**Metrics with SD Weyl tensor from Painlevé-VI**

In a Comment published last year (Class.Quant.Grav.8 (1991) 1049), I wrote down an autonomous system of six ODEs, any solution of which determines a diagonal Bianchi-type-IX metric with a self-dual (SD) Weyl tensor and vanishing scalar curvature. Call these half-conformally-flat, scalar-flat metrics. This class of metrics includes vacuum examples, scalar-flat Kähler examples and a class conformally related to some Einstein metrics with non-zero scalar-curvature. At Roger's birthday meeting last year, I described how this system could be boiled down to a single, second-order non-linear ODE which Chazy (Acta Math.34 (1911) 317) in a throw-away aside asserted was a 'transformée algebrique' of Painlevé-VI 'curieuse en raison de son élégance'. I can now, with help from Fokas and Ablowitz (J.Math.Phys.23 (1982) 2033), see how to do this transformation and this is what I shall describe here.

If the Bianchi-type-IX metric is written in the usual way as

$$ds^2 = \omega_1\omega_2\omega_3 dt^2 + \frac{\omega_2\omega_3\sigma_1{}^2}{\omega_1} + \frac{\omega_3\omega_1\sigma_2{}^2}{\omega_2} + \frac{\omega_1\omega_2\sigma_3{}^2}{\omega_3} \qquad 1$$

where each $\omega_i$ is a function only of 'time' t, then the autonomous system which I had is

$$\begin{aligned}\dot\omega_1 &= -\omega_2\omega_3 + \omega_1(a_2 + a_3)\\ \dot\omega_2 &= -\omega_3\omega_1 + \omega_2(a_3 + a_1)\\ \dot\omega_3 &= -\omega_1\omega_2 + \omega_3(a_1 + a_2)\end{aligned} \qquad 2$$

$$\begin{aligned}\dot a_1 &= -a_2a_3 + a_1(a_2 + a_3)\\ \dot a_2 &= -a_3a_1 + a_2(a_3 + a_1)\\ \dot a_3 &= -a_1a_2 + a_3(a_1 + a_2)\end{aligned} \qquad 3$$

where a dot denotes d/dt.

The second trio of equations is discussed by Ablowitz and Clarkson (in 'Solitons, Nonlinear Evolution Equations and Inverse Scattering' LMS Lecture Note Series 149) and called by them the 'Chazy system' following the solution given by Chazy (C.R.Acad.Sci.150 (1910)456). It was also solved by Brioschi (C.R.Acad.Sci. t.XCII (1881) 1389) and I will describe his method here.

First introduce X,Y by

$$X = a_1 - a_2 \qquad Y = a_3 - a_1 \qquad 4a$$

now take the differences between successive pairs of the equations in (3) to find

$$a_1 = \frac{\dot X + \dot Y}{2(X + Y)} \qquad a_2 = \frac{\dot Y}{2Y} \qquad a_3 = \frac{\dot X}{2X} \;. \qquad 4b$$

Introduce a new dependent variable x by

$$Y = \frac{\dot{x}}{2x} \qquad \text{5a}$$

when it follows from (3) that

$$X = \frac{\dot{x}}{2(1-x)} \qquad \text{5b}$$

and from (4b) and (5) that

$$a_1 = \frac{1}{2}\left(\frac{\ddot{x}}{\dot{x}} - \frac{\dot{x}}{x} + \frac{\dot{x}}{1-x}\right)$$

$$a_2 = \frac{1}{2}\left(\frac{\ddot{x}}{\dot{x}} - \frac{\dot{x}}{x}\right)$$

$$a_3 = \frac{1}{2}\left(\frac{\ddot{x}}{\dot{x}} + \frac{\dot{x}}{1-x}\right)$$

What is left of (3) is the 3rd-order ODE for x(t):

$$\frac{\dddot{x}}{\dot{x}} = \frac{3}{2}\frac{\ddot{x}^2}{\dot{x}^2} - \frac{\dot{x}^2}{2}\left(\frac{1}{x^2} + \frac{1}{x(1-x)} + \frac{1}{(1-x)^2}\right) \qquad 6$$

This is the equation satisfied by the reciprocal of the Elliptic Modular Function. (Chazy's solution of the system (3) led to what Ablowitz and Clarkson call the Chazy equation, which in turn is solved by a ratio of hypergeometric functions.)

