Endomorphisms $S : \otimes^2 \overline{H^1(PT^+, \mathcal{O}(-2))} \hookleftarrow$

We write H for the Hilbert space completion $\overline{H^1(PT^+, \mathcal{O}(-2))}$ with respect to $<\ |\ >$ of $H^1(PT^+, \mathcal{O}(-2))$, the space of analytic positive frequency free zero rest–mass (z.r.m.) fields on Minkowski space with finite L^2–norm on the momentum space light cone [1,2]. To a given basis

$$\mathcal{B} = \{\underset{|}{A}, \underset{|}{B}, \underset{|}{C}, \underset{|}{D}\} \subset T^* \cong \mathbf{C}^4 \tag{1}$$

of dual twistor space we can associate functions $e^{\mathbf{i}}_{\mathcal{B}}$ of homogeneity -2 on T as follows:

$$e^{\mathbf{i}}_{\mathcal{B}}(\overset{|}{Z}) = (\overset{C}{\underset{Z}{|}})^c (\overset{D}{\underset{Z}{|}})^d / (\overset{A}{\underset{Z}{|}})^{1+a} (\overset{B}{\underset{Z}{|}})^{1+b} \in H^0(PT - \{\overset{A}{\underset{Z}{|}} = 0\} - \{\overset{B}{\underset{Z}{|}} = 0\}, \mathcal{O}(-2))$$

$$for\ \mathbf{i} = (a,b,c,d) \in \mathbf{I} = \big\{(k,l,m,n) \in \mathbf{N}^4 | k + l - m - n = 0\big\}\,,\ \overset{|}{Z} \in T\,. \tag{2}$$

If the projective lines $\underset{++}{AB}$, $\underset{++}{CD}$ lie in PT^-, PT^+ resp. these functions give rise to a linearly independent set of states

$$\{|e^{\mathbf{i}}_{\mathcal{B}} >\}_{\mathbf{i} \in \mathbf{I}} \subset H^1(PT - \{\overset{A}{\underset{Z}{|}} = 0\} \cap \{\overset{B}{\underset{Z}{|}} = 0\}, \mathcal{O}(-2)) \tag{3}$$

with dense span in H[2], i.e. they form a Hilbert space basis. An arbitrary element $|f> \in H$ has a unique expansion

$$|f> = \sum_{\mathbf{I}} < e^{\mathcal{B}}_{\mathbf{i}} \,|\, f > |\, e^{\mathbf{i}}_{\mathcal{B}} > \tag{4}$$

where $\{< e^{\mathcal{B}}_{\mathbf{i}} \,|\,\}_{\mathbf{i} \in \mathbf{I}}$ is the basis dual to (3) which is conveniently defined via the basis in T dual to $\mathcal{B}$ and the twistor transform [4].

1) $\mathcal{B}$–independence

Algebraically, i.e. up to the definition of a topology on $H \otimes H$ we can define endomorphisms $S : H \otimes H \longrightarrow H \otimes H$

$$S\Big(|\, e^{\mathbf{i}}_{\mathcal{B}} > \otimes\, |\, e^{\mathbf{j}}_{\mathcal{B}} >\Big) \quad = \sum_{(\mathbf{k},\mathbf{l}) \in \mathbf{I}^2} S^{\mathbf{ij}}_{\mathbf{kl}}(\mathcal{B})\, |\, e^{\mathbf{k}}_{\mathcal{B}} > \otimes\, |\, e^{\mathbf{l}}_{\mathcal{B}} > . \tag{5}$$

Continuity w.r.t. the chosen topology is reflected in appropriate convergence conditions on the coefficients $S^{\mathbf{ij}}_{\mathbf{kl}}(\mathcal{B})$. We assume them to be smooth functions of $\mathcal{B}$ in a neighbourhood of a fixed basis $\mathcal{B}_0$. The (infinitesimal) action of $GL_4(\mathbf{C})$ on T^* then induces an action I on H:

$$g \in GL_4(\mathbf{C}) : \mathcal{B} \mapsto g\mathcal{B} \ \rightsquigarrow\ |e^{\mathbf{i}}_{\mathcal{B}} > \mapsto |e^{\mathbf{i}}_{g\mathcal{B}} > =: I_g\, |e^{\mathbf{i}}_{\mathcal{B}} > . \tag{6}$$

