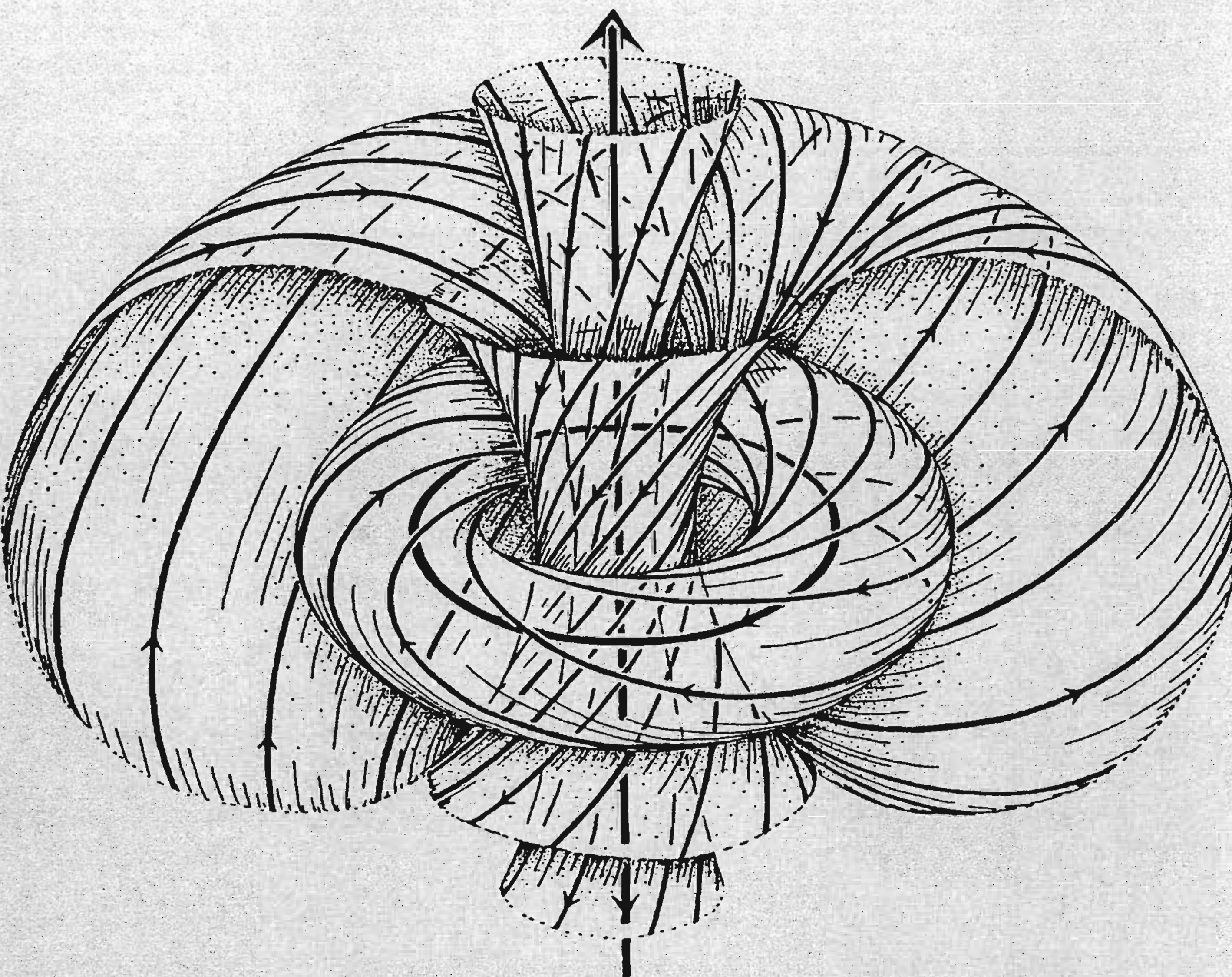
Twistor Newsletter (no 35: 8 Dec., 1992)



Mathematical Institute Oxford, England

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This is the first announcement of a

TWISTOR THEORY CONFERENCE

23rd-25th August 1993, in Devon.

Speakers include

T N Bailey (Edinburgh), M G Eastwood (Adelaide), C LeBrun (SUNY), S A Merkulov (Odense), H Pedersen (Odense), R Penrose (Oxford), K P Tod (Oxford), N M J Woodhouse (Oxford).

Place and Time

The conference will take place in our Faculty of Agriculture, Food, and Land Use, Seale Hayne, Newton Abbot, Devon. It starts after breakfast on the 23rd and ends at lunchtime on the 25th.

Registration

The registration fee will be of the order of £25, and full board for the conference will cost about £90. We are grateful to the London Mathematical Society for supporting this conference.

Contact for further information

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(maths063/slw)

Gravitational Collapse of the Wave Function: New Thoughts

A desire for a non-local basis for physical reality, as a means of making sense of the puzzling non-locality that occurs with Einstein-Podolsky-Rosen-type quantum measurements, was one powerful original motivation underlying twistor theory. According to this "twistor" view, it will not be possible to provide an adequate understanding of the state-vector reduction that quantum measurement induces, in ordinary space-time terms. In TN 26, p.25, I indicated a (somewhat fanciful) picture of how a twistor-type picture might conceivably provide some kind of resolution of the non-local puzzles that state-vector reduction confronts us with, but a properly coherent viewpoint must await appropriate fundamental changes in our present theory.

Despite the fact that we are a long way from a coherent theory, I believe that it has been possible to make some new progress which is consistent with the motivational ideas that I have described elsewhere (cf. TN 22, p.1-3, and "The Emperor's New Mind" Chapter 8) — that state-vector reduction should be a gravitational phenomenon. But the new suggestions that I am presenting here differ quite significantly from the specific proposals that I made earlier.

The starting point for this development is an expression that had some relevance to my earlier ideas, namely the integral that describes the symplectic form $\{\underset{1}{K}, \underset{2}{K}\}$ on the space of spin 2 massless fields in Minkowski space, whose linearized curvatures are given by

$$K_{abcd} = \psi_{ABCD}\,\varepsilon_{A'B'}\varepsilon_{C'D'} + \varepsilon_{AB}\varepsilon_{CD}\,\overline{\psi}_{A'B'C'D'}.$$

