta}\,\delta(v)$$

in a suitable spin frame (o^A pointing along generators of $\mathcal{C}$). (Nutku 1990.)

A simple example is given by a "snapping cosmic string" (also described, in a different C^0 way by Gleiser & Pullin 1989), where $f(\zeta) = \zeta^{1+\varepsilon}$. Here, the "wires" at the north and south poles arise from a deficit-angle identification for a cosmic string in **M** which snaps, in $\mathcal{M}$, emitting a gravitational wave along $\mathcal{C}$

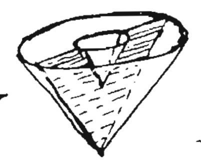


— or else in $\hat{\mathbf{M}}$, where here the cosmic string is created by the gravitational wave along $\mathcal{C}$

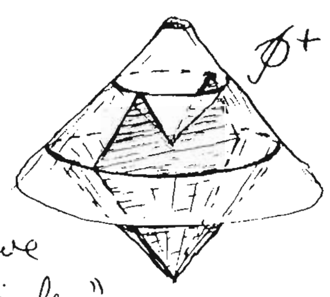


— or, analogously, we could snap or create a so-called "rotating" cosmic string if we allow ε to be complex.

An interesting consideration is that of energy balance, e.g. in the situation

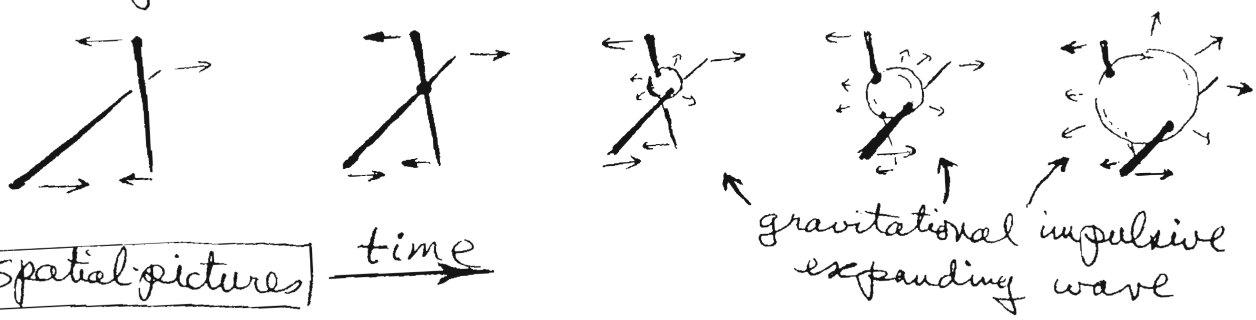


, where two string segments separate with the speed of light, having previously been joined as one string that was created with the emission of the first of the two gravitational wave bursts. Here the gravitational energy in the waves is infinite owing to an angular divergence, though the time-integral of energy flux along each generator of $\mathscr{I}^+$ is proportional to the length of the string segments.

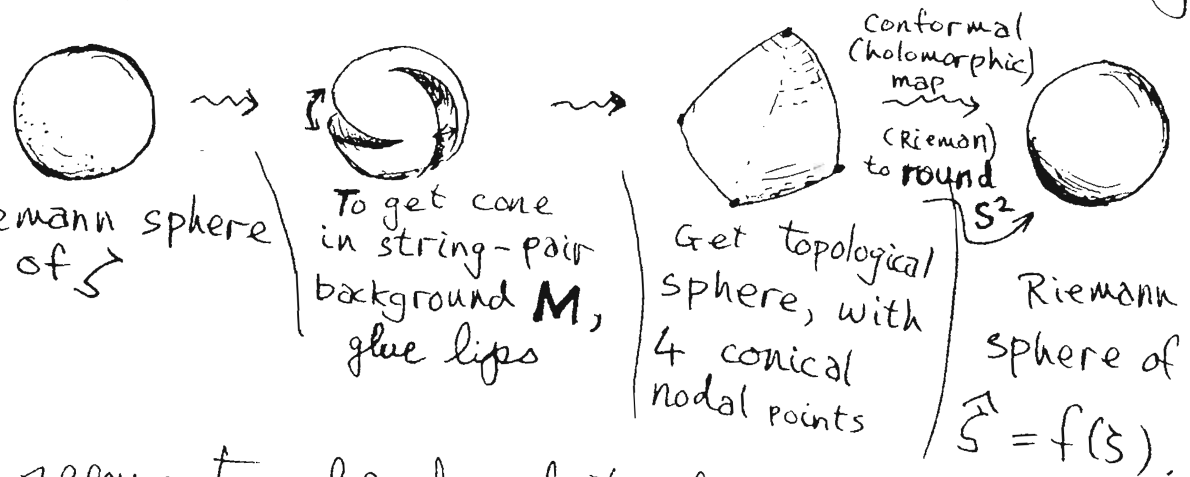


To make sense of all this, we must observe that the string segments have "particles" at their ends, where for a string of positive tension (and positive mass) the leading particle has a negative mass that decreases (becomes more negative) with time and the trailing one has a positive mass that increases with time; for a string of negative tension (i.e. positive pressure and hence negative mass) it is the other way around. (All the masses for these "particles" are inertial masses; they have zero rest-mass.)

A more involved situation is provided by a pair of strings that collide



Here the function h is obtained in the following way:



It seems to be hard to find f explicitly for this case, but a Riemann theorem ensures that f exists.

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Yavuz Nutku & Roger [illegible]

Kinking and Causality

Andrew Chamblin and Roger Penrose

Introduction

Recently, there has been some speculation along the following lines:

Suppose M is a compact spacetime, with

$$\partial M \cong \Sigma \ncong \emptyset$$

(Σ may be single three-manifold or the disjoint union of several). Let v be a timelike vector field with respect to the Lorentz metric, and let $\text{kink}(\partial M; v)$ denote the kinking number of v with respect to ∂M (see [1] or [2]).

Recently, there has been some suspicion that there may be a relation between the topology of ∂M, along with the value of $\text{kink}(\partial M; v)$, and the existence of closed timelike curves in M. In particular it has been conjectured that if $\partial M \cong S^3$ and $\text{kink}(\partial M; v) = 0$, then there must exist closed timelike curves in M (M assumed to be space and time orientable).

In this paper, we show that the above conjecture is false (by counterexample). In fact, we prove the more general

Proposition 1 Let Σ be any closed, orientable three-manifold, $n \in \mathbb{Z}$ an arbitrary integer. Then there exists a compact causal spacetime M with $\partial M \cong \Sigma$ and $\text{kink}(\partial M; v) = n$, where v is a timelike vector field.

Proposition 2 If M is compact and causality violating, with $\partial M \cong \Sigma \ncong \emptyset$, then there exists a continuous deformation of the metric on M such that the new spacetime with deformed metric does not possess closed timelike curves.

(Note: Deforming the metric does not alter the kinking number).

The proofs of Propositions 1 and 2 draw on the idea of the counterexample.

Construction of Counterexample

To construct the example, consider the manifold

$$M \cong \mathbb{CP}^2 \# (S^1 \times S^3) \tag{1}$$

where $\#$ denotes "connected sum". Let $e(M)$ = "Euler number of M", then $e(M) = e(\mathbb{CP}^2) + e(S^1 \times S^3) - 2$.

