Twistor description for Weyl's class of type D vacuum space-times

Thomas von Schroeter

In his article in TN 34, Nick Woodhouse derived the patching matrix for the Ward transform of the general anti-self-dual type D vacuum metric, the essential component of which turns out to be rational of degree 2 in a single complex variable w. As we know from other previously calculated examples, a simple form of the patching matrix in terms of rational components of low degree is also a characteristic feature of the *real* type D vacuum solutions.

In this note, I shall describe how, for the *Weyl solutions* among the type D vacuum metrics i.e. the ones which can be given in the form

$$ds^2 = f(z,r)dt^2 - \frac{r^2 d\theta^2}{f(z,r)} - \Omega^2(z,r)\left(dz^2 + dr^2\right) \tag{1}$$

our observation can be shown to arise from the existence of a Killing spinor of valence 2 - that is, a spinor field X_{AB} such that

$$\nabla_{A'(A}X_{BC)} = 0. \tag{2}$$

As this is true for all type D vacuum space-times (Walker & Penrose 1970), we expect that we will be able to generalize the argument given here.

Our strategy is to use the Yang-Mills twistor description of stationary axisymmetric vacuum space-times (Fletcher & Woodhouse 1990), in which the space-time M splits into a product of the orbits of the two Killing vectors $\partial/\partial t$ and $\partial/\partial\theta$ and a two-dimensional manifold Σ with co-ordinates (z,r), the space of orbits. For the metric (1), the patching matrix is simply $P(w) = \mathrm{diag}(1/f(w,0), f(w,0))$ and thus all we need to determine is the restriction of f to the axis (or horizon), $\{r=0\}$. If we write $J = \mathrm{diag}(-r^2/f, f)$ for the induced metric on the space of Killing vectors (i.e. $T\Sigma^\perp$), equation (2) translates into

$$\begin{aligned} F &= \mathrm{i} *F, & (3)\\ dA &= -\tfrac{3}{2}A\wedge J^{-1}dJ, & (4)\\ D_{(\alpha}A_{\beta)} &= \tfrac{1}{2}A_{(\alpha}J^{-1}\partial_{\beta)}J - \Omega^2\xi\,\delta_{\alpha\beta} & (5)\end{aligned}$$

where $F_{ab} = \epsilon_{A'B'}X_{AB}$ and $A = A_{\alpha j}dx^\alpha$ is the one-form on Σ with values in the dual of the space of Killing vectors that corresponds to X_{AB} via

$$(F_{ab}) = \begin{pmatrix} 0 & A_{\alpha j} \\ -A_{\alpha j} & 0 \end{pmatrix}.$$

Here, the matrix decomposition corresponds to the splitting $TM = T\Sigma^\perp \oplus T\Sigma$, $*F$ is the 4-space dual of F, d and D are, respectively, the operators of exterior and covariant differentiation on Σ, the indices α and β label elements of $T\Sigma$, j those of $T\Sigma^\perp$ and ξ is a complicated expression in J, A and DA with no further relevance for our purpose. (Note that (3)-(5) remain true also in the general case where J is no longer diagonal.)

If we satisfy (3) by putting

$$A = \left(-\frac{ir}{f} *\beta,\ \beta\right)$$

with $\beta =: \Omega^2\sqrt{f}\,(u\mathrm{d}z + v\mathrm{d}r)$ a one-form on Σ and $*\beta$ now denoting its *two*-space dual on Σ, then (5) implies that $h(w) := u + iv$ is holomorphic in $w = z + ir$, and (4) is equivalent to

$$\mathrm{d}\left(f^{-3/2}\beta\right) = 0 \ \text{ and } \ \mathrm{d}\left(r^{-2}\sqrt{f} *\beta\right) = 0.$$

As these equations are real, the real and the imaginary part of β will satisfy them too, and thus u and v can be taken to be real functions. As a consequence of the vacuum field equations, the conformal factor is related to f by

$$-i\partial_w \log\left(\Omega^2 f\right) = r\left(\partial_w \log f\right)^2,$$

and since J is a solution of Yang's form of the ASDYM equations, $\lambda = \log f$ has to satisfy

$$\lambda_r + r\left(\lambda_{zz} + \lambda_{rr}\right) = 0$$

(Fletcher & Woodhouse 1990).