This is given (up to a factor) by

$$\{\underset{1}{K}, \underset{2}{K}\} = \int_\Sigma \Big(\underset{1}{\psi}_{ABCD}\,\underset{2}{\overline{\eta}}_{A'}{}^{BCD} + \underset{1}{\overline{\psi}}_{A'B'C'D'}\,\underset{2}{\eta}_{A}{}^{B'C'D'}\Big)\, d^3x^{AA'}$$

taken over a (normally spacelike) 3-surface Σ. Integrating by parts we find, schematically

$$\int \underset{1}{\psi}\,\underset{2}{\overline{\eta}} = -\int \underset{1}{\gamma}\,\underset{2}{\overline{\chi}} = \int \underset{1}{\chi}\,\underset{2}{\overline{\gamma}} = -\int \underset{1}{\eta}\,\underset{2}{\overline{\psi}}$$ (whence $\{\underset{1}{K}, \underset{2}{K}\} = -\{K\ K?\}$

where we have a (Dirac) chain of potentials

$$\nabla_{BB'}\eta_A{}^{B'C'D'} = \chi_{AB}{}^{C'D'}, \quad \nabla^{AA'}\eta_A{}^{B'C'D'} = 0,$$

$$\nabla_{CC'}\chi_{AB}{}^{C'D'} = \gamma_{ABC}{}^{D'}, \quad \nabla^{AA'}\chi_{AB}{}^{C'D'} = 0$$

$$\nabla_{DD'}\gamma_{ABC}{}^{D'} = \psi_{ABCD}, \quad \nabla^{AA'}\gamma_{ABC}{}^{D'} = 0$$

$$\nabla^{AA'}\psi_{ABCD} = 0$$

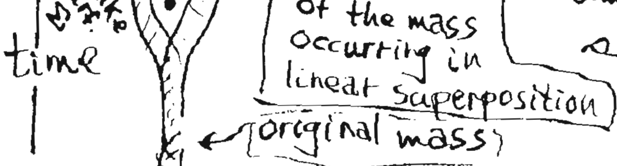
, all spinors η, χ, γ, ψ being symmetric.

N.B. $\gamma_{\dots}$ are linearized spin-coefficients and $\Gamma \sim \gamma + \bar{\gamma}$ (Christoffel). Also $h \sim \chi + \bar{\chi}$ (lin. metric)

The symplectic form { , } is closely related to the scalar product ⟨ | ⟩, but for that a positive/negative frequency split must be made. In fact $\langle K|K\rangle = i\{K, JK\}$ where J multiplies the positive-frequency part of K by i and the negative-frequency part by $-i$.

The idea underlying my earlier proposal was that for a quantum state consisting of a superposition of two significantly different $\overset{1}{\psi}$ and $\overset{2}{\psi}$ fields, state-vector reduction would take place when the difference between these states reached roughly the (longitudinal) "one-graviton level". This might be measured in terms of the "graviton number" in the difference field $\overset{1}{\psi} - \overset{2}{\psi}$, which might be obtained as an appropriate integral over a spacelike hypersurface. The view had been that some measure of longitudinal graviton number would accordingly be required, but this leads to profound problems, owing to the fact that the concept is not gauge invariant. The present idea is to use something closer to the symplectic form and to try to examine the Newtonian limit — since most situations of wave-function collapse occur when gravitational fields are weak and velocities small.

Consider a simple "Schrödinger's cat" type of situation in which a quantum measurement is to be achieved by moving a mass from one location to another. We thus have a superposition of the mass in one place with the mass in another. The question, according to the present point of view is: at what stage, as the masses are moved apart, should collapse occur? It appears that in the Newtonian limit, it makes more sense to do something like using a *timelike* hypersurface



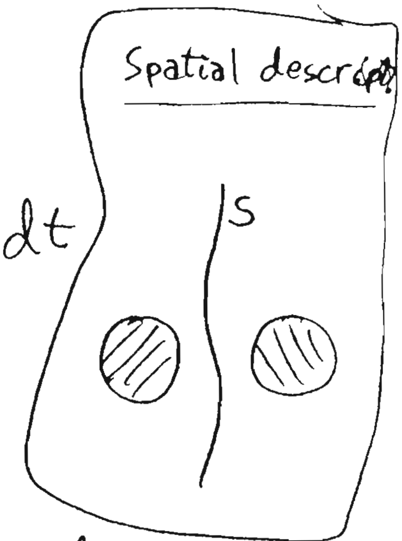
between the two masses and to perform the symplectic form integral for the two fields on that hypersurface. We take collapse to occur roughly when the region of integration is large enough that the integral reaches order unity. The integral can be written

$$\int_\Sigma \gamma_1 \bar{\chi}_2 + \int_\Sigma \bar{\gamma}_1 \chi_2 \sim \int_\Sigma (\Gamma_1 h_2 - \Gamma_2 h_1)$$

which in Newtonian terms becomes

$$\int_t^{t+T}\left(\int_S (\phi_2 \vec{\nabla}\phi_1 - \phi_1 \vec{\nabla}\phi_2)\cdot d\vec{x}\right)dt$$

$$= T\left(\int_S (\quad '' \quad)\cdot d\vec{x}\right).$$



Here T is the ~~total~~ time that the surface S persists for, so for the integral to be order unity we require $T\ (=\sim \text{collapse time}) = \dfrac{1}{\int_S (\phi_2\vec{\nabla}\phi_1 - \phi_1\vec{\nabla}\phi_2)}$.

Note that $\phi_2\vec{\nabla}\phi_1 - \phi_1\vec{\nabla}\phi_2$ is divergence-free in vacuum, so the integral is unchanged as S is moved continuously without crossing matter. When matter density ρ is present we see $\vec{\nabla}\cdot(\phi_2\vec{\nabla}\phi_1 - \phi_1\vec{\nabla}\phi_2) = -4\pi\phi_2\rho_1 + 4\pi\rho_2\phi_1$, so moving S across one mass, we find that our expression is just the gravitational energy of one mass in the gravitational field of the other ($\times 4\pi$). The collapse time this suggests is the reciprocal of this energy.

As it stands, this cannot be quite right, but it suggests a much more plausible expression, namely the energy that it would take to move the two masses apart from their initial positions of coincidence (gravitational energy only). It is worthwhile to consider the Newtonian expression in the case of a uniform sphere, mass m, radius a, distance between centres b. We get, with $\lambda = b/2a$

$$\text{Energy to separate} = \begin{cases} \dfrac{m^2 G}{a}\left(6/5 - 2\lambda^2 + \tfrac{3}{2}\lambda^3 - \tfrac{1}{5}\lambda^5\right) & (0 \le \lambda \le 1) \\[2ex] \dfrac{m^2 G}{2a\lambda} & (1 \le \lambda). \end{cases}$$

This is $\sim \dfrac{m^2 G}{a}$. In natural units ($G = c = 1 = \hbar$; gram $= 10^5$, sec $= 10^{43}$, cm $= 10^{33}$) we get: for a neutron, $T \sim 10^{60}$ = Hubble time
" " droplet of water of $a = 10^{-5}$ cm, $T = 10^{48}$ = day; $a = 10^{-4}$ cm, $T = 10^{43}$ = sec;
$a = 10^{-3}$ cm, $T = 10^{38} = 10^{-5}$ sec. All this seems fairly plausible.

