On the symmetries of the reduced self-dual Yang-Mills equations

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Introduction

One of the remarkable features of reductions of the self-dual Yang-Mills equations to systems in two dimensions is that the symmetry group of the reduced equations (in the context of space-time symmetries) is much larger than one might have expected, often being infinite dimensional. A priori, one would expect the symmetry group of the reduced equations to be just the projection of those conformal symmetries in 4-dimensions that normalize the invariance group that one is reducing by. In Hitchin (1987) it was observed that the reductions of the self-dual Yang-Mills equations on Euclidean 4-dimensional space by two translations are actually conformally invariant in the infinite dimensional sense in the residual 2-dimensional space (a priori one would only expect the equations to be invariant under the 2-dimensional Euclidean group plus scalings). In Mason & Sparling (1992) it was observed that reductions of self-dual Yang-Mills by 2 translations spanning a 2-plane on which the metric has rank one also has an infinite dimensional symmetry group at least when the gauge group is $SL(2)$—nonlinear analogues of the Galilean group in so called $(1+0)$-dimensions as opposed to just the linear Galilean group in 2-dimensions..

The purpose of this note is to clarify the geometry underlying this result and state it independently of the gauge group. I also discuss two other examples of this phenomena, one being the reduction by symmetries spanning a totally null ASD 2-plane where the symmetry group is the whole diffeomorphism group (rather than just $GL(2)$), and the other being the reduction by two rotations (or a rotation and a translation) in which the symmetry group is the hyperbolic group in 2-dimensions ($SL(2,\mathbb{R})$).

An important corollary of Hitchins result is that it makes it possible to transfer the equations to a general Riemann surface where they considerably enrich the theory of holomorphic vector bundles. The above results give alternate ways of transferring different reductions of the self-dual Yang-Mills equations to 2-dimensional surfaces endowed with different geometric structures.

The Yang-Mills Higgs equation on a Riemann surface

First a brief review of Hitchin's equations. We will use $(z, w, \tilde{z}, \tilde{w})$ as coordinates on $\mathbf{R}^4$ that are independent and real for signature $(2,2)$ or complex with $\tilde{z} = \bar{z}$ etc. for Euclidean signature. We start with the Lax pair formulation of the self-dual Yang-Mills equations.

The self-dual Yang-Mills equations are the compatibility conditions for the pair of operators:

$$L_0 = D_z - \lambda D_{\tilde{w}}, \quad L_1 = D_w + \lambda D_{\tilde{z}}.$$

where $\lambda \in \mathbf{C}$ is an auxilliary complex parameter and D_z is the covariant derivative of some Yang-Mills connection in the direction $\partial/\partial z$.

For Hitchin's equations we start in Euclidean signature and impose symmetries in the $\partial/\partial w$ and $\partial/\partial\bar{w}$ directions. In an invariant gauge (i.e. one in which the gauge potentials are independent of $(w, \bar{w})$, $D_w = \partial/\partial w + \bar{\Phi}'$ and cc. and we can throw away the derivatives with respect to $(w, \bar{w})$ to leave the pair of operators (with a litte rearrangement):

$$L_0 = D_z - \lambda\Phi', \quad L_1 = D_{\bar{z}} + \frac{1}{\lambda}\bar{\Phi}'.$$

We can make this more geometric by multiplying L_0 by dz and L_1 by $d\bar{z}$ and defining $\Phi = \Phi' dz$. We then obtain the form valued operator:

$$L = dz L_0 + d\bar{z} L_1 = D - \lambda\Phi + \frac{1}{\lambda}\bar{\Phi}.$$

The Yang-Mills Higgs equation on a Riemann surface are the consistency conditions for these operators:

$$D^2 = \Phi \wedge \bar{\Phi}, \quad D\Phi = 0, \quad D\bar{\Phi} = 0.$$

Where D is the covariant exterior derivative. These equations are invariant under the conformal group in two dimensions as they only require a bundle with connection and a complex structure to define Φ and $\bar{\Phi}$. One solution will be transformed to another if $z \mapsto z'(z)$ and Φ and D pull back.

Alternatively, these equations depend only on the $*$-operator on 1-forms on the quotient space of the symmetries. The data consists of a connection D on a bundle, E and a section $\Gamma = \Phi + \bar{\Phi}$ of $\Omega^1 \otimes End(E)$. The operator L is

$$L = D + \left(-\lambda \frac{1 - i*}{2} + \frac{1 + i*}{2\lambda}\right)\Gamma.$$

So the field equations arising from the consistency conditions of this operator are invariant under the diffeomorphisms preserving $*$, i.e. the conformal transformations in 2-dimensions.