Since $e(\mathbb{CP}^2) = 3$ and $e(S^1 \times S^3) = 0$ we find

$$e(M) = 1 \tag{2}$$

Define the manifold M' by

$$M' \cong M - D^4 , \tag{3}$$

where D^4 is a four-ball. Then

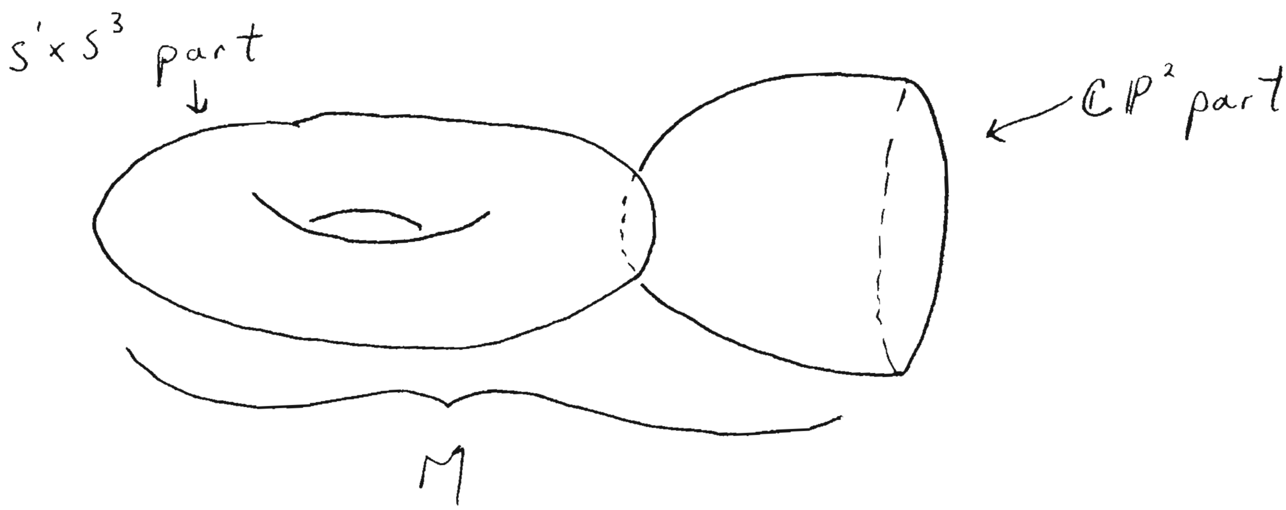
$$e(M') = 0 \tag{4}$$

Thus, we can put a nonvanishing vector field v on M' which has zero kinking on $\partial M' \cong S^3$, i.e.,

$$\text{kink}(\partial M'; v) = 0 \tag{5}$$

Now, one may suppose that there are closed timelike curves in M'; in fact, if v is outward normal on $\partial M'$ there <u>must</u> exist closed timelike curves (by a standard argument). However, we shall now show that we can always "cut" all of the closed timelike curves in $M' \cong M - D^4$ by choosing the D^4 that we remove from M cleverly. We shall do this "choosing" in an essentially constructive manner.

Hence, take $M \cong \mathbb{CP}^2 \# (S^1 \times S^3)$ as above and let v be a vector field on M, i.e., visually:



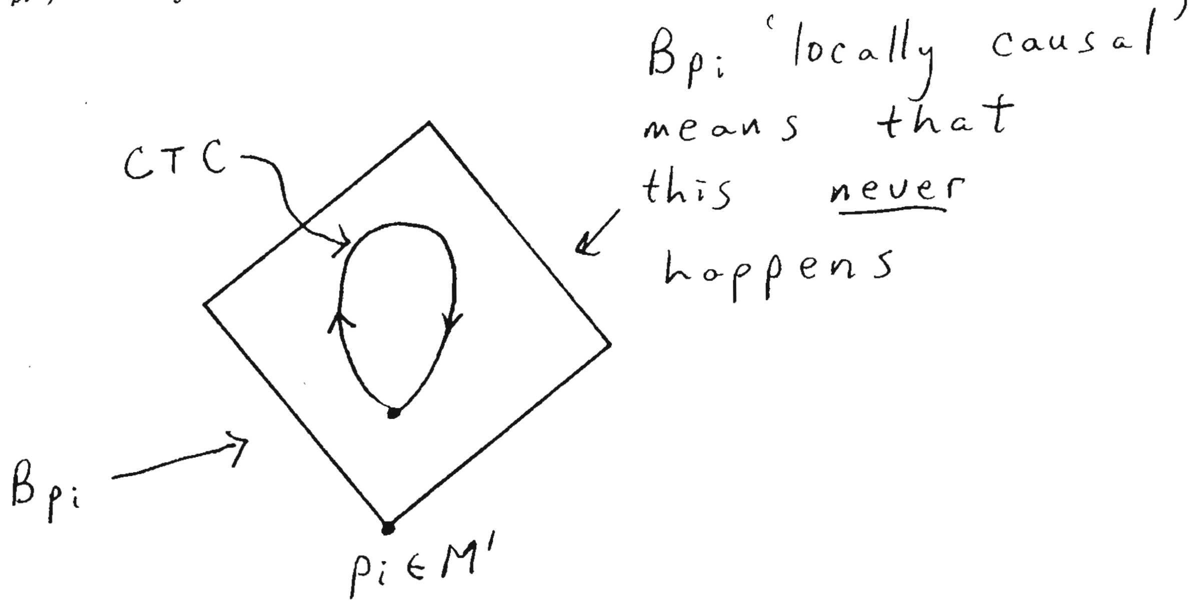
Remove a ball D^4 from around the singular point of v, so that

$$\partial D^4 \cong \partial M' \cong S^3.$$

Now, we can cover M' with a finite number of sets B_{pi} of the form

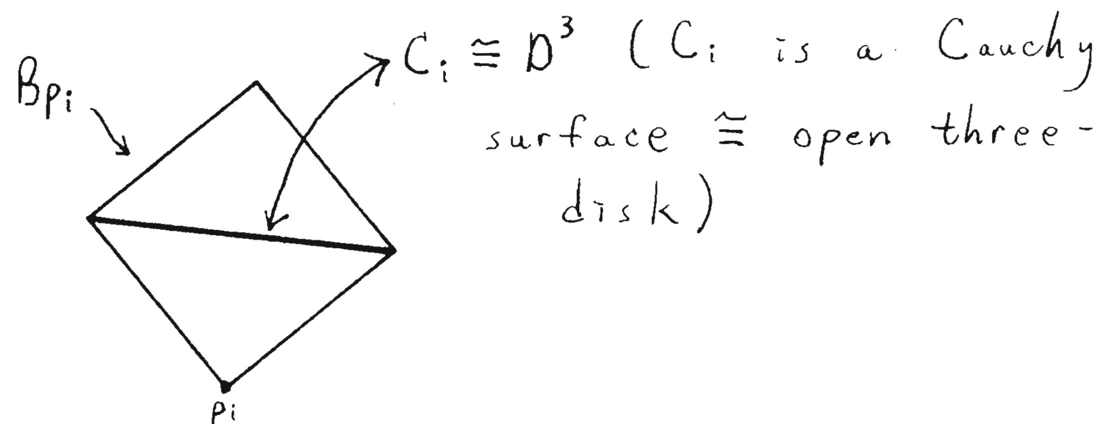
$$B_{pi} = \{x \in I^{+}(p_i) \cap I^{-}(q) \mid q \in I^{+}(p_i)\}$$

Furthermore, we can take the sets in this finite cover to be fine enough that they are all locally causal (i.e., no closed timelike curve, or CTC, lies entirely in any one of the B_{pi}s). Visually:

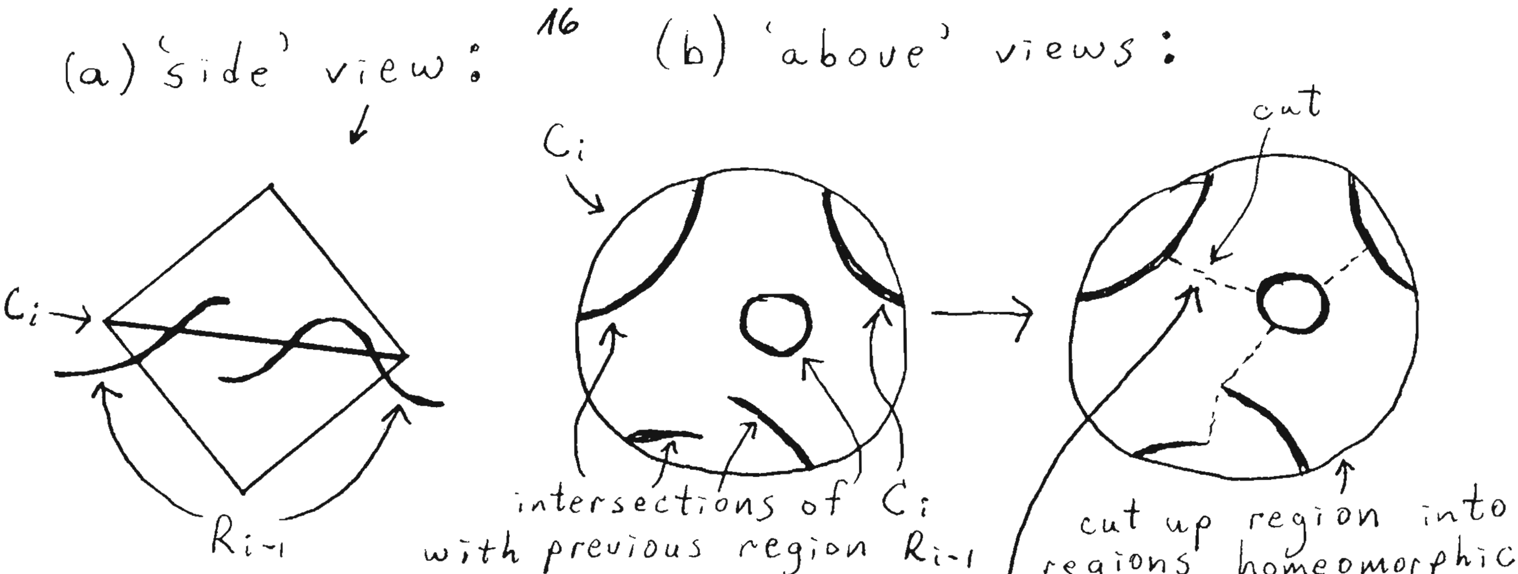


Now, the crucial idea of the construction depends upon our ability to cut all of the CTCs by removing a finite number of four - balls. That we can do this is reasonably intuitively obvious, but we justify this construction more rigorously as follows.

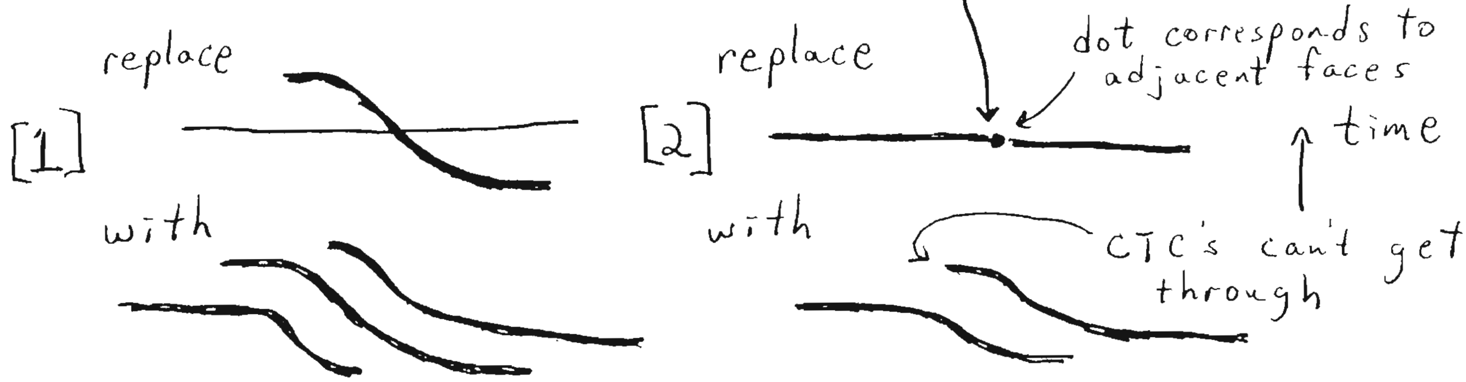
Begin by successively removing the '$t = 0$' Cauchy surface, C_i from each of our locally causal covering sets B_{pi}, as shown:



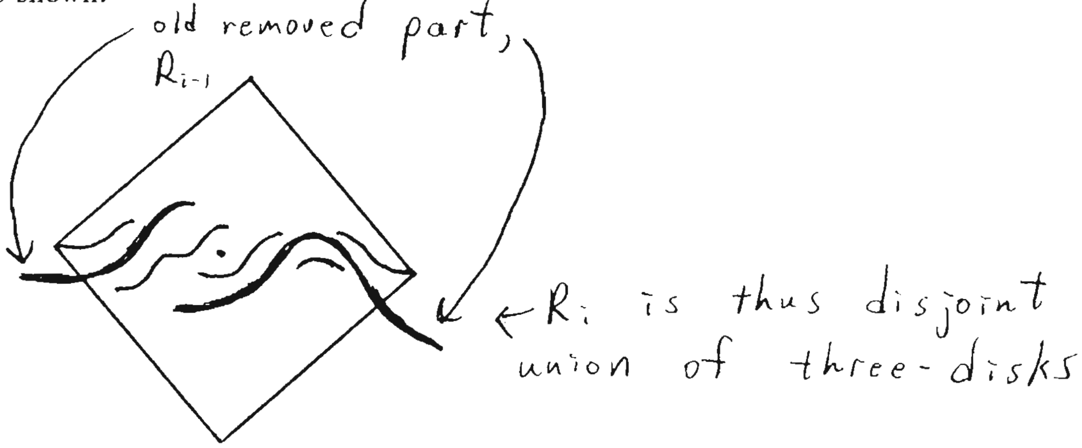
Now, at each stage C_i may already be intersected by a previously removed part (assumed to be a union of three-disks), R_{i-1}, so subdivide to get a covering of what's left by three-disks, (D^3s), as shown:



Next, modify C_i according to the following two rules:

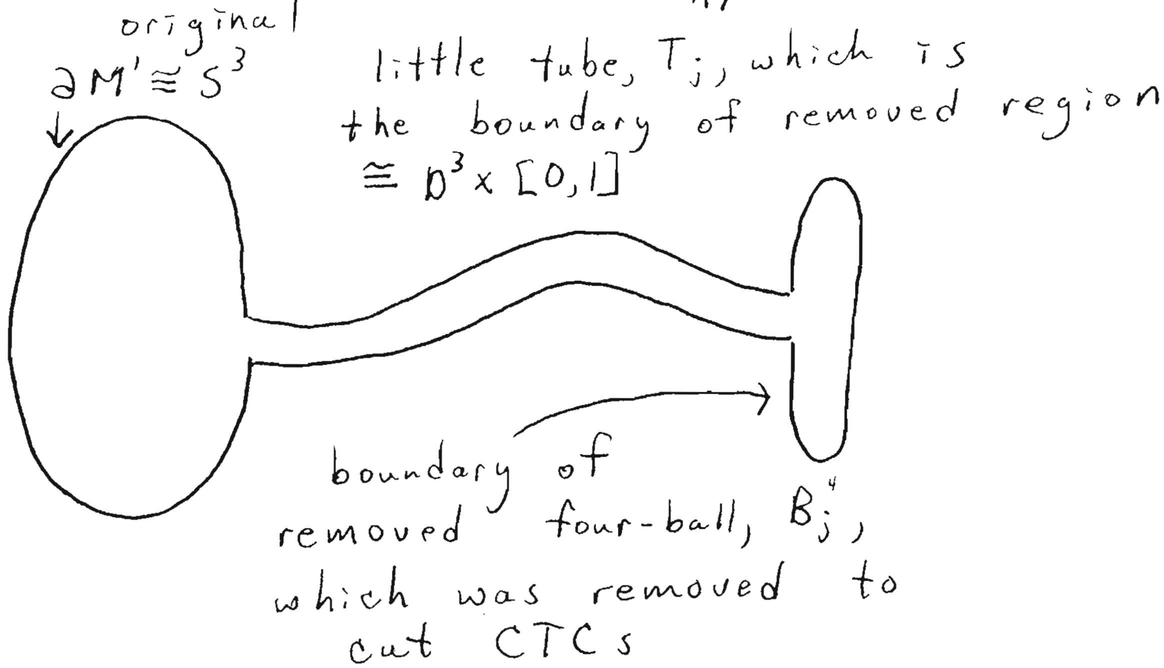


Adjoin the result to R_{i-1} to get R_i, which is thus given as a disjoint union of three-balls, D^3_j, as shown:



Finally, thicken out the D^3_js to get disjoint four-balls B^4_js which clearly cut the CTCs.