Thanks especially to Ted Newman and Abhay Ashtekar. ~ R Penrose

Metrics with SD Weyl tensor from Painlevé-VI

In a Comment published last year (Class.Quant.Grav.8 (1991) 1049), I wrote down an autonomous system of six ODEs, any solution of which determines a diagonal Bianchi-type-IX metric with a self-dual (SD) Weyl tensor and vanishing scalar curvature. Call these half-conformally-flat, scalar-flat metrics. This class of metrics includes vacuum examples, scalar-flat Kähler examples and a class conformally related to some Einstein metrics with non-zero scalar-curvature. At Roger's birthday meeting last year, I described how this system could be boiled down to a single, second-order non-linear ODE which Chazy (Acta Math.34 (1911) 317) in a throw-away aside asserted was a 'transformée algebrique' of Painlevé-VI 'curieuse en raison de son élégance'. I can now, with help from Fokas and Ablowitz (J.Math.Phys.23 (1982) 2033), see how to do this transformation and this is what I shall describe here.

If the Bianchi-type-IX metric is written in the usual way as

$$ds^2 = \omega_1\omega_2\omega_3 dt^2 + \frac{\omega_2\omega_3\sigma_1{}^2}{\omega_1} + \frac{\omega_3\omega_1\sigma_2{}^2}{\omega_2} + \frac{\omega_1\omega_2\sigma_3{}^2}{\omega_3} \qquad 1$$

where each ω_i is a function only of 'time' t, then the autonomous system which I had is

$$\begin{aligned}\dot{\omega}_1 &= -\omega_2\omega_3 + \omega_1(a_2 + a_3)\\ \dot{\omega}_2 &= -\omega_3\omega_1 + \omega_2(a_3 + a_1)\\ \dot{\omega}_3 &= -\omega_1\omega_2 + \omega_3(a_1 + a_2)\end{aligned} \qquad 2$$

$$\begin{aligned}\dot{a}_1 &= -a_2a_3 + a_1(a_2 + a_3)\\ \dot{a}_2 &= -a_3a_1 + a_2(a_3 + a_1)\\ \dot{a}_3 &= -a_1a_2 + a_3(a_1 + a_2)\end{aligned} \qquad 3$$

where a dot denotes d/dt.

The second trio of equations is discussed by Ablowitz and Clarkson (in 'Solitons, Nonlinear Evolution Equations and Inverse Scattering' LMS Lecture Note Series 149) and called by them the 'Chazy system' following the solution given by Chazy (C.R.Acad.Sci.150 (1910)456). It was also solved by Brioschi (C.R.Acad.Sci. t.XCII (1881) 1389) and I will describe his method here.

First introduce X,Y by

$$X = a_1 - a_2 \qquad Y = a_3 - a_1 \qquad 4a$$

now take the differences between successive pairs of the equations in (3) to find

$$a_1 = \frac{\dot{X} + \dot{Y}}{2(X + Y)} \qquad a_2 = \frac{\dot{Y}}{2Y} \qquad a_3 = \frac{\dot{X}}{2X} \; . \qquad 4b$$

Introduce a new dependent variable x by

$$Y = \frac{\dot{x}}{2x} \qquad \text{5a}$$

when it follows from (3) that

$$X = \frac{\dot{x}}{2(1-x)} \qquad \text{5b}$$

and from (4b) and (5) that

$$a_1 = \frac{1}{2}\left(\frac{\ddot{x}}{\dot{x}} - \frac{\dot{x}}{x} + \frac{\dot{x}}{1-x}\right)$$

$$a_2 = \frac{1}{2}\left(\frac{\ddot{x}}{\dot{x}} - \frac{\dot{x}}{x}\right)$$

$$a_3 = \frac{1}{2}\left(\frac{\ddot{x}}{\dot{x}} + \frac{\dot{x}}{1-x}\right)$$

What is left of (3) is the 3rd-order ODE for x(t):

$$\frac{\dddot{x}}{\dot{x}} = \frac{3}{2}\frac{\ddot{x}^2}{\dot{x}^2} - \frac{\dot{x}^2}{2}\left(\frac{1}{x^2} + \frac{1}{x(1-x)} + \frac{1}{(1-x)^2}\right) \qquad 6$$

This is the equation satisfied by the reciprocal of the Elliptic Modular Function. (Chazy's solution of the system (3) led to what Ablowitz and Clarkson call the Chazy equation, which in turn is solved by a ratio of hypergeometric functions.)

We have solved (3). To deal with (2), we define new variables Ω_i by

$$\omega_1 = 2\Omega_1(XY)^{1/2} \; ; \; \omega_2 = 2\Omega_2(X(X+Y))^{1/2} \; ; \; \omega_3 = 2\Omega_3(Y(X+Y))^{1/2} \qquad 7$$

These definitions are motivated by consideration of (4b). When we substitute (7) into (2) we obtain the new system:

$$\dot{\Omega}_1 = -2\Omega_2\Omega_3(X+Y)$$
$$\dot{\Omega}_2 = -2\Omega_3\Omega_1 Y$$
$$\dot{\Omega}_3 = -2\Omega_1\Omega_2 X \qquad 8$$

The difficulty with (8) is the presence of X and Y. With the aid of (5), we now change the independent variable from t to x. Using a prime for d/dx, we find (8) becomes

$$\Omega_1' = \frac{-\Omega_2\Omega_3}{x(1-x)} \; ; \; \Omega_2' = \frac{-\Omega_3\Omega_1}{x} \; ; \; \Omega_3' = \frac{-\Omega_1\Omega_2}{(1-x)} \qquad 9$$

This system has arisen elsewhere in connection with integrable systems: Fokas et al (Phys.Lett. A115 (1986) 329) obtain it from a 3-wave interaction, which is known to be a reduction of SD Yang-Mills, and Dubrovin (Functional Analysis and its Applications 24 (1990) 280) obtains it from the equations for a 3-dimensional diagonal metric to be flat.

Before solving (9), we write the original metric (1) in terms of the Ω_i making use of (5) and (7). The result is

$$ds^2 = \frac{\Omega_1\Omega_2\Omega_3 x}{x(1-x)}\left[\frac{dx^2}{x(1-x)} + \frac{\sigma_1{}^2}{\Omega_1{}^2} + \frac{(1-x)\sigma_2{}^2}{\Omega_2{}^2} + \frac{x\sigma_3{}^2}{\Omega_3{}^2}\right] \qquad 10$$

The part in square brackets strongly resembles the conformal metric given by Nigel Hitchin in his seminar in Oxford last year (15/10/91) and obtained by him via a direct twistor-space-construction of Einstein, half-conformally-flat Bianchi-type-IX metrics. The conformal factor in (10) can be viewed as the factor necessary to make the scalar curvature vanish.

The system (9) has a first-integral namely

$$\Omega_2{}^2 + \Omega_3{}^2 - \Omega_1{}^2 = 2\gamma, \text{ constant} \qquad 11$$

which is the residue of a more-complicated looking first-integral of the original combined system (2,3) which can be obtained by working back from (11) through (4a) and (7).

