Integrable Systems and Curved Twistor Spaces

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One of the ways in which the self-dual Einstein equations may be understood is as a two dimensional chiral model with the gauge fields taking values in the Lie algebra $sdiff(\Sigma^2)$ of volume preserving diffeomorphisms of the 2-surface Σ^2 [1]. Moreover, since $sl(2,\mathbb{C})$ is a subalgebra of $sdiff(\Sigma^2)$, solutions of certain integrable systems associated with $sl(2,\mathbb{C})$ may be encoded within the geometry of the nonlinear graviton [2]. This description breaks down for higher rank algebras, which are not subalgebras of $sdiff(\Sigma^2)$. However, by generalising the algebras such a description may be achieved. Another reason for studying integrable systems with infinite dimensional gauge groups is that the equations often simplify, and in some cases even linearise [3].

Let $\{\ ,\ \}$ be a generalised Poisson bracket acting on some manifold $\mathcal{N}$, satisfying the conditions:

- $\{f,g\} = -\{g,f\}$ (antisymmetry)
- $\{f,gh\} = \{f,g\}h + \{f,h\}g$ (derivation)
- $\{f,\{g,h\}\} + \text{cyclic} = 0$ (Jacobi identity)

With respect to a basis $x^i, i = 1, \dots, \dim\mathcal{N}$, one may take*

$$\{f,g\} = \sum_{i,j} G^{ij}(x)\frac{\partial f}{\partial x^i}\frac{\partial g}{\partial x^j}, \tag{1}$$

where $G^{ij}(x)$ is constrained by the equations

$$\begin{gathered} G^{ij} + G^{ji} = 0 \\ \sum_{l=1}^{\dim\mathcal{N}} G^{li}\frac{\partial G^{kj}}{\partial x^l} + G^{lj}\frac{\partial G^{ik}}{\partial x^l} + G^{lk}\frac{\partial G^{ji}}{\partial x^l} = 0 \end{gathered} \tag{2}$$

Given such a structure one may define an associated Lie algebra Ham of Hamiltonian vector fields. Let $L_f \in Ham$, where

$$L_f = \sum_{i,j} G^{ij}(x)\frac{\partial f}{\partial x^i}\frac{\partial}{\partial x^j}$$

The Lie bracket for the algebra may be defined in two different, but equivalent, ways:

* Such generalised Poisson structures were first studied by Sophus Lie.

• Regard L_f and L_g as differential operators, and define the Lie bracket for the algebra by $[L_f, L_g] = L_f L_g - L_g L_f$,

• Regard L_f and L_g as vector fields on $\mathcal{N}$ and define the Lie bracket for the algebra be the Lie bracket of vector fields $[L_f, L_g]_{Lie}$.

In both cases $[L_f, L_g] = L_{\{f,g\}}$. The fact that this forms a Lie algebra follows trivially from (1) and (2). The idea now is to study the self-dual Yang-Mills equations with gauge potentials taking values in this infinite dimensional Lie algebra.

Let $y^{AA'}$ be spinor coordinates for $\mathbb{C}^4$ (or perhaps $\mathbf{R}^{2+2}$ etc. depending on a choice of reality condition). The self-dual Yang-Mills equations are the compatibility condition for the otherwise overdetermined linear system:

$$\mathcal{L}_A \Psi = \pi^{A'} \left\{ \frac{\partial}{\partial y^{AA'}} + A_{AA'} \right\} \Psi , \quad A, A' = 0, 1 , \quad \pi^{A'} \in \mathbf{CP}^1 . \tag{3}$$

The $A_{AA'}(y)$ are Lie algebra valued functions known as gauge potentials. In what follows it will be assumed that these take values in the Lie algebra *Ham* constructed above. Thus the $A_{AA'}$'s are represented by vector fields $A_{AA'} \leftrightarrow L_{f_{AA'}}$, where the functions $f_{AA'}$ depend on both the coordinates on $\mathbb{C}^4$ and on $\mathcal{N}$.

With this, the linear operators $\mathcal{L}_A$ are now vector fields on $\mathbb{C}^4 \oplus \mathcal{N}$,

$$\mathcal{L}_A = \pi^{A'} \left\{ \frac{\partial}{\partial y^{AA'}} + \sum_{i,j} G^{ij}(x) \frac{\partial f_{AA'}}{\partial x^i} \frac{\partial}{\partial x^j} \right\} . \tag{4}$$

Owing to the equivalent definition of the Lie bracket, the self-duality equations are a special case of the (Frobenius) integrability conditions for the distribution (4), i.e. $[\mathcal{L}_0, \mathcal{L}_1]_{Lie} = 0$. The integral surfaces of this distribution may be regarded as curved twistor surfaces, and the space of such surfaces as a curved twistor space, fibred over the Riemann sphere.

The converse construction involves studying an appropriate Riemann-Hilbert problem for the infinite dimensional group. Similiar ideas have been applied to the $SU(\infty)$-Toda equations in [4], which also developes the notion of a τ-function for this system and its associated hierarchy.

As mentioned at the beginning of this article, Mason showed [2] showed that one could give a curved twistor space construction to certain integrable systems associated with $sl(2,\mathbb{C})$ by embedding it in the algebra $sdiff(\Sigma^2)$. The same is true for any finite dimensional Lie algebra g. Let the structure constants for the Lie algebra g, with respect to some basis $e^i, i = 1, \ldots, \dim g$ be $c^{ij}{}_k$, so $[e^i, e^j] = \sum_k c^{ij}{}_k e^k$. From this one may define a generalised Poisson bracket by setting

$$G^{ij}(x) = \sum_k c^{ij}{}_k x^k$$

(the conditions (2) are automatically satisfied due to the properties of the structure functions), and let the associated infinite dimensional Lie algebra of Hamiltonian vector field be denoted by $\tilde{g}$. The original Lie algebra is now a subalgebra of $\tilde{g}$, since

$$[L_{x^i}, L_{x^j}] = \sum_k c^{ij}{}_k L_{x^k}\,.$$

Thus any solution to the self-dual Yang-Mills equations with a finite dimensional algebra may be encoded within the structure of a curved twistor space by first embedding g in $\tilde{g}$.

Another approach is to use a deformation of $sdiff(\Sigma^2)$ known as the Moyal algebra [5], in which higher order derivatives are present. This leads to some interesting results, but a direct geometrical interpretation of the results is absent.

References

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