Hence, we can cut all of the CTC's with a finite number of such four - balls. We now connect each of these 'cut out regions' to the original deleted region (i.e., where D^4 was) via 'little tubes' $T_j \cong S^3 \times [0,1]$; that is, we cut out a little tube leading from the old boundary of M' ($\partial M' \cong \partial D^4$) to the new boundary component formed by removing B^4_j, as shown:



Call the new manifold obtained after such a finite sequence of operations 'N'. Then clearly

$$\partial N \cong S^3$$

since the total topology of the removed regions

$$R \cong D^4 \cup T_1 \cup T_2 \cup ... \cup T_n \cup B_1^4 \cup B_2^4 \cup ... \cup B_n^4$$

is still D^4, and $\partial D^4 \cong S^3$. Furthermore, v is still global and nonvanishing on N, and $e(N) = 0$; hence, $\text{kink}(\partial N; v) = 0$.

Thus, N is a causal spacetime (which is orientable) for which $\partial N \cong S^3$ and $\text{kink}(\partial N; v) = 0$; hence, N constitutes a counterexample to the conjecture mentioned in the introduction.

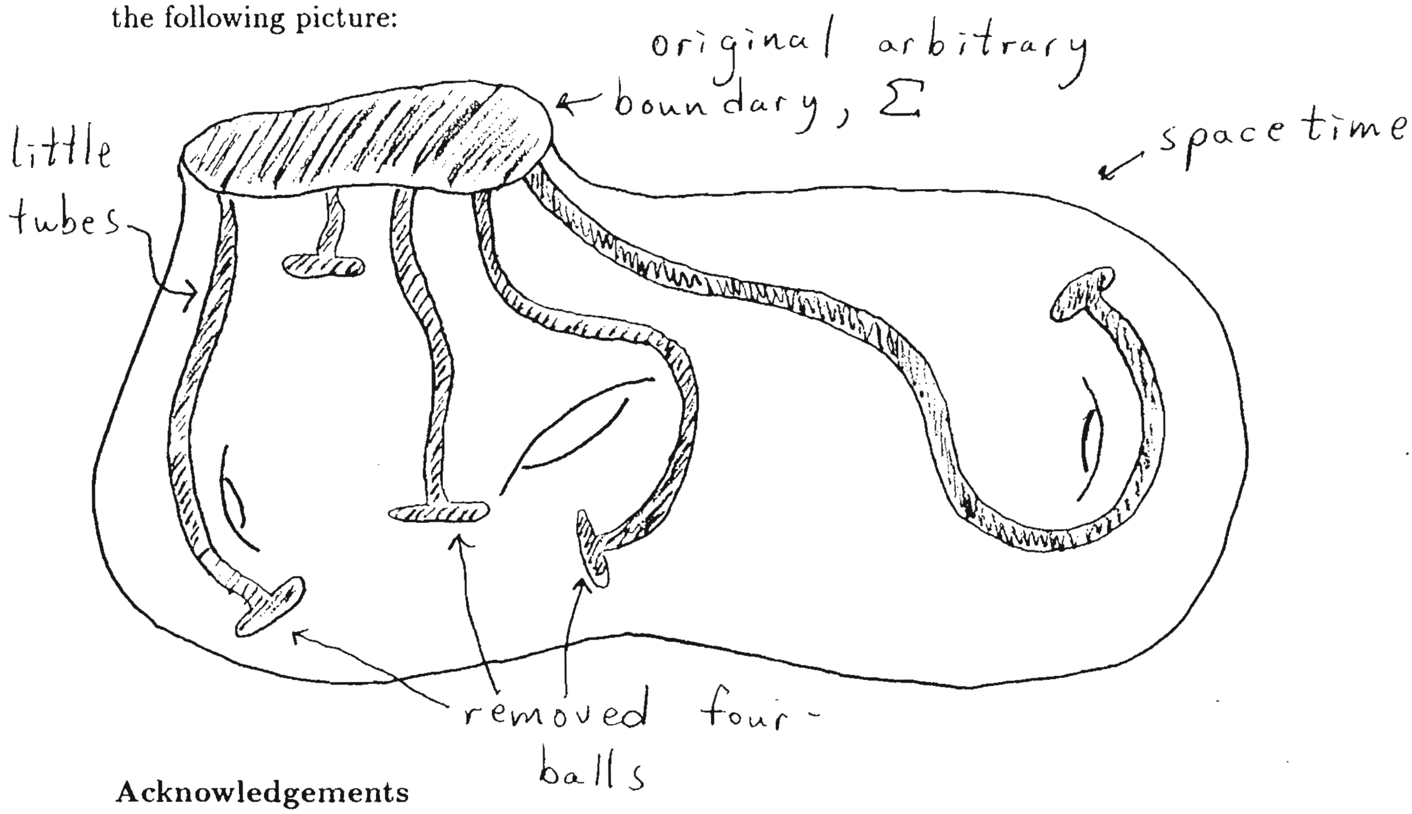
Proof of the general statement

To prove the more general Propositions (stated in the introduction) we simply generalize the above construction.

That is, let Σ be any three-manifold (or perhaps disjoint union of three-manifolds) and $n \in \mathbb{Z}$ any integer. Then we can always find a Lorentz manifold M (with timelike vector v) such that $\partial M \cong \Sigma$ and $\text{kink}(\partial M; v) = n$. This follows from the general formula

$$e(M) = \Sigma i_v \; + \; \text{kink}(\partial M; v) \tag{6}$$

(see [2]). If M should happen to possess CTC's we can always do the above construction and "cut" them by removing a finite number of four - balls B_j^4 and connecting these four - balls to the original boundary by removing little (nonintersecting) tubes T_j. (For Proposition 2, we simply continuously retract the T_js and the $B_j^4 s$, dragging the metric with them). The fundamental idea of this paper, then, is represented in the following picture:



Acknowledgements

The authors wish thank Dr. R.P.A.C. Newman for helpful discussions. Also thanks to Jo Ashbourn for help in preparing this paper.

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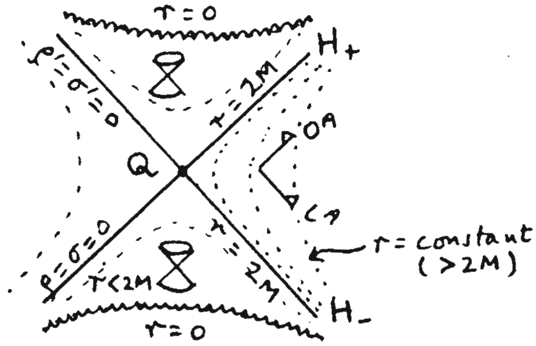
GROWING THE KERR GEOMETRY FROM SEED
PART ONE: FIRST CHOOSE A SUITABLE SEED

We take as our starting point Kruskal's map of the (analytically extended) Schwarzschild geometry.



In the diagram, θ and ϕ are suppressed. The future and past horizons (H_+ and H_-) appear in the diagram as null lines, but they are actually null hypersurfaces with "cylindrical" topology S^2xR. In each case the null generators are all parallel to each other (ρ, σ = 0 on H_+, ρ', σ' = 0 on H_-). The intersection Q of the two hypersurfaces is a spacelike 2-surface, which (in this case) is a sphere of radius 2M.

The geometry of the spacetime may be regarded as being determined by <u>initial data</u> on Q and on H_+ and H_-, the null hypersurfaces emanating orthogonally from Q. The data which is required consists of Ψ_0 on H_+, Ψ_4 on H_-, and $\rho, \sigma, \rho', \sigma'$ on Q, together with the intrinsic geometry of Q and its "complex curvature" K (the real part of which is half the Gaussian curvature of Q). If the spacetime is to correspond to a stationary black hole (with H_+ and H_- of constant area) all of the data must be zero except for K, which in this case is the same as $-\Psi_2$. Thus any stationary black hole corresponds to an especially simple set of initial data. Hawking [1] used the fact that the only non-zero datum is Ψ_2 to show that a rotating, stationary black hole must be axisymmetric (there has to be a second Killing vector, distinct from the time-translation Killing vector, which at H_+ and H_- points along the generators of the horizons).

