Some new scalar-flat Kähler and hyper-Kähler metrics

We know from the work of Claude Lebrun (J.Diff.Geom. 34 (1991) 223-253) that any scalar-flat Kähler metric with a Killing vector (or 'S'-action) arises from a solution \( u(x,y,z) \) of the equation

\[
    u_{xx} + u_{yy} + (e^u)_{xx} = 0
\]

which is variously known as the 'SU(∞) Toda field equation' or 'Boyer-Finley equation'. What is more Claude tells us in detail how to go from the metric to the function \( u \), though given a solution \( u \) there is a choice to be made on the way back to the metric. In particular, given \( u \) there is a choice which leads back to a hyper-Kähler metric.

Henrik Pedersen and Yat Sun Poon (Class.Quant.Grav. 7 (1990) 1707) found a scalar-flat Kähler metric of Bianchi-type-IX, in the terminology of relativity. This means that the metric has a 3-parameter group of isometries, isomorphic to SU(2) and transitive on 3-surfaces. In particular then the Pedersen-Poon (PP) metric comes from a solution of (1) and it is possible to follow Lebrun's direction to find out which. It turns out that the PP-metric arises from an ansatz for (1) which has a simple generalisation to a wider class of solutions of (1). Thus the PP-metric can be generalised to a wider class of scalar-flat Kähler or hyper-Kähler metrics, and this is what I want to describe here.

The idea, with the benefit of hindsight, is to seek a solution of (1) for which \( u \) is constant on central ellipsoids. In other words, define \( u \) implicitly by the equation

\[
    Q(u,x,y,z) \equiv X^*M(u)X = 1
\]

where \( X^* = (x,y,z) \) and \( M(u) \) is a (symmetric, positive-definite) matrix function of \( u \), to be found. Differentiate (2) implicitly and substitute into (1), then what remains is a second-order matrix ODE in \( u \) which can be integrated once. To write the resulting first-order equation out, first define the 3x3 matrix

\[
    g = \text{diag}(1,1,e^u)
\]

then the equation we want turns out to be

\[
    VM_u = MgM
\]

using a subscript \( u \) to denote \( \frac{d}{du} \)

where \( V \) is an integral of

\[
    V_u = \text{htr}(gM)
\]

Because of (5), equation (4) is still quite complicated, but there is a dramatic simplification if we work with minus the inverse of \( M \).
set \( N = -N^{-1} \) then (4) and (5) reduce to
\[
V_{N,u} = g
\]
and
\[
V_u = -\text{tr}(gN^{-1}).
\]
From (3) and (6), the off-diagonal terms in \( N \) are constant so set
\[
N = \begin{pmatrix}
\lambda & \mu & \nu \\
\nu & b & \lambda \\
\mu & \lambda & c \\
\end{pmatrix}
\]
where \( \lambda, \mu, \nu \) are constants, then (6) in components reduces to the 3 equations
\[
a_u = b_u = 1/V ; \quad c_u = a^u/V
\]
At once, from (9)
\[
a-b = 2\zeta \text{ constant},
\]
where we retain the convention of using Greek letters for constants. Define a new constant \( \eta \) by
\[
\eta^2 = \zeta^2 + v^2
\]
then a convenient parametrisation, in terms of a new variable \( w \), turns out to be
\[
a = \eta \left( \frac{w+1}{w-1} \right) + \zeta \quad \quad b = \eta \left( \frac{w+1}{w-1} \right) - \zeta
\]
Next we note from (7) that
\[
V^2 \text{det}N = \xi \text{ constant}
\]
Our aim is to obtain an equation for \( w \). We solve (10) and (11) for \( c \) in terms of \( w, V \) and constants, then eliminate \( V \) in favour of \( w_u \) using (9) and (10). This gives \( c \) in terms of \( w, w_u \) and constants and then the last part of (9) gives a second-order ODE for \( w \). A change of independent variable
\[
z = e^w
\]
ensures this final ODE to be written as
\[
\frac{d^2w}{dz^2} = \left( \frac{1}{2w} + \frac{1}{w-1} \right) \left( \frac{dw}{dz} \right)^2 - \frac{1}{z} \frac{dw}{dz} + (w-1)^2 (aw + \beta) + \gamma w + \delta w(w+1)
\]
where \( \alpha, \beta, \gamma, \delta \) are constants which can be expressed in terms of the constants which we already have i.e. \( \lambda, \mu, \nu, \xi, \eta, \zeta \). Painlevé-buffs will recognise the ODE in (13) as Painlevé-V (P-V for short), with the usual conventions. In fact, the constant \( \delta \) turns out to be zero in this case, and Fokas and Ablowitz (J.Math.Phys. 23 2033 (1982)) explain how to transform this special case of P-V to P-III.
Given a solution $w(z)$ of (13), we can work our way back to the matrix $N$ and so to $M$ and to $u(x,y,z)$. To be able to write down a scalar-flat Kähler metric, we have still to solve a linear PDE, but to write down a hyper-Kähler metric, there are no more choices to be made: given $u(x,y,z)$ the metric is immediate. In particular, given the function $u(x,y,z)$ appropriate to the PP-metric, there is a hyper-Kähler metric with the same $u$ (apparently a new one, but it won’t be or Bianchi-type-IX unless it’s already known!).

The special case of this procedure when the matrix $M$ in (2) is diagonal, or equivalently when the constants $\lambda, \mu, \nu$ in (8) are all zero, is the one which leads to the PP-metric, which was already known to involve P-III. What is obscure in this approach is why the PP solution of (1), which must lead to a scalar-flat Kähler metric with an $S^1$-action, actually leads to a far more symmetric metric with an SU$(2)$-action. This raises the possibility that these more general solutions of (1) also lead to scalar-flat Kähler metrics with SU$(2)$-action. If these exist, and I am grateful to Andrew Dancer for the suggestion that they do, then their metrics will look like

$$\text{ds}^2 = F dt^2 + A_{ij} \sigma^i \sigma^j$$

where $A_{ij}$ is a matrix function of $t$, and the $\sigma^i$ are a basis of left-invariant one-forms for $S^3$. The question is whether one can relate the matrix $A$ to the matrix $M$ of (2). Certainly for the PP-metric, when $M$ and $A$ are both diagonal and $t$ in (14) is proportional to $u$, this is possible.

One could also try different generalisations of the ansatz (2): for example with hyperboloids in place of ellipsoids (ie with an indefinite $M(u)$ rather than a positive-definite one) which presumably leads to a Bianchi-type-VIII metric, or with paraboloids or even with planes (This case, which is fairly easy to do, leads to a Riccati equation in place of (13)). And, of course, free with every solution of (1) one obtains a hyper-Kähler metric too.

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