The Galilean analogue

If, in $(2,2)$ signature, we impose one null symmetry along $\partial/\partial\tilde{w}$ and one non-null symmetry along $\partial/\partial z - \partial/\partial\tilde{z}$ we obtain the Lax pair:

$$L_0 = D_x - \lambda\Phi, \quad L_1 = D_w + \lambda(D_x + \Psi)$$

where $x = (z + \tilde{z})$ and we have reorganized the covariant derivative in the x direction to include part of the Higgs field associated to the symmetry in the $\partial_z - \partial_{\tilde{z}}$ direction. We can again perform the above trick, multiplying L_0 by dx and L_1 by dw to and adding together to obtain

$$L = dxL_0 + dwL_1 = D + \lambda(\Gamma + dwD_x)$$

where $\Gamma = -\Phi dx + \Psi dw$. To write this more geometrically, we introduce a degenerate $*$-operator that can be thought of as a map from 1-forms to 1-forms:

$$* = dw\frac{\partial}{\partial x}, \quad \alpha \mapsto \alpha(\frac{\partial}{\partial x})dw.$$

The operator L then becomes:

$$L = D + \lambda(*D + \Gamma).$$

The field equations arising from the consistency equations for this system are:

$$D^2 = 0, \quad D\Gamma = 0, \quad D * \Gamma + \Gamma \wedge \Gamma = 0$$

where D above is acting as the covariant exterior derivative so that the equations are all 2-form equations. Geometrically these equations determine a flat connection D on a bundle E, together with a section Γ of $\Omega^1 \otimes End(E)$.

It is clear, now, that the field equations arising from the consistency conditions for this operator will be invariant under diffeomorphisms of $\mathbf{R}^2$ preserving the degenerate $*$–operator, $dw \otimes \partial/\partial x$. These are the nonlinear Galilean transformations referred to previously:

$$(w, x) \mapsto (w', x') = (h(w), (\partial_w h(w))x + g(w))$$

where $h(w)$ and $g(w)$ are free functions except that $\partial_w h \neq 0$.

These equations embed the nonlinear Schrodinger and KdV equations and most of their generalizations to higher rank gauge groups (the Drinfeld Sokolov hierarchies etc.) into a galilean invariant system. At least in the $SL(2)$ case, this coordinate freedom is completely fixed by the reduction to KdV and NLS.

The totally null case

In the case where the symmetries span an anti self-dual null 2–plane we obtain the linear system

$$L = D + \lambda\Gamma$$

where again D is a flat connection on a bundle E and Γ is again a section of $\Omega^1 \otimes End(E)$. The field equations are now:

$$D^2 = 0, \quad D\Gamma = 0, \quad \Gamma \wedge \Gamma = 0.$$

These equations are now invariant under the full 2-dimensional diffeomorphism group (preserving the 'zero' $*$–operator).

These equations are therefore 'topological' and indeed are another way of writing the Wess-Zumino-Witten equations (Strachan 1992). Their reductions include the n–wave equations and those parts of the Drinfeld-Sokolov hierarchies not obtainable from the Galilean reductions. These further reductions require that their exists coordinates and a gauge in which the components of Γ are constant. In the $SL(3)$ case one can fix the coordinate freedom by using these additional conditions.

Stationary axisymmetric systems

In Fletcher & Woodhouse (1990) it was observed that the reduction of SDYM by 2 rotations gave the same field equations as the reduction by a translation and a rotation. This fact alone endows the 2 rotation reduction with one unexpected symmetry, the residual translation symmetry. However, more is true. These equations are invariant under $SL(2,\mathbf{R})$ the group of motions of the residual space preserving a hyperbolic metric. While this was in some sense clear from the reduced twistor correspondence in Woodhouse & Mason (1988), it was difficult to see on space-time.

To see this we impose a rotational invariance with respect to θ in the $w = y\exp(i\theta)$ plane and set $x = z + \bar{z}$ and impose a symmetry in the $\partial_z - \partial_{\bar{z}}$ direction. We obtain the linear system:

$$D_x - iA + \lambda e^{i\theta}(D_y + \frac{i}{y}(\partial_\theta - B)), \quad e^{-i\theta}(D_y - \frac{i}{y}(\partial_\theta - B)) - \lambda(D_x + iA).$$

We cannot just throw away the ∂_θ as there is explicit dependence on θ in the operators. This is connected with the fact that the Lie derivative of a spinor and hence λ along ∂_θ is not zero. To work independently of θ and to avoid derivatives with respect to the 'spectral parameter' we must use, instead of λ the parameter

$$\gamma = \frac{ye^{i\theta}\lambda}{2} + x - \frac{y}{2e^{i\theta}\lambda}$$

as this the simplest function on the spin bundle that is both invariant and constant along the twistor distribution.