For the Schwarzschild geometry, the initial data, the "seed" of the geometry, is simply the above-mentioned sphere, with $\Psi_2 = -1/8M^2$ (no imaginary part).

We note that at Q the curvature spinor is algebraically special (of type D):

$$\Psi_{ABCD} = 6\Psi_2 \, o_{(A} o_B \iota_C \iota_{D)}$$

(Ψ_1 and Ψ_3 vanish as well as Ψ_0 and Ψ_4 because $\rho, \sigma, \rho', \sigma'$ are zero). In the case of Schwarzschild the type D property extends from the "seed" throughout the whole spacetime. We are going to investigate the question: What other "seeds" grow into spacetimes that are type D?

We obtain a necessary condition for this by looking at the GHP equations for a type D spacetime and picking out the ones that can be applied "intrinsically" to Q. These are

$$ð\Psi_2 = 3\tau\Psi_2 \qquad \text{and} \qquad ð\tau = \tau^2$$

together with their primed versions. (Note that Ψ_2 has spin-weight 0 since $\Psi_2' = \Psi_2$). Letting $X = \Psi_2^{-1/3}$, we have

$$ðX = -X\tau \qquad \text{and} \qquad ð^2 X = 0 .$$

Thus we are seeking a surface Q whose complex curvature to the power of −1/3 satisfies $ð^2 X = ð'^2 X = 0$. (This problem has already been considered in a similar context by Ludvigsen [2]. However, he needed to impose an extra, arbitrary condition, namely $\oint \mathrm{Im}[X]dS = 0$, in order to arrive at the Kerr horizon. This does not seem to be necessary: see paragraph (5) below.)

Note that $\mu = \mathrm{Re}[X]$ is a real solution of $ð^2\mu = 0$. Now $ð^2\mu = 0$ can be solved on any surface, but a real solution (not a constant) gives rise to an isometry of Q. Using μ as one of the coordinates it turns out that the metric must have the form

$$ds^2 = d\mu^2/F(\mu) + F(\mu)\, d\phi^2 .$$

The Gaussian curvature of such a metric is

$$G = -\tfrac{1}{2}\, d^2F/d\mu^2 .$$

If the real part of X is constant the argument fails but we can use the imaginary part for μ, instead. If both real and imaginary parts are non-constant, they both give rise to isometries. Assuming that Q is not a sphere, both isometries must be the same, and in all cases we have that X must be of the form

$$X = A + B\mu,$$

with A and B complex constants (there is no implication, at this stage, that B must be pure imaginary: c.f.[2]). Next, we equate the above expression for G with the real part of $-2\Psi_2$:

$$-\tfrac{1}{2}d^2F/d\mu^2 = -2\,\mathrm{Re}[1/X^3] = -2\,\mathrm{Re}[1/(A + B\mu)^3] .$$

Integrating (and disallowing B = 0, the case of the sphere), we get

$$F = 2\,\mathrm{Re}[1/(B^2(A + B\mu))] + C\mu + D$$

with C and D real constants. This may be described as the local solution of the problem. At this point there are 6 real degrees of freedom. All but 2 of these degrees of freedom can be removed by applying appropriate global conditions, as follows.

1) We want $F(\mu)$ to have two zeroes, corresponding to the North and South poles of the surface Q.

2) We want ϕ to range from 0 to 2π. This can be achieved by replacing μ by a real constant times μ.

3) A coordinate transformation $\mu \to \mu$ + constant gives rise to a new F with different A and D, but to the same geometry. So without loss of generality we may take the zeroes of F to be at $\mu = \pm\mu_0$.

4) The Gauss-Bonnet theorem, generalized to the complex curvature, tells us

$$\oint \Psi_2 \, dS = -2\pi.$$

This amounts to two real conditions on A, B, C and D.

5) The condition $\oint G dS = 4\pi$ fails to exclude the possibility of equal and opposite <u>conical singularities</u> at the poles of our axisymmetric surface. This possibility must be excluded by one extra condition, namely, $dF/d\mu = -2$ at $\mu = \mu_0$.

When these constraints are worked through in detail we are left with an F of the form

$$F = \mu_0 \, (1 + u^2) \, (\mu_0{}^2 - \mu^2)/(\mu_0{}^2 + u^2\mu^2)$$

with u and μ_0 real parameters. Making the substitutions

$$\mu = \mu_0 \cos\theta; \quad u = a/r_+; \quad \mu_0 = r_+{}^2 + a^2,$$

we can bring the metric to the form

$$ds^2 = (r_+{}^2+a^2\cos^2\theta) \, d\theta^2 + (r_+{}^2+a^2)^2 \sin^2\theta \, d\phi^2/(r_+{}^2+a^2\cos^2\theta)$$

which is indeed the metric of the horizon of a Kerr black hole with mass $M = (r_+{}^2+a^2)/2r_+$ and angular momentum $J = aM$. Note that any spatial cross-section of H_+ or H_- has the same shape as Q.

We are now in a position to "grow" the geometry out from Q using the radial Newman-Penrose (or GHP) equations.

George Burnett-Stuart

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Many thanks to R.P. and especially Paul Tod for pointing me towards reference [2].

The Bogomolny Hierarchy and Higher Order Spectral Problems.

I.A.B.Strachan

The starting point for the construction and solution of a wide range of integrable models is to write the equation as the integrability condition for the otherwise overdetermined linear system (where $\lambda \in \mathbb{CP}^1$ is the spectral parameter):

$$\begin{aligned} \partial_x s &= -U(\lambda).s\,, \\ \partial_t s &= -V(\lambda).s\,. \end{aligned} \tag{1}$$

The integrability conditions for (1) is

$$\partial_x V - \partial_t U + [U,V] = 0\,, \tag{2}$$

and equating powers of λ (if U and V are polynomial in λ) yields the equation in question. Many of those systems which are known to have a twistorial description (such as the KdV, mKdV, NLS, SG and N-wave equations) arise from a so-called first order spectral problem, with

$$\begin{aligned} U &= \lambda A + Q(x,t)\,, \\ V &= \sum_i \lambda^i A_i(x,t)\,. \end{aligned}$$

In this article the matrices will be taken to be $sl(2,\mathbb{C})$-valued, with* $A = \begin{pmatrix} \kappa & 0 \\ 0 & -\kappa \end{pmatrix}$ and $Q = \begin{pmatrix} 0 & p \\ q & 0 \end{pmatrix}$, i.e. $A \in \boldsymbol{h}$ and $Q \in \boldsymbol{k}$, where $\boldsymbol{h}$ is the Cartan subalgebra and $\boldsymbol{k}$ is the complement. A higher order spectral problem is one for which U and V are general polynomial functions, namely:

$$\begin{aligned} U &= \lambda^p A + \lambda^{p-1} Q_1 + \ldots + Q_p\,, \\ V &= \lambda^n V_0 + \lambda^{n-1} V_1 + \ldots + V_n\,. \end{aligned}$$

The simplest example ($p = 2, n = 4$ and $Q_2 = 0$) results in the derivative Non-Linear Schrödinger (or DNLS) equation. The purpose of this article is two-fold: firstly to show how such systems are nothing more than a reduction of the Bogomolny hierarchy introduced in [1], and secondly to generalise these systems to $(2+1)$-dimensions while retaining their integrability.

$\star$ In the terminology of [1], the fields are of type β; type α fields will not be considered here.