Using the first-integral, it is relatively straightforward to reduce the system (9) to a second-order non-linear ODE for, say, Ω_3. The problem is that this ODE is quadratic in the second-derivative of Ω_3 and so cannot be one of the Painlevé equations. It is this ODE which is given by Chazy, as described in the first paragraph, and this is where I was stuck until I came across the paper of Fokas and Ablowitz. By following their procedure one is led, after some calculation, to make a change of independent variable to z via the transformation

$$x = \frac{4\sqrt{z}}{(1+\sqrt{z})^2} \qquad 12$$

and then to express Ω_3 in terms of a new dependent variable v via the transformation

$$\Omega_3 = \frac{zv_z}{v} - \frac{v}{2(z-1)} - \frac{1}{2} + \frac{z}{2v(z-1)} \qquad 13$$

The remarkable thing is that now v satisfies Painlevé-VI (as given eg by Ince or by Ablowitz and Clarkson) with the parameter values $(\alpha,\beta,\gamma,\delta) = (1/8,-1/8,\gamma,(1-2\gamma)/2)$ with γ as in (11), and in fact (13) can be inverted so that all solutions of (9) arise this way.

Fokas and Ablowitz, and also Dubrovin give some particular solutions of Painlevé-VI in terms of solutions of hypergeometric functions. Reversing the above, these will lead to particular metrics.

It should be possible, following the lead of my other article in this issue of Twistor Newsletter, to find non-diagonal generalisations of (1) in terms of Painlevé-VI with more general values of the parameters.

Paul Tod

Some new scalar-flat Kähler and hyper-Kähler metrics

We know from the work of Claude Lebrun (J.Diff.Geom. 34 (1991) 223-253) that any scalar-flat Kähler metric with a Killing vector (or 'S^1-action') arises from a solution u(x,y,z) of the equation

$$u_{xx} + u_{yy} + (e^u)_{zz} = 0 \qquad 1$$

which is variously known as the 'SU(∞) Toda field equation' or 'Boyer-Finley equation'. What is more Claude tells us in detail how to go from the metric to the function u, though given a solution u there is a choice to be made on the way back to the metric. In particular, given u there is a choice which leads back to a hyper-Kähler metric.

Henrik Pedersen and Yat Sun Poon (Class.Quant.Grav.7 (1990) 1707) found a scalar-flat Kähler metric of Bianchi-type-IX, in the terminology of relativity. This means that the metric has a 3-parameter group of isometries, isomorphic to SU(2) and transitive on 3-surfaces. In particular then the Pedersen-Poon (PP) metric comes from a solution of (1) and it is possible to follow Lebrun's direction to find out which. It turns out that the PP-metric arises from an ansatz for (1) which has a simple generalisation to a wider class of solutions of (1). Thus the PP-metric can be generalised to a wider class of scalar-flat Kähler or hyper-Kähler metrics, and this is what I want to describe here.

The idea, with the benefit of hindsight, is to seek a solution of (1) for which u is constant on central ellipsoids. In other words, define u implicitly by the equation

$$Q(u,x,y,z) \equiv X^t M(u) X = 1 \qquad 2$$

where $X^t = (x,y,z)$ and M(u) is a (symmetric, positive-definite) matrix function of u, to be found. Differentiate (2) implicitly and substitute into (1), then what remains is a second-order matrix ODE in u which can be integrated once. To write the resulting first-order equation out, first define the 3x3 matrix

$$g = \mathrm{diag}(1,1,e^u) \qquad 3$$

then the equation we want turns out to be

$$VM_u = MgM \qquad 4$$

using a subscript u to denote $\frac{d}{du}$

where V is an integral of

$$V_u = \tfrac{1}{2}\mathrm{trace}(gM) \qquad 5$$

Because of (5), equation (4) is still quite complicated, but there is a dramatic simplification if we work with minus the inverse of M:

set $N = -M^{-1}$ then (4) and (5) reduce to

$$VN_\omega = g \qquad 6$$

and

$$V_\omega = -\tfrac{1}{2}\mathrm{trace}(gN^{-1}). \qquad 7$$

From (3) and (6), the off-diagonal terms in N are constant so set

$$N = \begin{pmatrix} a & \nu & \mu \\ \nu & b & \lambda \\ \mu & \lambda & c \end{pmatrix} \qquad 8$$

where λ, μ, ν are constants, then (6) in components reduces to the 3 equations

$$a_\omega = b_\omega = 1/V \;;\; c_\omega = e^\omega/V \qquad 9$$

At once, from (9)

$$a-b = 2\zeta \quad \text{constant},$$

where we retain the convention of using Greek letters for constants. Define a new constant η by

$$\eta^2 = \zeta^2 + \nu^2$$

then a convenient parametrisation, in terms of a new variable w, turns out to be

$$a = \eta\left(\frac{w+1}{w-1}\right) + \zeta \qquad b = \eta\left(\frac{w+1}{w-1}\right) - \zeta \qquad 10$$

Next we note from (7) that

$$V^2\det N = \xi \text{ constant} \qquad 11$$

Our aim is to obtain an equation for w. We solve (10) and (11) for c in terms of w, V and constants, then eliminate V in favour of w_ω using (9) and (10). This gives c in terms of w, w_ω and constants and then the last part of (9) gives a second-order ODE for w. A change of independent variable

$$z = e^\omega \qquad 12$$

enables this final ODE to be written as

$$\frac{d^2w}{dz^2} = \left(\frac{1}{2w} + \frac{1}{w-1}\right)\left(\frac{dw}{dz}\right)^2 - \frac{1}{z}\frac{dw}{dz} + \frac{(w-1)^2(\alpha w + \beta)}{z^2}\frac{}{w} + \frac{\gamma w}{z} + \frac{\delta w(w+1)}{w-1} \qquad 13$$

where $\alpha, \beta, \gamma, \delta$ are constants which can be expressed in terms of the constants which we already have ie $\lambda, \mu, \nu, \xi . \eta . \zeta$. Painlevé-buffs will recognise the ODE in (13) as Painlevé-V (P-V for short), with the usual conventions. In fact, the constant δ turns out to be zero in this case, and Fokas and Ablowitz (J.Math.Phys. 23 2033 (1982)) explain how to transform this special case of P-V to P-III.

Given a solution $w(z)$ of (13), we can work our way back to the matrix N and so to M and to $u(x,y,z)$. To be able to write down a scalar-flat Kähler metric, we have still to solve a linear PDE, but to write down a hyper-Kähler metric, there are no more choices to be made: given $u(x,y,z)$ the metric is immediate. In particular, given the function $u(x,y,z)$ appropriate to the PP-metric, there is a hyper-Kähler metric with the same u (apparently a new one, but it won't be of Bianchi-type-IX unless its already known!).

The special case of this procedure when the matrix M in (2) is diagonal, or equivalently when the constants λ,μ,ν in (8) are all zero, is the one which leads to the PP-metric, which was already known to involve P-III. What is obscure in this approach is why the PP solution of (1), which must lead to a scalar-flat Kähler metric with an S^1-action, actually leads to a far more symmetric metric with an SU(2)-action. This raises the possibility that these more general solutions of (1) also lead to scalar-flat Kähler metrics with SU(2)-action. If these exist, and I am grateful to Andrew Dancer for the suggestion that they do, then their metrics will look like

$$ds^2 = Fdt^2 + A_{ij}\sigma^i\sigma^j \qquad 14$$

where A_{ij} is a matrix function of t, and the σ^i are a basis of left-invariant one-forms for S^3. The question is whether one can relate the matrix A to the matrix M of (2). Certainly for the PP-metric, when M and A are both diagonal and t in (14) is proportional to u, this is possible.

