Exceptionally Vile Invariants

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Regard $\mathbf{P}^n$ as a homogeneous space for $\mathrm{SL}(n+1,\mathbf{R})$ and, for a particular homogeneous bundle over $\mathbf{P}^n$, consider the problem of constructing all density valued differential invariants on the bundle which are polynomial in the jets. We will call such objects ***projective invariants***. An example for $n = 1$ is given by the formula,

$$wf\nabla\nabla f - (w-1)\nabla f\nabla f$$

where f has weight w. Here ∇, (often written $\eth$ and called edth) is a local flat affine connection. Of course $\mathbf{P}$ does not have a unique such connection but rather a family of them related by transformation formulae, $\nabla \mapsto \nabla f + w\Upsilon f$ where Υ satisifies $\nabla\Upsilon = \Upsilon^2$ (see e.g. [1] for the corresponding formulae on $\mathbf{P}^n$). The point is that this differential operator is invariant under these transformations. (An analogous, and more familiar, problem is to find conformally invariant differential equations for flat conformal structures. The standard model for the latter is S^n as a homogeneous space for $\mathrm{SO}(n+1,1)$. Of course for $n = 1$ these are the *same* problem.)

Recall that a function f of weight w on $\mathbf{P}^n$ corresponds to a function on $\mathbf{R}^{n+1}$ which is homogeneous of degree w, i.e. $f(\lambda X^A) = \lambda^w f(X^A)$, where X^A are the standard coordinates on $\mathbf{R}^{n+1}$. So a good trick for proliferating many projective invariants on $\mathbf{P}^n$ is simply to write down affine invariants on $\mathbf{R}^{n+1}$ and then regard these as invariants on $\mathbf{P}^n$ by simply insisting that f be homogeneous of some weight. For example if $n = 1$ then the following is an affine invariant on $\mathbf{R}^{n+1}$,

$$\epsilon^{AC}\epsilon^{BD}\partial_A\partial_B f\partial_C\partial_D f$$

where $\partial_A := \partial/\partial X^A$. The standard representation theory of $\mathrm{SL}(n+1,\mathbf{R})$, due to Weyl and others, tells us scalar valued affine invariants on $\mathbf{R}^{n+1}$ are always linear combinations of such complete contractions. If, in the last formula, we now restrict f to be homogeneous of weight w then we obtain a projective invariant. In fact for $w \neq 1$ this is precisely the invariant mentioned earlier. Invariants which arise this way are called ***Weyl invariants***.

Since it is evidently possible to list all Weyl invariants, it is interesting to ask if all projective invariants are Weyl. It turns out [7] that if the weight w of f is non-integral or negative integral then all invariants are Weyl. However for the remaining case with w non-negative integral, to which we now restrict our attention, we shall see that it is easy to write down some invariants which are not Weyl. Such will be called *exceptional* invariants or, since these are the troublemakers of the invariant world and with a view to homophony, *vile* invariants. Even for the simple case of densities on $\mathbf{P}^n$ it is not known how to sort out the Weyl invariants from the vile invariants. However, there is a simpler, yet very important, problem on which much progress has been made. For each weight w there is a linear invariant differential operator $\mathcal{O}(w) \to \mathcal{O}_{\underbrace{(ab\cdots d)}_{w+1}}(w)$;

in terms of a local affine connection ∇_a on $\mathbf{P}^n$, this is given by $f \mapsto \nabla_a\nabla_b\cdots\nabla_d f$. This operator splits the jet bundle. So instead of looking for invariants on the jets of $\mathcal{O}(w)$ one can look for invariants on the"slightly smaller" space which, at a particular point of $\mathbf{P}^n$, is the jets of $\mathcal{O}(w)$ modulo the kernel of this operator. There are analogous conformal and CR versions of this latter problem too [4,6] and they are geometrical equivalents of some difficult algebraic problems first posed and discussed by Fefferman [5]. On $\mathbf{P}^n$, I have completely solved this problem [7]. It turns out that, in this case, there are non-vanishing exceptional invariants. For example, if $w = 1$, then $\nabla_a\nabla_b f$ is invariant and therefore when $n = 2$ one can construct the projective invariant

$$\epsilon^{ac}\epsilon^{bd}\nabla_a\nabla_b f\nabla_c\nabla_d f.$$

Since its homogeniety with respect to f is just two it cannot be a Weyl invariant (the construction of which requires that a ϵ^{ABC} be used). The general situation is well characterised by this example; in n dimensions the exceptional invariants are always constructed by a contraction of $w+1$ $\epsilon^{ab\cdots d}$'s into an n-fold juxtaposition of the linear operator with itself.

