Some new scalar-flat Kähler and hyper-Kähler metrics

We know from the work of Claude Lebrun (J.Diff.Geom. 34 (1991) 223-253) that any scalar-flat Kähler metric with a Killing vector (or 'S^1-action') arises from a solution u(x,y,z) of the equation

$$u_{xx} + u_{yy} + (e^u)_{zz} = 0 \qquad 1$$

which is variously known as the 'SU(∞) Toda field equation' or 'Boyer-Finley equation'. What is more Claude tells us in detail how to go from the metric to the function u, though given a solution u there is a choice to be made on the way back to the metric. In particular, given u there is a choice which leads back to a hyper-Kähler metric.

Henrik Pedersen and Yat Sun Poon (Class.Quant.Grav.7 (1990) 1707) found a scalar-flat Kähler metric of Bianchi-type-IX, in the terminology of relativity. This means that the metric has a 3-parameter group of isometries, isomorphic to SU(2) and transitive on 3-surfaces. In particular then the Pedersen-Poon (PP) metric comes from a solution of (1) and it is possible to follow Lebrun's direction to find out which. It turns out that the PP-metric arises from an ansatz for (1) which has a simple generalisation to a wider class of solutions of (1). Thus the PP-metric can be generalised to a wider class of scalar-flat Kähler or hyper-Kähler metrics, and this is what I want to describe here.

The idea, with the benefit of hindsight, is to seek a solution of (1) for which u is constant on central ellipsoids. In other words, define u implicitly by the equation

$$Q(u,x,y,z) \equiv X^t M(u) X = 1 \qquad 2$$

where $X^t = (x,y,z)$ and M(u) is a (symmetric, positive-definite) matrix function of u, to be found. Differentiate (2) implicitly and substitute into (1), then what remains is a second-order matrix ODE in u which can be integrated once. To write the resulting first-order equation out, first define the 3x3 matrix

$$g = \mathrm{diag}(1,1,e^u) \qquad 3$$

then the equation we want turns out to be

$$VM_u = MgM \qquad 4$$

using a subscript u to denote $\frac{d}{du}$

where V is an integral of

$$V_u = \tfrac{1}{2}\mathrm{trace}(gM) \qquad 5$$

Because of (5), equation (4) is still quite complicated, but there is a dramatic simplification if we work with minus the inverse of M:

set $N = -M^{-1}$ then (4) and (5) reduce to

$$VN_\omega = g \qquad 6$$

and

$$V_\omega = -\tfrac{1}{2}\mathrm{trace}(gN^{-1}). \qquad 7$$

From (3) and (6), the off-diagonal terms in N are constant so set

$$N = \begin{pmatrix} a & \nu & \mu \\ \nu & b & \lambda \\ \mu & \lambda & c \end{pmatrix} \qquad 8$$

where λ, μ, ν are constants, then (6) in components reduces to the 3 equations

$$a_\omega = b_\omega = 1/V \; ; \; c_\omega = e^\omega/V \qquad 9$$

At once, from (9)

$$a-b = 2\zeta \quad \text{constant,}$$

where we retain the convention of using Greek letters for constants. Define a new constant η by

$$\eta^2 = \zeta^2 + \nu^2$$

then a convenient parametrisation, in terms of a new variable w, turns out to be

$$a = \eta\left(\frac{w+1}{w-1}\right) + \zeta \qquad b = \eta\left(\frac{w+1}{w-1}\right) - \zeta \qquad 10$$

Next we note from (7) that

$$V^2 \det N = \xi \;\; \text{constant} \qquad 11$$

Our aim is to obtain an equation for w. We solve (10) and (11) for c in terms of w, V and constants, then eliminate V in favour of w_ω using (9) and (10). This gives c in terms of w, w_ω and constants and then the last part of (9) gives a second-order ODE for w. A change of independent variable

$$z = e^\omega \qquad 12$$

enables this final ODE to be written as

$$\frac{d^2w}{dz^2} = \left(\frac{1}{2w} + \frac{1}{w-1}\right)\left(\frac{dw}{dz}\right)^2 - \frac{1}{z}\frac{dw}{dz} + \frac{(w-1)^2(\alpha w + \beta)}{z^2 \; w} + \frac{\gamma w}{z} + \frac{\delta w(w+1)}{w-1} \qquad 13$$

where $\alpha, \beta, \gamma, \delta$ are constants which can be expressed in terms of the constants which we already have ie $\lambda, \mu, \nu, \xi . \eta . \zeta$. Painlevé-buffs will recognise the ODE in (13) as Painlevé-V (P-V for short), with the usual conventions. In fact, the constant δ turns out to be zero in this case, and Fokas and Ablowitz (J.Math.Phys. 23 2033 (1982)) explain how to transform this special case of P-V to P-III.

Given a solution $w(z)$ of (13), we can work our way back to the matrix N and so to M and to $u(x,y,z)$. To be able to write down a scalar-flat Kähler metric, we have still to solve a linear PDE, but to write down a hyper-Kähler metric, there are no more choices to be made: given $u(x,y,z)$ the metric is immediate. In particular, given the function $u(x,y,z)$ appropriate to the PP-metric, there is a hyper-Kähler metric with the same u (apparently a new one, but it won't be of Bianchi-type-IX unless its already known!).

The special case of this procedure when the matrix M in (2) is diagonal, or equivalently when the constants λ, μ, ν in (8) are all zero, is the one which leads to the PP-metric, which was already known to involve P-III. What is obscure in this approach is why the PP solution of (1), which must lead to a scalar-flat Kähler metric with an S^1-action, actually leads to a far more symmetric metric with an SU(2)-action. This raises the possibility that these more general solutions of (1) also lead to scalar-flat Kähler metrics with SU(2)-action. If these exist, and I am grateful to Andrew Dancer for the suggestion that they do, then their metrics will look like

$$ds^2 = Fdt^2 + A_{ij}\sigma^i\sigma^j \qquad 14$$

where A_{ij} is a matrix function of t, and the σ^i are a basis of left-invariant one-forms for S^3. The question is whether one can relate the matrix A to the matrix M of (2). Certainly for the PP-metric, when M and A are both diagonal and t in (14) is proportional to u, this is possible.

One could also try different generalisations of the ansatz (2): for example with hyperboloids in place of ellipsoids (ie with an indefinite M(u) rather than a positive-definite one) which presumably leads to a Bianchi-type-VIII metric, or with paraboloids or even with planes (This case, which is fairly easy to do, leads to a Riccati equation in place of (13)). And, of course, free with every solution of (1) one obtains a hyper-Kähler metric too.

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Paul Tod