We demand invariance of (5) under the action (6). Let for example $g_\epsilon \in GL_4(\mathbf{C})$ be given by

$$g_\epsilon : \{\underset{|}{A}, \underset{|}{B}, \underset{|}{C}, \underset{|}{D}\} \mapsto \{\underset{|}{A}, \underset{|}{B}, \underset{|}{C} + \epsilon \underset{|}{A}, \underset{|}{D}\} \,. \tag{7}$$

Applying g_ϵ to both sides of (5) and comparing coefficients of ϵ we find

$$S\left(\overset{A}{\underset{\partial_C}{|}} \, |e^{\mathrm{i}}_{\mathcal{B}} > \otimes \, |e^{\mathrm{j}}_{\mathcal{B}} >\right) = \overset{A}{\underset{\partial_C}{|}} \, S\left(|e^{\mathrm{i}}_{\mathcal{B}} > \otimes \, |e^{\mathrm{j}}_{\mathcal{B}} >\right) \tag{8}$$

and in general we obtain

$$\left[\overset{X}{\underset{\partial_Y}{|}}, S\right] = 0 \ \ for\ \underset{|}{Y} \in \mathcal{B}\ and\ arbitrary\ \underset{|}{X}. \tag{9}$$

The differential operators are understood to act on $H \otimes H$ via their action on the parameters $\mathcal{B}$ of $e^{\mathrm{i}}_{\mathcal{B}}$. This action is the infinitesimal version of (6). It carries over to the cohomology classes (3) — see [3]. (9) places severe consistency restrictions on the $S^{\mathrm{ij}}_{\mathrm{kl}}(\mathcal{B})$. For example, since every $|e^{\mathrm{i}}_{\mathcal{B}} > \otimes \, |e^{\mathrm{j}}_{\mathcal{B}} >$ can be obtained as finite linear combination

$$|e^{\mathrm{i}}_{\mathcal{B}} > \otimes \, |e^{\mathrm{j}}_{\mathcal{B}} > = \sum_k a_k \left(\prod_{l_k}^{n_k} \overset{X^{l_k}}{\underset{\partial Y^{l_k}}{|}}\right) |e^{(m,0,m,0)}_{\mathcal{B}} > \otimes \, |e^{0}_{\mathcal{B}} > \Bigg|_{\underset{|}{X}^{l_k} \in \mathcal{B}} \tag{10}$$

where $\underset{|}{Y}^{l_k} \in \mathcal{B}$, $a_k \in \mathbf{Q}$ and $m \in \mathbf{N}$ is sufficiently big, we observe that it is enough to give

$$S\left(|e^{m}_{\mathcal{B}} > \otimes \, |e^{0}_{\mathcal{B}} >\right) \in H \otimes H \ ; \ \ e^m := e^{(m,0,m,0)} \tag{11}$$

for (almost) all $m \in \mathbf{N}$ in order to fix $S : \otimes^2 H \to \otimes^2 H$.

2) Conformal invariance

Looking at the induced action of the subgroup $SU(2,2) \subset GL_n(\mathbf{C})$ (which is a $4:1$ cover of the conformal transformations of space–time) we can investigate the condition on the coefficients in (5) for S to be a 'conformally' invariant map:

$$\begin{aligned} S^{\mathrm{ij}}_{\mathrm{kl}}(\mathcal{B}) &= \ < e^{\mathcal{B}}_{\mathrm{k}} | \otimes < e^{\mathcal{B}}_{\mathrm{l}} | \ S \ | e^{\mathrm{i}}_{\mathcal{B}} > \otimes \, | e^{\mathrm{j}}_{\mathcal{B}} > \\ &= \ < e^{\mathcal{B}}_{\mathrm{k}} | \otimes < e^{\mathcal{B}}_{\mathrm{l}} | (I_g)^* S (I_g) | e^{\mathrm{i}}_{\mathcal{B}} > \otimes \, | e^{\mathrm{j}}_{\mathcal{B}} > \ = S^{\mathrm{ij}}_{\mathrm{kl}}(g\mathcal{B}) \end{aligned} \tag{12}$$