We have solved (3). To deal with (2), we define new variables $\Omega_i$ by

$$\omega_1 = 2\Omega_1(XY)^{1/2} \; ; \; \omega_2 = 2\Omega_2(X(X+Y))^{1/2} \; ; \; \omega_3 = 2\Omega_3(Y(X+Y))^{1/2} \qquad 7$$

These definitions are motivated by consideration of (4b). When we substitute (7) into (2) we obtain the new system:

$$\dot{\Omega}_1 = -2\Omega_2\Omega_3(X+Y)$$
$$\dot{\Omega}_2 = -2\Omega_3\Omega_1 Y$$
$$\dot{\Omega}_3 = -2\Omega_1\Omega_2 X \qquad 8$$

The difficulty with (8) is the presence of X and Y. With the aid of (5), we now change the independent variable from t to x. Using a prime for d/dx, we find (8) becomes

$$\Omega_1' = \frac{-\Omega_2\Omega_3}{x(1-x)} \; ; \; \Omega_2' = \frac{-\Omega_3\Omega_1}{x} \; ; \; \Omega_3' = \frac{-\Omega_1\Omega_2}{(1-x)} \qquad 9$$

This system has arisen elsewhere in connection with integrable systems: Fokas et al (Phys.Lett. A115 (1986) 329) obtain it from a 3-wave interaction, which is known to be a reduction of SD Yang-Mills, and Dubrovin (Functional Analysis and its Applications 24 (1990) 280) obtains it from the equations for a 3-dimensional diagonal metric to be flat.

Before solving (9), we write the original metric (1) in terms of the $\Omega_i$ making use of (5) and (7). The result is

$$ds^2 = \frac{\Omega_1\Omega_2\Omega_3 x}{x(1-x)}\left[\frac{dx^2}{x(1-x)} + \frac{\sigma_1^2}{\Omega_1^2} + \frac{(1-x)\sigma_2^2}{\Omega_2^2} + \frac{x\sigma_3^2}{\Omega_3^2}\right] \qquad 10$$

The part in square brackets strongly resembles the conformal metric given by Nigel Hitchin in his seminar in Oxford last year (15/10/91) and obtained by him via a direct twistor-space-construction of Einstein, half-conformally-flat Bianchi-type-IX metrics. The conformal factor in (10) can be viewed as the factor necessary to make the scalar curvature vanish.

The system (9) has a first-integral namely

$$\Omega_2^2 + \Omega_3^2 - \Omega_1^2 = 2\gamma, \text{ constant} \qquad 11$$

which is the residue of a more-complicated looking first-integral of the original combined system (2,3) which can be obtained by working back from (11) through (4a) and (7).

Using the first-integral, it is relatively straightforward to reduce the system (9) to a second-order non-linear ODE for, say, $\Omega_3$. The problem is that this ODE is quadratic in the second-derivative of $\Omega_3$ and so cannot be one of the Painlevé equations. It is this ODE which is given by Chazy, as described in the first paragraph, and this is where I was stuck until I came across the paper of Fokas and Ablowitz. By following their procedure one is led, after some calculation, to make a change of independent variable to z via the transformation

$$x = \frac{4\sqrt{z}}{(1+\sqrt{z})^2} \qquad 12$$

and then to express $\Omega_3$ in terms of a new dependent variable v via the transformation

$$\Omega_3 = \frac{zv_z}{v} - \frac{v}{2(z-1)} - \frac{1}{2} + \frac{z}{2v(z-1)} \qquad 13$$

The remarkable thing is that now v satisfies Painlevé-VI (as given eg by Ince or by Ablowitz and Clarkson) with the parameter values $(\alpha,\beta,\gamma,\delta) = (1/8,-1/8,\gamma,(1-2\gamma)/2)$ with $\gamma$ as in (11), and in fact (13) can be inverted so that all solutions of (9) arise this way.

Fokas and Ablowitz, and also Dubrovin give some particular solutions of Painlevé-VI in terms of solutions of hypergeometric functions. Reversing the above, these will lead to particular metrics.

It should be possible, following the lead of my other article in this issue of Twistor Newsletter, to find non-diagonal generalisations of (1) in terms of Painlevé-VI with more general values of the parameters.

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