The following method to generate the matrices $Q_1, \ldots, Q_p, V_0, \ldots, V_n$ for these higher order problems is due to Crumey [2]. Let

$$u = \lambda^p . A \,,$$
$$v = \lambda^n . A \,.$$

These trivially satisfy (2). However, this equation is gauge invariant, so if $\omega(x,t)$ is a λ-dependent gauge transformation (often called a 'dressing transformation'), defined by $\omega = \exp \sum_{i=1}^{\infty} \omega_i \lambda^{-i}$ with $\omega_i \in sl(2, \mathbb{C})$, then U and V, defined by

$$U = \omega u \omega^{-1} - \omega_x \omega^{-1} \,,$$
$$V = \omega v \omega^{-1} - \omega_t \omega^{-1} \,,$$

will also satisfy (2). Assuming that ω is chosen so that U and V involve only non-negative powers of λ yields, on projecting onto positive (including the λ^0 term) and negative powers of λ, the equations

$$U = (\lambda^p \omega A \omega^{-1})_+ \,, \qquad V = (\lambda^n \omega A \omega^{-1})_+ \,,$$
$$\omega_x \omega^{-1} = (\lambda^p \omega A \omega^{-1})_- \,, \qquad \omega_t \omega^{-1} = (\lambda^p \omega A \omega^{-1})_- \,.$$

These simplify further by decomposing ω as $\omega = h.k$, where $h = \sum_{i=1}^{\infty} h_i(x,t)\lambda^{-i}$, $h_i(x,t) \in \boldsymbol{h}$ and $k = \sum_{i=1}^{\infty} k_i(x,t)\lambda^{-i}$, $k_i(x,t) \in \boldsymbol{k}$. One then has

$$U = (\lambda^p k A k^{-1})_+ \,, \qquad V = (\lambda^n k A k^{-1})_+ \,.$$

Let A_{n-i} denote the coefficient of λ^{-i} in the expansion of kAk^{-1} (the reason for this skew choice will become apparent later), i.e.

$$A_{n-i} = \sum_{r=1}^{i} \frac{1}{r!} \sum_{(\{s_j\}:\sum s_j = i)} [k_{s_1}, [k_{s_2}, \ldots, [k_{s_r}, A] \ldots]] \,.$$

From this proceadure one obtains the general form of the functions U and V. The matrices $k_1, \ldots, k_p$ are matrix valued fields. The integrable equation itself (which connects the time evolution of these fields with their spacial derivatives), together with the remaining matrices, may be found using the above equations, or equivalently, equation (2).

Having found the general form of U and V it remains to show how these are contained within the Bogonolny hierarchy. Assuming $m \equiv n - p \geq 0$, the matrix V may be written in the form

$$V = \lambda^m . U + \sum_{i=0}^{m-1} \lambda^i A_i \,,$$

and hence the original system (1) may be rewritten as

$$\begin{aligned} \partial_x s &= -\Big\{ \sum_{i=0}^{p} \lambda^i A_{m+i} \Big\} s \,, \\ \partial_t s - \lambda^m \partial_x s &= -\Big\{ \sum_{i=0}^{m-1} \lambda^i A_i \Big\} s \,. \end{aligned} \tag{3}$$

Recall [1] that given the minitwistor space $\mathcal{O}(n)$, the line bundle over the Riemann sphere of Chern class $n \geq 1$, the Ward construction gives rise to the linear system

$$\{[\partial_{z_i} + A_i] - \lambda[\partial_{z_{i+1}} + B_{i+1}]\} s = 0 \,, \qquad i = 0 \,, \ldots , n-1 \,,$$

where A_i and B_{i+1} are $sl(2,\mathbb{C})$-valued gauge potentials. With the symmetry generated by ∂_{z_n}, together with $B_i = 0 \,, i = 1, \ldots, , n-1 \,, B_n \equiv A_{n+1} \,, A_0 = A$, relabelling $z_0 = t \,, z_m = x$, and eliminating the other variables results in (3) :

$$\left.\begin{aligned} [\partial_t - \lambda\partial_{z_1}]s &= -A.s \\ &\vdots \\ [\partial_{z_{m-1}} - \lambda\partial_x]s &= -A_{m-1}.s \end{aligned}\right\} \Rightarrow [\partial_t - \lambda^m \partial_x]s = -\Big\{ \sum_{i=0}^{m-1} \lambda^i A_i \Big\}.s \,,$$

$$\left.\begin{aligned} [\partial_x - \lambda\partial_{z_{m+1}}]s &= -A_m.s \\ &\vdots \\ [\partial_{z_{n-1}} - \lambda\partial_{z_n}]s &= -[A_{n-1} + \lambda A_n].s \end{aligned}\right\} \Rightarrow \partial_x s = -\Big\{ \sum_{i=0}^{p} \lambda^i A_{m+i} \Big\}.s \,.$$

Thus these higher order spectral problems may all be embedded within the Bogomolny hierarchy. Solutions of the simplest example, that of the DNLS equation, correspond to bundles over the space $\mathcal{O}(4)$ with certain symmetries.

These systems have an elegant generalisation to $(2+1)$-dimensions [3]. By replacing the term $\lambda^m \partial_x$ in (3) by $\lambda^m \partial_y$ one naturally obtains examples of $(2+1)$-dimensional integrable systems. Thus the DNLS equation has the following generalisation:

$$
\begin{aligned}
i\partial_t\psi &= \partial_{xy}\psi + 2i\partial_x[V.\psi]\,, \\
\partial_x V &= \partial_y|\psi|^2\,.
\end{aligned}
$$

These may be given a twistorial description by introducing a weighted twistor space defined by $\mathbb{T}_{m,p} = \{(Z_0, Z_1, Z_2, Z_3)\}/\sim$, where Z_0, Z_1 are coördinates on the Riemann sphere, Z_2, $Z_3 \in \mathbb{C}$, and $\sim$ is the equivalence relation

$$(Z_0, Z_1, Z_2, Z_3) \sim (\mu Z_0, \mu Z_1, \mu^m Z_2, \mu^p Z_3)\,, \quad \forall\mu \in \mathbb{CP}^1\,.$$

Reimposing the symmetry $\partial_x = \partial_y$ corresponds to factoring out by a non-vanishing holomorphic vector field on $\mathbb{T}_{m,p}$ to recover $\mathcal{O}(m+p)$, exactly analogous to the construction of the minitwistor space $\mathcal{O}(2)$ from standard twistor space.

Ian Strachan

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Cohomology of the scalar product diagram in higher dimensions

As an example for the cohomological treatment of twistor diagrams in higher dimensions (cf. [1]) we discuss the scalar product of helicity $-(1+\frac{r}{2})$ massless fields "based on a line $\mathcal{L}$". Essentially we can just adapt [2] to the case of arbitrary dimensions. We also refer to [2] for notation.
Choosing homogeneous coordinates for $\mathbf{C}P^n, n > 3$, we let π_n be the forgetful map $\pi_n : \mathbf{C}P^n - \mathbf{C}P^{n-4} \to \mathbf{C}P^3$. We write $\mathcal{L}^{n-2} := \pi_n^{-1}(\mathcal{L}^1) \cup \mathbf{C}P^{n-4}$ where $\mathcal{L}^1$ is a line in $\mathbf{C}P^3$. The fibration

$$\begin{array}{ccccc} \mathbf{C}^{n-3} & \to & \mathbf{C}P^n - \mathcal{L}^{n-2} & \ni & [(Z^0,\ldots,Z^3,\ldots,Z^n)] \\ & & \pi_n \downarrow & & \downarrow \\ & & \mathbf{C}P^3 - \mathcal{L}^1 & \ni & [(Z^0,\ldots,Z^3)] \end{array} \tag{1}$$

induces an injection

$$\pi_n^* : H^1(\mathbf{C}P^3 - \mathcal{L}^1; \mathcal{O}(r)) \hookrightarrow H^1(\mathbf{C}P^n - \mathcal{L}^{n-2}; \mathcal{O}(r)), \quad r \in \mathbf{Z}, \tag{2}$$

in the following way: $(\pi_n^{-1}U_1, \pi_n^{-1}U_2)$ is a Stein cover for $\mathbf{C}P^n - \mathcal{L}^{n-2}$ if (U_1, U_2) is a Stein cover for $\mathbf{C}P^3 - \mathcal{L}^1$. If f_{12} is a Čech representative for $f \in H^1(\mathbf{C}P^3 - \mathcal{L}^1; \mathcal{O}(r))$ then $f_{12} \circ \pi_n$ is a representative for $\pi_n^* f$.
As in the case $n = 3$ we have

$$\begin{array}{ll} H^k(\mathbf{C}P^n; \mathcal{O}(r)) = 0, & \text{if } 0 < k < n, \\ H^0(\mathbf{C}P^n - \mathcal{L}^{n-2}; \mathcal{O}(r)) = H^0(\mathbf{C}P^n; \mathcal{O}(r)), & \\ H^k(\mathbf{C}P^n - \mathcal{L}^{n-2}; \mathcal{O}(r)) = 0 & \text{for } k \geq 2. \end{array} \tag{3}$$