One could also try different generalisations of the ansatz (2): for example with hyperboloids in place of ellipsoids (ie with an indefinite M(u) rather than a positive-definite one) which presumably leads to a Bianchi-type-VIII metric, or with paraboloids or even with planes (This case, which is fairly easy to do, leads to a Riccati equation in place of (13)). And, of course, free with every solution of (1) one obtains a hyper-Kähler metric too.

Acknowledgement
I gratefully acknowledge that this calculation was precipitated by Andrew Dancer's suggestion to me that the PP-metrics have non-diagonal generalisations.

Paul Tod

11

A 'twistor transform' for complex manifolds with connection

by Dominic Joyce, Christ Church.

In this note we will briefly describe the geometry of a class of complex manifolds, to be called *complex-flat manifolds*, that have a connection ∇ satisfying a curvature condition given in §1, which is the curvature condition satisfied by the Levi-Civita connection of a Kähler manifold. The structure has a sort of twistor transform: in §2, ∇ will be used to define an almost complex structure J on the tangent bundle of X, and it will be shown that J is integrable exactly when the curvature condition holds.

It therefore gives a miniature picture of the Penrose transform for conformal 4-manifolds, where the Cartan conformal connection is used to define a complex structure on a bundle, and the integrability condition is a condition on the conformal curvature. In §3 we give some examples of complex-flat manifolds.

1. Connections, curvature and complex structures

We begin by recalling how to decompose tensors relative to a complex structure I. Let X be a complex manifold, with complex structure I, which will be written with indices as I^k_j with respect to some real coordinate system $(x^1,\ldots,x^{2n})$. Let $K = K^{a\cdots}_{\cdots}$ be a tensor on X, taking values in $\mathbb{C}$. Here a is a contravariant index of K, and any other indices of K are represented by dots. The Greek characters $\alpha,\beta,\gamma,\delta,\epsilon$, and the starred characters $\alpha^*,\beta^*,\gamma^*,\delta^*,\epsilon^*$, will be used in place of the Roman indices a,b,c,d,e respectively. They are tensor indices with respect to $(x^1,\ldots,x^{2n})$ in the normal sense, and their use is actually a shorthand indicating a modification to the tensor itself.

Define $K^{\alpha\cdots}_{\cdots} = (K^{a\cdots}_{\cdots} + iI^a_j K^{j\cdots}_{\cdots})/2$ and $K^{\alpha^*\cdots}_{\cdots} = (K^{a\cdots}_{\cdots} - iI^a_j K^{j\cdots}_{\cdots})/2$. In the same way, if b is a covariant index on a complex-valued tensor $L^{\cdots}_{b\cdots}$, define $L^{\cdots}_{\beta\cdots} = (L^{\cdots}_{b\cdots} - iI^j_b L^{\cdots}_{j\cdots})/2$ and $L^{\cdots}_{\beta^*\cdots} = (L^{\cdots}_{b\cdots} + iI^j_b L^{\cdots}_{j\cdots})/2$. Then $K^{\alpha\cdots}_{\cdots}$ and $L^{\cdots}_{\beta\cdots}$ are the components of K and L that are *complex linear* w.r.t. I, and the starred versions are the components that are *complex antilinear* w.r.t. I. These operations are projections, and satisfy $K^{a\cdots}_{\cdots} = K^{\alpha\cdots}_{\cdots} + K^{\alpha^*\cdots}_{\cdots}$ and $L^{\cdots}_{b\cdots} = L^{\cdots}_{\beta\cdots} + L^{\cdots}_{\beta^*\cdots}$. The complex decomposition of a real-valued tensor is *self-adjoint*. This means that changing round starred and unstarred indices has the same effect as complex conjugation. All the tensors we deal with will be self-adjoint.

Let ∇ be a torsion-free connection on X satisfying $\nabla I = 0$. The connection will be written in the usual way as Γ^a_{bc}, relative to the coordinate system $(x^1,\ldots,x^{2n})$. In this fixed coordinate system, Γ may be decomposed into components relative to I as in the previous subsection, but as Γ is not a tensor this decomposition does depend on the coordinate system. Therefore, we shall consider only coordinate systems $(x^1,\ldots,x^{2n})$ with the property that I is constant in coordinates, i.e. $\partial I^a_b/\partial x^c = 0$ for all a,b,c.

As $\nabla I = 0$ we have $\Gamma^a_{bc} = \Gamma^\alpha_{\beta c} + \Gamma^{\alpha^*}_{\beta^* c}$, and as ∇ is torsion-free $\Gamma^a_{bc} = \Gamma^a_{cb}$. Together these imply that $\Gamma^a_{bc} = \Gamma^\alpha_{\beta\gamma} + \Gamma^{\alpha^*}_{\beta^*\gamma^*}$. Now the curvature $R^a{}_{bcd}$ of ∇ is given by $R^a{}_{bcd} = \partial\Gamma^a_{bd}/\partial x^c - \partial\Gamma^a_{bc}/\partial x^d + \Gamma^a_{jc}\Gamma^j_{bd} - \Gamma^a_{jd}\Gamma^j_{bc}$. Substituting in for Γ gives $R^a{}_{bcd} = R^\alpha{}_{\beta cd} + R^{\alpha^*}{}_{\beta^* cd}$.

Because ∇ is torsion-free, R satisfies the Bianchi identity $R^a{}_{bcd} + R^a{}_{cdb} + R^a{}_{dbc} = 0$, and thus $R^\alpha{}_{\beta\gamma^*\delta^*} + R^\alpha{}_{\gamma^*\delta^*\beta} + R^\alpha{}_{\delta^*\beta\gamma^*} = 0$. But from above the last two terms are zero, and so $R^\alpha{}_{\beta\gamma^*\delta^*} = 0$, and similarly $R^{\alpha^*}{}_{\beta^*\gamma\delta} = 0$. Therefore

$$R^a{}_{bcd} = R^\alpha{}_{\beta\gamma\delta} + R^\alpha{}_{\beta\gamma^*\delta} + R^\alpha{}_{\beta\gamma\delta^*} + R^{\alpha^*}{}_{\beta^*\gamma^*\delta^*} + R^{\alpha^*}{}_{\beta^*\gamma^*\delta} + R^{\alpha^*}{}_{\beta^*\gamma\delta^*}. \quad (1)$$

Now by [Bo], Lemma 5, the curvature tensor of a Kähler manifold satisfies

$$R^{a}{}_{bcd} = R^{\alpha^*}{}_{\beta^*\gamma\delta^*} + R^{\alpha^*}{}_{\beta^*\gamma^*\delta} + R^{\alpha}{}_{\beta\gamma\delta^*} + R^{\alpha}{}_{\beta\gamma^*\delta}. \tag{2}$$

So the curvature of the Levi-Civita connection of a Kähler metric satisfies (2), whereas the curvature of a torsion-free $GL(n,\mathbb{C})$- connection ∇ only need satisfy (1), which is weaker. A torsion-free connection ∇ will be called *complex-flat* if $\nabla I = 0$ and its curvature satisfies (2). In fact, as the curvature of ∇ already satisfies (1) it is necessary and sufficient that it should satisfy the additional condition $R^{\alpha}{}_{\beta\gamma\delta} = 0$.