By adapting the methods in [7] and invoking some new tricks Bailey, Eastwood and Graham [2] were able to solve the corresponding problems for flat CR structures and odd dimensional conformal S^n. There are no exceptional invariants in the CR case but for the conformal S^n case there are. Here again it turns out that operators which are homogeneous of degree n in the argument density f are exceptional while all others are Weyl. However, in [2] the authors posed the question of whether the exceptionals were, as in the projective case, constructed purely from juxtapositions of the linear invariant $\underbrace{\nabla_{(a}\nabla_b\cdots\nabla_{d)_0}}_{w+1} f$, where w is the (non-negative integral) weight of f.

More recently I was investigating the same question for invariants of vector fields on $\mathbf{P}^n$ and discovered a means of generating exceptionals which are rather more vile. Here is a simple example. Consider the problem of constructing invariants of the module which is jets of vectors v^a of weight 0, at some point modulo the kernel of the linear invariant differential operator. In this case this operator is trace-free$(\nabla_a\nabla_b v^c)$ and is given in terms of $\mathbf{R}^{n+1}$ objects by $v^C_{AB} := \partial_A\partial_B v^C$, where v^C satisfies (the divergence free "gauge" condition) $\partial_C v^C = 0$. Note that, as well as being trace free, v^C_{AB} is annihilated upon contraction with X^A, since v^A is homogeneous of degree 1 with respect to X^A. Now let β_A be a covector in R^{n+1} which satisfies $X^A\beta_A = 1$. So β_A is homogeneous of degree -1 and is only defined up to transformations $\beta_A \mapsto \hat{\beta}_A = \beta_A + \Upsilon_A$ where $X^A\Upsilon_A = 0$. Consider now the object

$$\partial_E\partial_F(v^E_{AB}v^F_{CD}\epsilon^{ACI}\epsilon^{BDJ}\beta_I\beta_J).$$

This is clearly an invariant provided it is independent of the choice of β_A. Indeed it is independent of β_A, and so an invariant, because $\epsilon^{ACI}\hat{\beta}_I = \epsilon^{ACI}\beta_I + X^{[A}\gamma^{C]}$ for some γ^C. When expanded out it is given by the formula

$$\epsilon^{cd}\epsilon^{ef}(\nabla_c\nabla_e\nabla_a v^b\nabla_d\nabla_f\nabla_b v^a + 2\nabla_c\nabla_e\nabla_b\nabla_a v^a\nabla_d\nabla_f v^b + \nabla_c\nabla_e\nabla_a v^a\nabla_d\nabla_f\nabla_b v^b)$$

and so is clearly non-zero. (Thanks to Michael Eastwood for helping check this expansion.) Furthermore this is certainly an exceptional invariant since it is homogeneous of degree just two with respect to v^A and is an example which is not simply a juxtapostion of the linear invariant with itself. Fortunately it turns out [8] that, for vectors on $\mathbf{P}^n$, all exceptional invariants can be constructed by a generalisation of the method used for this example. So the exceptional invariants can now be listed as readily as Weyl invariants. These methods work for many other similar modules. For example an important application of these new results and the earlier ones mentioned is for the job of listing all invariants of projective structures (i.e. the *curvature invariants* of a projective manifold). There is an algebraic problem which arises in this context, analogous to the ones alluded to above, which I have now solved. This will appear in [9]. Toby Bailey and I [3] have shown that these arguments also work for the conformal case.

References

[1] T. N. Bailey, M. G. Eastwood and A. R. Gover. Thomas's structure bundle for conformal, projective and related structures. *preprint* (1992).

[2] T. N. Bailey, M. G. Eastwood and C. R. Graham. Invariant theory for conformal and CR geometry. *preprint* (1992).

[3] T. N. Bailey and A. R. Gover. Exceptional invariants in the parabolic invariant theory of conformal geometry. *In preparation.*

[4] M. G. Eastwood and C. R. Graham. Invariants of conformal densities. *Duke Math. Jour.* **63** (1991), 633–671.

[5] C. Fefferman. Parabolic invariant theory in complex analysis. *Adv. in Math.* **31** (1979), 131–262.

[6] C. Fefferman and C. R. Graham. Conformal invariants. In: *Élie Cartan et les Mathématiques d'Aujourdui.* Astérisque (1985), 95–116.

[7] A. R. Gover. Invariants on projective space. *submitted to J.A.M.S.* (1991).

[8] A. R. Gover. Invariant theory for a parabolic subgroup of $\mathrm{SL}(n+1,\mathbf{R})$. *In preparation.*

[9] A. R. Gover and C. R. Graham. Invariant theory for projective geometries. *In Preparation.*