Finally, eliminating Ω and h and expanding f near the axis (or horizon), one obtains a remarkably simple ODE for $f_0(z) := f(z, 0)$, namely

$$3f_0^{(4)}f_0'' - 4\left(f_0'''\right)^2 = 0$$

of which the general solution can be reduced to

$$f_0(z) = \begin{cases} az^{-1} + b + cz & \text{if } f_0''' \neq 0 \\ dz^2 + e & \text{if } f_0''' = 0 \end{cases}$$

by using the freedom $z \mapsto z + \text{const.}$ (here, a, b, c, d and e are real constants). As a constant overall factor in f can be absorbed into $\mathrm{d}t$ and $\mathrm{d}\theta$, one should think of both sets of parameters as homogeneous co-ordinates labelling a projective space of solutions, which is two-dimensional in the case $f_0''' \neq 0$ and one-dimensional in the case $f_0''' = 0$.

Examples:

- Flat space with time translation and rotation:

$$\mathrm{d}s^2 = \mathrm{d}t^2 - r^2\mathrm{d}\theta^2 - \mathrm{d}r^2 - \mathrm{d}z^2$$

  and hence $f \equiv 1$, $\Omega \equiv 1$, thus $a = c = 0$, $b = 1$.

- Schwarzschild space-time with time translation and rotation:

$$\mathrm{d}s^2 = \left(1 - \frac{2m}{R}\right)\mathrm{d}t^2 - \left(1 - \frac{2m}{R}\right)^{-1}\mathrm{d}R^2 - R^2\left(\mathrm{d}\psi^2 + \sin^2\psi\,\mathrm{d}\theta^2\right).$$

  The Weyl co-ordinates are $z = (R - m)\cos\psi$ and $r = \sqrt{R^2 - 2mR}\sin\psi$ and one finds $a = -2m$, $b = 1$, $c = 0$.

- Kinnersley's metric IV A (Kinnersley 1969) can be transformed to

$$ds^2 = (x^2 + a^2)\left[Cv^2 dt^2 + 2dt dv\right] - \tfrac{1}{2}\Delta^{-1} dx^2 - 2\Delta d\theta^2$$

  where

$$\Delta(x) = \frac{2mx + C(a^2 - x^2)}{2(x^2 + a^2)}.$$

  As $r = v\sqrt{2C\Delta(x^2 + a^2)}$ and $z = v(Cx - m)$, $v = 0$ is just a single point on Σ and, in order to evaluate f on $\{r = 0\}$, we have to put $\Delta = 0$. We obtain

$$d = \frac{2}{C}\left(1 + \frac{m}{\sqrt{m^2 + C^2 a^2}}\right) \quad \text{and} \quad e = 0.$$

- The vacuum C metric (Ehlers & Kundt 1962). Kinnersley & Walker (1970) give it in the form

$$ds^2 = A^{-2}(x + y)^{-2}\left(F dt^2 - F^{-1} dy^2 - G^{-1} dx^2 - G d\theta^2\right)$$

  where

$$F(y) = -1 + y^2 - 2mAy^3 \quad \text{and} \quad G(x) = 1 - x^2 - 2mAx^3.$$

  One finds $r = A^{-2}(x+y)^{-2}\sqrt{FG}$ and $z = A^{-2}(x+y)^{-2}[mAxy(y-x) - xy - 1]$. The patching matrix can be adapted to different parts of $\{r = 0\}$. For the physically relevant one, Fletcher (1990) found

$$a = 2, \quad b = -\frac{2m}{A}(\beta_1 + \beta_2), \quad \text{and} \quad c = \frac{m^2}{A^2}\beta_1\beta_2,$$

  where β_1 and β_2 are roots of F and G (both polynomials have the same roots and all three of them are real provided $m^2A^2 < 1/27$).

It is not yet clear to me what the most general case with $f_0''' = 0$ (i.e. d and e arbitrary) corresponds to.

References

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