for all $g \in SU(2,2)$. Hence we conclude that the $S^{\mathrm{ij}}_{\mathrm{kl}}(\underset{|}{A}, \underset{|}{B}, \underset{|}{C}, \underset{|}{D})$ have to be $SU(2,2)$–invariant, i.e. functions of $\underset{\sqcup\!\sqcup\!\sqcup}{ABCD}$. A consideration of homogeneities then shows that in fact they have to be constants. Together with observation (11) this implies that a conformally invariant $S : H \otimes H \hookleftarrow$ is determined by

$$S\left(|e^i_B> \otimes |e^0_B>\right) = \sum_{k+l=i} S^{i\,0}_{k\,l}\, |e^k_B> \otimes |e^l_B>. \tag{13}$$

There are still relations among the $S^{i\,0}_{k\,l}$ due to the condition (9). In fact one gets for example

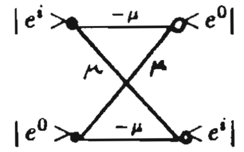
$$S_{i-k\,k}^{\;i\;\,0} = \sum_{l=0}^{k} (-1)^{k+l} \binom{i}{k} \binom{k}{l} S^{i-l\,0}_{i-l\,0} \tag{14}$$

giving S in terms of the (up to convergence conditions) free data $\{S^{i\,0}_{i\,0}\}_{i\in\mathbb{N}}$.

Applications

3) Conformally invariant twistor diagrams

For the single and double (and presumably also for higher order) box diagrams it is quite easy to compute the coefficients $S^{i\,0}_{0\,i}$ which, in an analogous way to (14), determine the respective maps completely. For the single box one finds

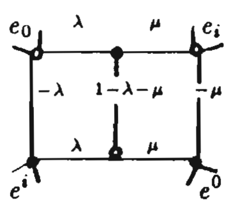


$$= (-1)^{\mu} \frac{\Gamma(1-\mu)\Gamma(1+i+\mu)}{\Gamma(2+i)} =: (s_\mu)^{i\,0}_{0\,i} \tag{15}$$

from which one can compute

$$\begin{aligned} s_\mu\left(|e^i> \otimes |e^0>\right) &= \sum_{k+l=i} (s_\mu)^{i\,0}_{k\,l}\, |e^k> \otimes |e^l> \\ &= \frac{(-1)^\mu}{1+i} \sum_{k+l=i} \frac{\Gamma(1+k-\mu)\Gamma(1+l+\mu)}{\Gamma(1+k)\Gamma(1+l)}\, |e^k> \otimes |e^l>. \end{aligned} \tag{16}$$

For the double box we get



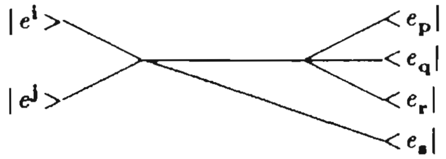
$$= \frac{\Gamma(1+\lambda+i)\Gamma(1+\mu+i)}{\Gamma(2+i)\Gamma(1+\lambda+\mu+i)} =: (s_{\lambda,\mu})^{i\,0}_{0\,i}. \tag{17}$$

Certain properties of the double box, such as derived for example in §2.4 of [5] are immediate from this expression.