2. The twistor transform

Let X be a complex manifold, with complex structure I, equipped with a torsion-free connection ∇ satisfying $\nabla I = 0$. The tangent bundle TX of X is naturally a complex manifold, with complex structure also denoted I. Using ∇, a second almost complex structure J will be defined upon the total space of TX, which will turn out to be integrable exactly when $R^{\alpha}{}_{\beta\gamma\delta} = 0$. So J is a complex structure if and only if ∇ is a complex-flat connection.

Let $x \in X$ and $y \in T_xX$. Then (x,y) is a point in TX. The tangent space $T_{(x,y)}(TX)$ splits naturally into a direct sum $H \oplus V$, where H is the horizontal subspace of the connection ∇ at (x,y), and V is the tangent space of the fibre of TX over x. Now V is closed under I as TX is a holomorphic bundle, and H is closed under I as $\nabla I = 0$. Let v be a vector in $T_{(x,y)}(TX)$. Under the splitting $T_{(x,y)}(TX) = H \oplus V$, we may write $v = (v_1, v_2)$. Define $Jv = (Iv_1, -Iv_2)$ for all vectors v, and for all $x \in X, y \in T_xX$. This defines an almost complex structure J on the total space of TX, commuting with I and projecting down to I on X.

We will write J out explicitly in terms of the connection components Γ, and calculate the Nijenhuis tensor N_J of J, which will give the condition for J to be integrable. Let $(x^1, \dots, x^{2n})$ be a coordinate system as in §1, for some open set $U \subset X$. Let $(y^1, \dots, y^{2n})$ be coordinates w.r.t. the basis $(\partial/\partial x^1, \dots, \partial/\partial x^{2n})$ for the fibres of TU. Then $(x^1, \dots, x^{2n}, y^1, \dots, y^{2n})$ are coordinates for TU. In these coordinates, J is

$$J\left(p^a \frac{\partial}{\partial x^a} + q^a \frac{\partial}{\partial y^a}\right) = I^a_b p^b \frac{\partial}{\partial x^a} - I^a_b q^b \frac{\partial}{\partial y^b} - 2 I^d_a \Gamma^a_{bc} y^b p^c \frac{\partial}{\partial y^d}.$$

Decomposing this expression w.r.t. I leads to some simplifications, as we may use the facts that $\Gamma^a_{bc} = \Gamma^\alpha_{\beta\gamma} + \Gamma^{\alpha^*}_{\beta^*\gamma^*}$ and $I^a_b = i\delta^\alpha_\beta - i\delta^{\alpha^*}_{\beta^*}$. So we have

$$\begin{aligned} J\left(p^a \frac{\partial}{\partial x^a} + q^a \frac{\partial}{\partial y^a}\right) = {} & ip^\alpha \frac{\partial}{\partial x^\alpha} - ip^{\alpha^*} \frac{\partial}{\partial x^{\alpha^*}} - iq^\alpha \frac{\partial}{\partial y^\alpha} + iq^{\alpha^*} \frac{\partial}{\partial y^{\alpha^*}} \\ & - 2i\Gamma^\alpha_{\beta\gamma} y^\beta p^\gamma \frac{\partial}{\partial y^\alpha} + 2i\Gamma^{\alpha^*}_{\beta^*\gamma^*} y^{\beta^*} p^{\gamma^*} \frac{\partial}{\partial y^{\alpha^*}}. \end{aligned}$$

Theorem. *The almost complex structure J is integrable if and only if $R^{\alpha}{}_{\beta\gamma\delta} = 0$.*

Proof. By the Newlander-Nirenberg theorem, a necessary and sufficient condition for the integrability of J is the vanishing of the Nijenhuis tensor N_J of J, which is given by $N_J(v,w) = [v,w] + J([Jv,w] + [v,Jw]) - [Jv,Jw]$. We shall evaluate N_J with $v = p^a\partial/\partial x^a + q^a\partial/\partial y^a$ and $w = r^a\partial/\partial x^a + s^a\partial/\partial y^a$, where p^a, q^a, r^a and s^a are constants independent of x^a, y^a. It is easy to see that $[v,w] = 0$. Using the fact that J acts as $-I$ on V, one calculates that

$$J([Jv,w]) = 2r^d \frac{\partial \Gamma^\alpha_{\beta\gamma}}{\partial x^d} y^\beta p^\gamma \frac{\partial}{\partial y^\alpha} + 2r^d \frac{\partial \Gamma^{\alpha^*}_{\beta^*\gamma^*}}{\partial x^d} y^{\beta^*} p^{\gamma^*} \frac{\partial}{\partial y^{\alpha^*}} + 2\Gamma^a_{bc} s^b p^c \frac{\partial}{\partial y^a},$$

$$J([v,Jw]) = -2p^d \frac{\partial \Gamma^\alpha_{\beta\gamma}}{\partial x^d} y^\beta r^\gamma \frac{\partial}{\partial y^\alpha} - 2p^d \frac{\partial \Gamma^{\alpha^*}_{\beta^*\gamma^*}}{\partial x^d} y^{\beta^*} r^{\gamma^*} \frac{\partial}{\partial y^{\alpha^*}} - 2\Gamma^a_{bc} q^b r^c \frac{\partial}{\partial y^a},$$