Let $X := \mathbf{C}P^n, U := \mathbf{C}P^n - \mathcal{L}^{n-2}$. The relative cohomology exact sequence

$$\begin{array}{ccccccc} & 0 & \to & H^0(X;\mathcal{O}(r)) & \overset{\cong}{\to} & H^0(U;\mathcal{O}(r)) & \to \\ \to & H^1(X,U;\mathcal{O}(r)) & \to & \underbrace{H^1(X;\mathcal{O}(r))}_{=0} & \to & H^1(U;\mathcal{O}(r)) & \to \\ \to & H^2(X,U;\mathcal{O}(r)) & \to & \underbrace{H^2(X;\mathcal{O}(r))}_{=0} & \to & H^2(U;\mathcal{O}(r)) & \to \end{array} \tag{4}$$

then gives us

$$\begin{array}{l} H^1(U;\mathcal{O}(r)) \cong H^2(X,U;\mathcal{O}(r)), \\ H^0(X,U;\mathcal{O}(r)) = H^1(X,U;\mathcal{O}(r)) = 0. \end{array} \tag{5}$$

Thus, by the Künneth formula,

$$\begin{array}{rl} & H^1(\mathbf{C}P^n - \mathcal{L}_1^{n-2}; \mathcal{O}(r)) \otimes H^1(\mathbf{C}P^{n*} - \mathcal{L}_2^{n-2*}; \mathcal{O}(r)) \\ \cong & H^4(\mathbf{C}P^n \times \mathbf{C}P^{n*}, \mathcal{L}_1^{n-2} \times \mathcal{L}_2^{n-2*}; \mathcal{O}(r,r)), \end{array} \tag{6}$$

as a straightforward extension of proposition 3.1. in [2].
The "higher dimensional propagator"

$$h^n_{-r} = \frac{(n+r)!\, \mathcal{D}^n Z \mathcal{D}^n W}{(2\pi i)^{n-3} (Z^i W_i)^{n+1+r}} \in H^0(\mathbf{C}P^n \times \mathbf{C}P^{n*} - \Sigma; \Omega^{2n}(-r,-r)) \tag{7}$$

(where $n+1+r>0$ and Σ is the singularity set of h^n_{-r})
together with a contour in

$$\begin{array}{ll} & H_{2n+4}(\mathbf{C}P^n \times \mathbf{C}P^{n*} - \Sigma, \mathcal{L}_1^{n-2} \times \mathcal{L}_2^{n-2*} - \Sigma; \mathbf{C}) \\ \overset{\text{Thom}}{\cong} & H_{2(n-2)}(\mathcal{L}_1^{n-2} \times \mathcal{L}_2^{n-2*} - \Sigma; \mathbf{C}) \end{array} \tag{8}$$

induces a continuous functional on (6) (see §3.2. in [2]) and hence via π_n^* (and its analogue for $\mathbf{C}P^{n*}$ which we again denote by π_n^*) a continuous functional

$$F_n : H^1(\mathbf{C}P^3 - \mathcal{L}_1^1; \mathcal{O}(r)) \otimes H^1(\mathbf{C}P^{3*} - \mathcal{L}_2^{1*}; \mathcal{O}(r)) \to \mathbf{C}. \tag{9}$$

If $\mathcal{L}_1^{n-2}$ and $\mathcal{L}_2^{n-2*}$ are in general position, i.e.

$$(\mathcal{L}_1^{n-2})^\perp \cap \mathcal{L}_2^{n-2*} = \emptyset, \tag{10}$$

then $V := \mathcal{L}_1^{n-2} \times \mathcal{L}_2^{n-2*} - \Sigma$ has the topology of

$$\{([Z],[W]) \in \mathbf{C}P^{n-2} \times \mathbf{C}P^{n-2*} \mid Z^i W_i \neq 0\}$$

which fibres over $\mathbf{C}P^{n-2}$ with contractible fibre $\mathbf{C}^{n-2}$:

$$\begin{array}{ccccc} \mathbf{C}^{n-2} & \to & V & \ni & ([Z],[W]) \\ & & \downarrow & & \downarrow \\ & & \mathbf{C}P^{n-2} & \ni & [Z] \end{array} \tag{11}$$

so that $H_*(V;\mathbf{C}) = H_*(\mathbf{C}P^{n-2};\mathbf{C})$. Therefore there is a unique contour $C \sim \mathbf{C}P^{n-2}$, $[C] \in H_{2(n-2)}(V;\mathbf{C})$, which associates the functional F_n of (9) to the kernel (7).

It remains to be seen that this functional does indeed coincide with the scalar product: We assume that the field

$$f_r^1 = \partial^* f_r^0 \in H^1(\mathbf{C}P^3 - \mathcal{L}_1^1; \mathcal{O}(r)) \tag{12}$$

is given as image under the Mayer-Vietoris map ∂^* of

$$f_r^0 \in H^0(\mathbf{C}P^3 - H_1^2 - H_2^2; \mathcal{O}(r)), \tag{13}$$

where the hyperplanes H_1^2, H_2^2 define $\mathcal{L}_1^1 = H_1^2 \cap H_2^2$, and similarly for g_r^1 based on $\mathcal{L}_2^{1*} = H_3^{2*} \cap H_4^{2*}$. Then $\pi_n^* f_r^1 = \partial_n^* \pi_n^* f_r^0$ (with ∂_n^* defined in the obvious way) and by theorem 1 of [3], carried over to higher dimensions, $F(f_r^1, g_r^1)$ can be evaluated as an integral

$$\int_{(S^1)^4 \times C} (\pi_n^* f_r^0)(Z)\,(\pi_n^* g_r^0)(W)\, h^n_{-r}(Z,W) \tag{14}$$

over an $(S^1)^4$-bundle. Let for example $H_{1,2}^2, H_{3,4}^{2*}$ be given by

$$\begin{array}{l} H_{i+1}^2 = \{[Z] \in \mathbf{C}P^3 \mid Z^i = 0\}, \\ H_{i+3}^{2*} = \{[W] \in CP^{3*} \mid W_i = 0\}, \end{array} \quad i = 0, 1; \tag{15}$$

and define $(S^1)^4 \times C$ in these coordinates as

$$\left\{\begin{array}{l} ([Z],[W]) \in \mathbf{C}P^n \times \mathbf{C}P^{n*} \mid \\ [Z] = [(\epsilon e^{i\phi_0}, \epsilon e^{i\phi_1}, ra_2, ra_3, \sqrt{1-r^2}a_4, \ldots, \sqrt{1-r^2}a_n)], \\ [W] = [(\epsilon e^{i\psi_0}, \epsilon e^{i\psi_1}, r\bar{a}_2, r\bar{a}_3, \sqrt{1-r^2}\bar{a}_4, \ldots, \sqrt{1-r^2}\bar{a}_n)]. \\ a_i = r_i e^{i\phi_i}; a_2\bar{a}_2 + a_3\bar{a}_3 = a_4\bar{a}_4 + \ldots + a_n\bar{a}_n = 1; \\ r, r_i \in [0,1]; \phi_i, \psi_i \in [0, 2\pi]. \end{array}\right\} \qquad (16)$$