4) Factorization of Feynman Diagrams

Extending the algebraic approach we try to use compositions of linear maps

$S : \otimes^m H \longrightarrow \otimes^n H$ (and their analytic continuations) to build up complex Feynman diagrams from simpler ones. Consider for example the conformally non-invariant diagram of ϕ^4-theory:



(18)

One finds (see (23) - (26)) that the corresponding map factorizes

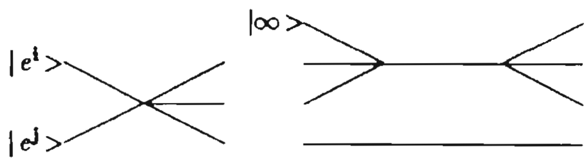
$$S : \otimes^2 H \longrightarrow \otimes^4 H$$
$$S(|e^i> \otimes |e^j>) \quad = \quad (S_2 \otimes 1)(|\infty> \otimes S_1(|e^i> \otimes |e^j>)) \ . \tag{19}$$

Here $S_1 : \otimes^2 H \longrightarrow \otimes^3 H$ and $S_2 : \otimes^3 H \longrightarrow \otimes^3 H$ are determined by

$$S_1(|e^a> \otimes |e^0>) \quad = \quad \frac{1}{1+a}\Bigg(\sum_{a=b+c+d} \underset{\sqcup}{AB} |e^b> \otimes |e^c> \otimes |e^d> \tag{20}$$
$$+ \sum_{a-1=b+c+d} \underset{\sqcup}{CB} |e^b> \otimes |e^c> \otimes |e^d>\Bigg)$$

$$S_2(|e^{(a,0,a,0)}> \otimes |e^{(b,0,0,b)}> \otimes |e^0>) \quad = \tag{21}$$
$$\frac{a!\, b!}{(a+b)!} \sum_{\substack{\sum a_i = a \\ \sum b_i = b}} \bigotimes_{i=1}^{3} \frac{(a_i+b_i)!}{a_i!\, b_i!} |e^{(a_i+b_i,0,a_i,b_i)}>$$

where one uses the appropriate version of (11) in (21). S_2 is conformally invariant and has an analytic continuation to $|\infty> \otimes H \otimes H$. The state $|\infty> \notin H$ corresponds to the constant field $\phi(x) \equiv 1$, i.e. it can be represented by a twistor function $(\overset{AB}{\underset{ZZ}{||}})^{-1}$ in the limit $\underset{++}{AB} \rightarrow \sqcap$, the line at infinity. In diagrammatic notation we can represent this factorization by



(22)

To arrive at this one uses two facts. First one assumes that in the space–time integral corresponding to (18)

$$\int \phi^{i}(x)\phi^{j}(x)\phi_{s}(x)\Delta_F(x-y)\phi_{p}(y)\phi_{q}(y)\phi_{r}(y)\, d^4x d^4y$$

$$= \int \phi^{i}(x)\phi^{j}(x)\phi_{s}(x)\Box_x^{-1}\left(\phi_{p}(x)\phi_{q}(x)\phi_{r}(x)\right) d^4x \qquad (23)$$

one can write

$$\Box_x^{-1}\left(\phi_{p}(x)\phi_{q}(x)\phi_{r}(x)\right) = \sum_{(k,l)\in I^2} a^{k\,l}_{p\,q\,r}\, \psi_{k}(x)\psi_{l}(x) \qquad (24)$$

i.e. that products of two free negative frequency z.r.m fields are dense in the space of general (L^2) negative frequency fields. Secondly, one observes that

$$(s_0 \otimes 1) \circ S_1 = S_1 \qquad (25)$$

where $s_0 = s_{\mu=0}$ from (16) corresponds to the integral of four (two + and two – frequency) z.r.m. fields. This enables one to write

$$\sum_{(k,l)} a^{k\,l}_{p\,q\,r}\ (<e_k| \otimes <e_l| \otimes <e_s|)\ S_1\ (|e^i> \otimes |e^j>)$$

$$= \sum_{(m,n,t)} (S_1)^{i\,j}_{m\,n\,t} \int \phi^{m}(x)\phi^{n}(x)\Box_x^{-1}\left(\phi_{p}(x)\phi_{q}(x)\phi_{r}(x)\right) d^4x <e_s|e^t> . \qquad (26)$$

It remains to be seen how such factorizations can be extended to general Feynman diagrams and whether they are reflected on the twistor diagram level. A rigorous treatment requires careful consideration of the various topologies involved.

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