and $[Jv, Jw] =$

$$\begin{aligned}
&\left(ip^\delta \frac{\partial}{\partial x^\delta} - ip^{\delta^*} \frac{\partial}{\partial x^{\delta^*}}\right)\left(-2i\Gamma^\alpha_{\beta\gamma} y^\beta r^\gamma \frac{\partial}{\partial y^\alpha} + 2i\Gamma^{\alpha^*}_{\beta^*\gamma^*} y^{\beta^*} r^{\gamma^*} \frac{\partial}{\partial y^{\alpha^*}}\right)\\
&- \left(ir^\delta \frac{\partial}{\partial x^\delta} - ir^{\delta^*} \frac{\partial}{\partial x^{\delta^*}}\right)\left(-2i\Gamma^\alpha_{\beta\gamma} y^\beta p^\gamma \frac{\partial}{\partial y^\alpha} + 2i\Gamma^{\alpha^*}_{\beta^*\gamma^*} y^{\beta^*} p^{\gamma^*} \frac{\partial}{\partial y^{\alpha^*}}\right)\\
&- 4\Gamma^\delta_{\beta\gamma} y^\beta p^\gamma \Gamma^\alpha_{\delta\epsilon} r^\epsilon \frac{\partial}{\partial y^\alpha} - 4\Gamma^{\delta^*}_{\beta^*\gamma^*} y^{\beta^*} p^{\gamma^*} \Gamma^{\alpha^*}_{\delta^*\epsilon^*} r^{\epsilon^*} \frac{\partial}{\partial y^{\alpha^*}}\\
&+ 4\Gamma^\delta_{\beta\gamma} y^\beta r^\gamma \Gamma^\alpha_{\delta\epsilon} p^\epsilon \frac{\partial}{\partial y^\alpha} + 4\Gamma^{\delta^*}_{\beta^*\gamma^*} y^{\beta^*} r^{\gamma^*} \Gamma^{\alpha^*}_{\delta^*\epsilon^*} p^{\epsilon^*} \frac{\partial}{\partial y^{\alpha^*}}\\
&- 2\Gamma^\alpha_{\beta\gamma} q^\beta r^\gamma \frac{\partial}{\partial y^\alpha} - 2\Gamma^{\alpha^*}_{\beta^*\gamma^*} q^{\beta^*} r^{\gamma^*} \frac{\partial}{\partial y^{\alpha^*}} + 2\Gamma^\alpha_{\beta\gamma} s^\beta p^\gamma \frac{\partial}{\partial y^\alpha} + 2\Gamma^{\alpha^*}_{\beta^*\gamma^*} s^{\beta^*} p^{\gamma^*} \frac{\partial}{\partial y^{\alpha^*}}.
\end{aligned}$$

Combining the above gives

$$N_J(v,w) = 4R^\alpha{}_{\beta\gamma\delta} y^\beta r^\gamma p^\delta \frac{\partial}{\partial y^\alpha} + 4R^{\alpha^*}{}_{\beta^*\gamma^*\delta^*} y^{\beta^*} r^{\gamma^*} p^{\delta^*} \frac{\partial}{\partial y^{\alpha^*}},$$

using the expression for R in §1. As this holds for all v, w and y^a for each fixed x, $N_J = 0$ identically if and only if $R^\alpha{}_{\beta\gamma\delta} = 0$. □

3. Examples

The simplest examples of complex-flat manifolds are Kähler manifolds, taking ∇ to be the Levi-Civita connection of the Kähler metric. However, there are many other examples of complex-flat manifolds with no compatible Kähler metric. We shall comment briefly on three such families. Firstly, using the work of [J] for hypercomplex manifolds it is possible to define a quotient construction for complex-flat manifolds analogous to the Kähler quotient. Starting with a flat complex-flat structure one may produce non-Kähler complex-flat structures by choosing a moment map not compatible with any Kähler metric.

Another way of constructing examples is to consider complex submanifolds of complex-flat manifolds. To induce a connection on the tangent bundle of a submanifold M of X we need a splitting $TX|_M = TM \oplus V$ for some vector bundle V; for the induced connection to be complex-flat, it turns out that V must be a holomorphic subbundle w.r.t. J. In the case, say, of projective varieties in $X = \mathbb{CP}^n$, there may be many different choices of V satisfying this condition, and each will give a distinct complex-flat connection on M.

Our final family of examples are hypercomplex manifolds. A hypercomplex manifold is a manifold M^{4n} with complex structures I_1, I_2 and I_3 satisfying $I_1 I_2 = I_3$. By [S], §6, there is a unique connection ∇ on M called the Obata connection, that is torsion-free and satisfies $\nabla I_j = 0$. We shall show that ∇ is a complex-flat connection for each of the complex structures I_1, I_2, I_3. Thus hypercomplex manifolds are examples of complex-flat structures that in general do not come from Kähler structures.

Proposition. *Let M and ∇ be as above. Then the curvature $R^a{}_{bcd}$ of ∇ satisfies $R^\alpha{}_{\beta\gamma\delta} = 0$ in the complex decomposition with respect to each complex structure I_j. Thus (M, I_j, ∇) is a complex-flat manifold.*

Proof. We shall prove the result for I_1, for by symmetry it then holds for I_2, I_3. As ∇ is torsion-free and $\nabla I_2 = 0$, from §1 the curvature R satisfies $R^a{}_{bcd} = R^\alpha{}_{\beta cd} + R^{\alpha^*}{}_{\beta^* cd}$ in the complex decomposition w.r.t. I_2, and so $R^a{}_{bcd} = -(I_2)^a_j (I_2)^k_b R^j{}_{kcd}$. Also, from §1 the component $R^{\alpha^*}{}_{\beta^* \gamma\delta}$ is zero in the complex decomposition w.r.t. I_1. Therefore

$$\begin{aligned} 0 &= (1 - iI_1)^a_p (1 + iI_1)^q_b (1 - iI_1)^r_c (1 - iI_1)^s_d R^p{}_{qrs} \\ &= (1 - iI_1)^a_p (1 + iI_1)^q_b (1 - iI_1)^r_c (1 - iI_1)^s_d (I_2)^p_j (I_2)^k_q R^j{}_{krs} \\ &= (I_2)^a_j (I_2)^k_b (1 + iI_1)^j_p (1 - iI_1)^q_k (1 - iI_1)^r_c (1 - iI_1)^s_d R^p{}_{qrs}, \end{aligned}$$

where $I_1 I_2 = -I_2 I_1$ is used in the last line. So

$$\frac{1}{16}(1 + iI_1)^a_p (1 - iI_1)^q_b (1 - iI_1)^r_c (1 - iI_1)^s_d R^p{}_{qrs} = R^\alpha{}_{\beta\gamma\delta} = 0 \tag{3}$$

in the complex decomposition w.r.t. I_1, which is the condition for (I_1, ∇) to be a complex-flat structure on M. □

Thus the results of §2 apply to hypercomplex manifolds, and lead to some new ideas about the Obata connection and complex structures on the tangent and cotangent bundles of a hypercomplex manifold.

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Exceptionally Vile Invariants

A. Rod Gover 12 August 1992

Regard $\mathbf{P}^n$ as a homogeneous space for $\mathrm{SL}(n+1,\mathbf{R})$ and, for a particular homogeneous bundle over $\mathbf{P}^n$, consider the problem of constructing all density valued differential invariants on the bundle which are polynomial in the jets. We will call such objects ***projective invariants***. An example for $n = 1$ is given by the formula,

$$wf\nabla\nabla f - (w-1)\nabla f\nabla f$$

where f has weight w. Here ∇, (often written $\eth$ and called edth) is a local flat affine connection. Of course $\mathbf{P}$ does not have a unique such connection but rather a family of them related by transformation formulae, $\nabla \mapsto \nabla f + w\Upsilon f$ where Υ satisifies $\nabla\Upsilon = \Upsilon^2$ (see e.g. [1] for the corresponding formulae on $\mathbf{P}^n$). The point is that this differential operator is invariant under these transformations. (An analogous, and more familiar, problem is to find conformally invariant differential equations for flat conformal structures. The standard model for the latter is S^n as a homogeneous space for $\mathrm{SO}(n+1,1)$. Of course for $n = 1$ these are the *same* problem.)