We could now just insert elementary states, based on $\mathcal{L}_1^1, \mathcal{L}_2^{1*}$, for $f_r^0(Z), g_r^0(W)$ to verify agreement with the scalar product on a dense subset (cf.[4]). Alternatively, using the fact that $\pi_n^* f_r^0, \pi_n^* g_r^0$ are constant along the fibres of π_n (i.e. they do not depend on the var iables $Z^4, \ldots, Z^n; W_4, \ldots, W_n$), one can try to reduce (14) to the familiar $\mathbf{C}P^n \times \mathbf{C}P^{n*}$-integral which is known to represent the scalar product in the case $r > -4$. Integrating out the $2n-7$ variables $r_5^2, \ldots, r_n^2; \phi_4, \ldots, \phi_n$ we are left with

$$\frac{(n+r)!}{(n-4)!} \int_{(S^1)^4 \times CP^1 \times [0,\infty]} \frac{f_r^0(Z) g_r^0(W) (Z^2 W_2 + Z^3 W_3)^{n-3} \mathcal{D}^3 Z \mathcal{D}^3 W\, t^{n-4} dt}{(Z^0 W_0 + Z^1 W_1 + (t+1)(Z^2 W_2 + Z^3 W_3))^{n+1+r}}$$

where $t = \frac{r^2}{1-r^2}$ and $(S^1)^4 \times \mathbf{C}P^1$ is the standard contour for the scalar product-twistor diagram in $\mathbf{C}P^3 \times \mathbf{C}P^{3*}$. $n-3$ partial integrations w.r.t. t give the standard integral for this diagram if $r > -4$. Of course it would be desirable to establish the n-independence of the functionals F_n (9) and other relations between representatives of functionals in different dimensions (see [1]) in more abstract terms, probably starting off from [5].

We note however that an increase in dimensions genuinely widens the possibility of representing such functionals by twistor diagrams without boundaries. Moreover, one could try to accommodate some additional space-time structure by a more essential use of the extra dimensions.

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Franz Müller

Abstracts

On Bell non-locality without probabilities: some curious geometry

by Roger Penrose,
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Abstract. In 1966, John Bell showed how Gleason's 1957 theorem can be used to demonstrate the incompatibility of the predictions of quantum theory with "non-contextual" hidden variable models. Later, Kochen and Specker independently found a set of 117 (unoriented) spatial directions that exhibited this incompatibility in a finite explicit way. Such configurations have been used (Heywood and Redhead 1983, Stairs 1983, Brown and Svetlichny 1990) as part of an EPR system, to show that the non-contextual assumption can be replaced by one of locality. This, like results obtained recently by Greenberger, Horne, Zeilinger (GHZ) and others illustrates a conflict between quantum mechanics and locality that shows up in yes/no constraints on the results of certain idealized experiments, no probabilities being involved. Kochen and Specker's original set of 117 directions, for a 3-state (spin 1) system , has recently been reduced to 33 by Peres (1990a) (and to 31 by Conway and Kochen). Peres has also exhibited a set of 24 Hilbert-space directions, with similar properties, for a 4-state system, these being the common eigenstates of sets of commuting operators among a set of 9 found by Peres (1990b) (and Mermin). In this article, I show how Peres's set of 33 directions can be directly visualized in terms of a geometrical configuration (three interpenetrating cubes) that appears in the Escher print "Waterfall". Using the Majorana description of general spin states, I also exhibit a quite different set of 33 idealized measurements that can be performed on a spin 1 system. These measurements are specified in terms of an explicit set of 18 oriented directions in space. The configuration involved in Peres's set of 24 Hilbert-space directions can be understood in terms of a 4-dimensional regular polytope known as the "24-cell", and they are, in principle, ideally suited to providing an EPR-type of GHZ non-locality without probabilities. Unfortunately, if each 4-state system is taken to be a spin 3/2 particle, no simple spatial geometrical description of the needed measurements seems to emerge. Instead, I provide an alternative configuration for spin 3/2, based on a regular dodecahedron, in which only 20 oriented directions are explicitly used.

Existence and Deformation Theory for Scalar-Flat Kähler Metrics on Compact Complex Surfaces

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and

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Abstract

Let M^4 be a compact complex 2-manifold which admits a Kähler metric whose scalar curvature has integral zero. Suppose, moreover, that $\pi_1(M)$ does not contain an Abelian subgroup of finite index. Then if M is blown up at sufficiently many points, the resulting complex manifold $\tilde{M}$ admits Kähler metrics with scalar curvature identically zero. The proof, which proceeds by deforming the explicit metrics constructed in [27], hinges on a remarkable relationship between Kodaira-Spencer theory and the Futaki invariant that arises via the Penrose transform. In the process, we point out a relationship between the existence problem for scalar-flat Kähler metrics and the parabolic stability of vector bundles in the sense of Seshadri [38].

1991 Mathematics Subject Classification. Primary: 53C55.
Secondary: 32G05, 32J15, 32L25, 53C25, 58E11, 58H15.
Running title: Scalar-Flat Kähler Surfaces

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POSITIVE EINSTEIN METRICS WITH SMALL $L^{n/2}$-NORM OF THE WEYL TENSOR

Michael Singer

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Abstract: A gravitational analogue is given of Min-Oo's gap theorem for Yang-Mills fields.

Keywords: Riemannian manifold, Einstein metric, Weyl tensor, L^p-norm, Sobolev constant, Euler characteristic.

MS classification: 53C.

INTRODUCTION

In this note we prove

Theorem 1. *Let M be a compact oriented n-manifold ($n = 2m \geq 4$) with non-vanishing Euler characteristic $\chi(M)$ and let g be a (Riemannian) positive Einstein metric on M with Weyl curvature W. Then there is a constant $\varepsilon > 0$, depending only upon n and $\chi(M)$, such that if $||W||_{L^{n/2}} < \varepsilon$, then $W = 0$ (and so M is isometric to a quotient of S^n with the standard metric).*

The Fröhlicher Spectral Sequence on a Twistor Space

Michael G. Eastwood   Michael A. Singer

1 Introduction

Associated to any compact self-dual four-manifold M is a compact complex three-dimensional manifold Z known as its twistor space [1,18]. Twistor spaces provide a source of interesting complex three-manifolds (cf. [6]). The purpose of this article is to investigate the Fröhlicher spectral sequence [9]

$$E_1^{p,q} = H^q(Z, \Omega^p) \Longrightarrow H^{p+q}(Z, \mathbf{C})$$

where Ω^p denotes the sheaf of holomorphic p-forms on Z. The Penrose transform [2,3,4,7,12] interprets the Dolbeault cohomology $H^q(Z, \Omega^p)$ in terms of differential equations on M. In this way, the Fröhlicher spectral sequence has differential-geometric consequences on M and vice versa.

We shall explain this interpretation and its consequences. For, example we shall show that $E_1 = E_\infty$ if and only if a certain conformally invariant system of linear differential equations has only constant solutions. The classical case in which $E_1 = E_\infty$ is when Z admits a Kähler metric. Hitchin [13] has shown that there are only two such twistor spaces, namely $\mathbf{CP}_3$ and the space of flags in $\mathbf{C}^3$. However, we shall construct other twistor spaces with $E_1 = E_\infty$. We shall show that if $E_1 \neq E_\infty$, then $E_2 = E_\infty$ and that this possibility does occur.

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* A special issue to celebrate the 60th birthday of Roger Penrose.

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Puzzle Page

Who wrote this? When was it published?

What does it mean?

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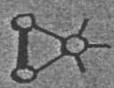
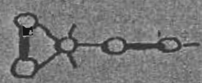
$j =$ 

$k =$ 

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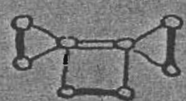
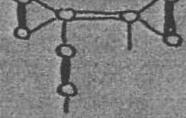
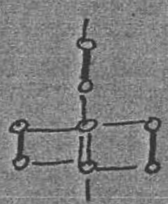
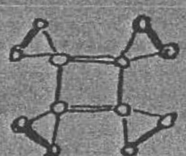
$B =$ 

$\vartheta =$ 

$\vartheta =$ 

$D_T = C =$ 

$$D_{AT-Bi} = A^2C - AB^2 + AB^2 = A^2C.$$

$$\Theta = \kappa^2 + \frac{A}{2}\lambda^2$$

$$J = \kappa j + \lambda k$$

$$f_F = F = \kappa f + \lambda g$$

$$g_F = \frac{1}{2}\Theta \cdot F\Theta$$

TWISTOR NEWSLETTER No. 34

Contents

Short contributions for TN 35 should be sent to

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