If we introduce the complex coordinate $\xi = x + iy$, a bit of massage yields the following form for the linear system:

$$2D_\xi + i\sqrt{\frac{\gamma - \bar{\xi}}{\gamma - \xi}}(A + \frac{i}{y}B), \quad 2D_{\bar{\xi}} + i\sqrt{\frac{\gamma - \xi}{\gamma - \bar{\xi}}}(A - \frac{i}{y}B).$$

In order to bring out the invariance properties of this system, we can first of all multiply the first operator by $d\xi$ and the second by $d\bar{\xi}$ and add them together. Then introduce homogeneous coordinates $\gamma_A = (\gamma_0, \gamma_1)$ with $\gamma = \gamma_1/\gamma_0$ and similarly for ξ_A. Define $\bar{\xi}_A$ to be the componentwise complex conjugate of ξ_A and denote the skew product $\gamma_1\xi_0 - \xi_1\gamma_0 = \gamma \cdot \xi$. The linear system then reduces, after some further massage, to:

$$2D + i\sqrt{\frac{\gamma\cdot\bar{\xi}}{i\xi\cdot\bar{\xi}\gamma\cdot\xi}}\Phi\xi\cdot d\xi + i\sqrt{\frac{\gamma\cdot\xi}{i\xi\cdot\bar{\xi}\gamma\cdot\bar{\xi}}}\bar{\Phi}\bar{\xi}\cdot d\bar{\xi}$$

where we have put $\Phi = (\sqrt{y}A + iB/\sqrt{y})/\xi_0$.

It can now be seen that the linear system is invariant under $SL(2,\mathbf{R})$; the Mobius transformations on ξ_A preserving the reality structure $\xi_A \mapsto \bar{\xi}_A$ and hence the hyperbolic metric $\xi\cdot d\xi \odot \bar{\xi}\cdot d\bar{\xi}/(i\xi\cdot\bar{\xi})^2$. The integrability conditions are equations for a connection D on a bundle E and a section $\Phi \in \Gamma(\mathcal{O}(-1)\otimes E)$ that is a dual spinor valued section of End(E).

The field equations are

$$D^2 = [\Phi,\bar{\Phi}]\frac{\xi\cdot d\xi\wedge\bar{\xi}\cdot d\bar{\xi}}{\xi\cdot\bar{\xi}},\quad \bar{\eth}\Phi = \frac{\bar{\Phi}}{2\xi\cdot\bar{\xi}},\quad \eth\bar{\Phi} = -\frac{\Phi}{2\xi\cdot\bar{\xi}}$$

where $\eth$ and $\bar{\eth}$ here denote the 'eth' operator and its complex conjugate, the (0,1) and (1,0) parts of the covariant derivative respectively.

So the 'Higgs fields' Φ and $\bar{\Phi}$ together constitute a Dirac field and satisfy the background coupled massive Dirac equation. Their commutator provides the curvature of the connection.

Remarks

Just as in Hitchin's case, one might hope to be able to transfer the other equations above to a Riemann surface also.

For the Galilean analogue, instead of endowing the Riemann surface with a complex structure, one might endow it with a measured foliation which corresponds to a limit of a complex structure (i.e. the space of measured foliations modulo certain equivalence relations is a good boundary for Teichmuller space). It turns out that this is *not* the same concept as the degenerate $*$–operator introduced above. They both determine a foliation of the Riemann surface, but the degenerate $*$–operator has an affine structure on the leaves, but no structure transverse to the leaves, whereas the measured foliation has a measure transverse to the leaves, but no structure on the leaves. Nevertheless, one might hope that one could prove an equivalence between the two, modulo diffeomorphisms in the global context as a kind of uniformization result. Even if this is feasible, it is still perhaps not clear that one can obtain a good existence theory for solutions of this equation as the linearized analogues of these equations have $*\Gamma$ covariant constant along the leaves of the foliation, a condition that will have no solutions when the leaves are dense.

Further analysis is required for the other cases. The totally null reduction will presumably not give rise to any difficulty as the equations are underdetermined anyway. This leaves the Hyperbolic case for which more analysis is required.

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