Recall that a function f of weight w on $\mathbf{P}^n$ corresponds to a function on $\mathbf{R}^{n+1}$ which is homogeneous of degree w, i.e. $f(\lambda X^A) = \lambda^w f(X^A)$, where X^A are the standard coordinates on $\mathbf{R}^{n+1}$. So a good trick for proliferating many projective invariants on $\mathbf{P}^n$ is simply to write down affine invariants on $\mathbf{R}^{n+1}$ and then regard these as invariants on $\mathbf{P}^n$ by simply insisting that f be homogeneous of some weight. For example if $n = 1$ then the following is an affine invariant on $\mathbf{R}^{n+1}$,

$$\epsilon^{AC}\epsilon^{BD}\partial_A\partial_B f\partial_C\partial_D f$$

where $\partial_A := \partial/\partial X^A$. The standard representation theory of $\mathrm{SL}(n+1,\mathbf{R})$, due to Weyl and others, tells us scalar valued affine invariants on $\mathbf{R}^{n+1}$ are always linear combinations of such complete contractions. If, in the last formula, we now restrict f to be homogeneous of weight w then we obtain a projective invariant. In fact for $w \neq 1$ this is precisely the invariant mentioned earlier. Invariants which arise this way are called ***Weyl invariants***.

Since it is evidently possible to list all Weyl invariants, it is interesting to ask if all projective invariants are Weyl. It turns out [7] that if the weight w of f is non-integral or negative integral then all invariants are Weyl. However for the remaining case with w non-negative integral, to which we now restrict our attention, we shall see that it is easy to write down some invariants which are not Weyl. Such will be called *exceptional* invariants or, since these are the troublemakers of the invariant world and with a view to homophony, *vile* invariants. Even for the simple case of densities on $\mathbf{P}^n$ it is not known how to sort out the Weyl invariants from the vile invariants. However, there is a simpler, yet very important, problem on which much progress has been made. For each weight w there is a linear invariant differential operator $\mathcal{O}(w) \to \mathcal{O}_{\underbrace{(ab\cdots d)}_{w+1}}(w)$;

in terms of a local affine connection ∇_a on $\mathbf{P}^n$, this is given by $f \mapsto \nabla_a \nabla_b \cdots \nabla_d f$. This operator splits the jet bundle. So instead of looking for invariants on the jets of $\mathcal{O}(w)$ one can look for invariants on the "slightly smaller" space which, at a particular point of $\mathbf{P}^n$, is the jets of $\mathcal{O}(w)$ modulo the kernel of this operator. There are analogous conformal and CR versions of this latter problem too [4,6] and they are geometrical equivalents of some difficult algebraic problems first posed and discussed by Fefferman [5]. On $\mathbf{P}^n$, I have completely solved this problem [7]. It turns out that, in this case, there are non-vanishing exceptional invariants. For example, if $w = 1$, then $\nabla_a \nabla_b f$ is invariant and therefore when $n = 2$ one can construct the projective invariant

$$\epsilon^{ac}\epsilon^{bd}\nabla_a\nabla_b f \nabla_c \nabla_d f.$$

Since its homogeniety with respect to f is just two it cannot be a Weyl invariant (the construction of which requires that a ϵ^{ABC} be used). The general situation is well characterised by this example; in n dimensions the exceptional invariants are always constructed by a contraction of $w+1$ $\epsilon^{ab\cdots d}$'s into an n-fold juxtaposition of the linear operator with itself.

By adapting the methods in [7] and invoking some new tricks Bailey, Eastwood and Graham [2] were able to solve the corresponding problems for flat CR structures and odd dimensional conformal S^n. There are no exceptional invariants in the CR case but for the conformal S^n case there are. Here again it turns out that operators which are homogeneous of degree n in the argument density f are exceptional while all others are Weyl. However, in [2] the authors posed the question of whether the exceptionals were, as in the projective case, constructed purely from juxtapositions of the linear invariant $\underbrace{\nabla_{(a}\nabla_b\cdots\nabla_{d)_0}}_{w+1} f$, where w is the (non-negative integral) weight of f.

More recently I was investigating the same question for invariants of vector fields on $\mathbf{P}^n$ and discovered a means of generating exceptionals which are rather more vile. Here is a simple example. Consider the problem of constructing invariants of the module which is jets of vectors v^a of weight 0, at some point modulo the kernel of the linear invariant differential operator. In this case this operator is trace-free$(\nabla_a\nabla_b v^c)$ and is given in terms of $\mathbf{R}^{n+1}$ objects by $v^C_{AB} := \partial_A\partial_B v^C$, where v^C satisfies (the divergence free "gauge" condition) $\partial_C v^C = 0$. Note that, as well as being trace free, v^C_{AB} is annihilated upon contraction with X^A, since v^A is homogeneous of degree 1 with respect to X^A. Now let β_A be a covector in R^{n+1} which satisfies $X^A\beta_A = 1$. So β_A is homogeneous of degree -1 and is only defined up to transformations $\beta_A \mapsto \hat{\beta}_A = \beta_A + \Upsilon_A$ where $X^A\Upsilon_A = 0$. Consider now the object

$$\partial_E\partial_F(v^E_{AB}v^F_{CD}\epsilon^{ACI}\epsilon^{BDJ}\beta_I\beta_J).$$

This is clearly an invariant provided it is independent of the choice of β_A. Indeed it is independent of β_A, and so an invariant, because $\epsilon^{ACI}\hat{\beta}_I = \epsilon^{ACI}\beta_I + X^{[A}\gamma^{C]}$ for some γ^C. When expanded out it is given by the formula

$$\epsilon^{cd}\epsilon^{ef}(\nabla_c\nabla_e\nabla_a v^b\nabla_d\nabla_f\nabla_b v^a + 2\nabla_c\nabla_e\nabla_b\nabla_a v^a\nabla_d\nabla_f v^b + \nabla_c\nabla_e\nabla_a v^a\nabla_d\nabla_f\nabla_b v^b)$$

and so is clearly non-zero. (Thanks to Michael Eastwood for helping check this expansion.) Furthermore this is certainly an exceptional invariant since it is homogeneous of degree just two with respect to v^A and is an example which is not simply a juxtapostion of the linear invariant with itself. Fortunately it turns out [8] that, for vectors on $\mathbf{P}^n$, all exceptional invariants can be constructed by a generalisation of the method used for this example. So the exceptional invariants can now be listed as readily as Weyl invariants. These methods work for many other similar modules. For example an important application of these new results and the earlier ones mentioned is for the job of listing all invariants of projective structures (i.e. the *curvature invariants* of a projective manifold). There is an algebraic problem which arises in this context, analogous to the ones alluded to above, which I have now solved. This will appear in [9]. Toby Bailey and I [3] have shown that these arguments also work for the conformal case.

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On the symmetries of the reduced self-dual Yang-Mills equations

L. J. Mason

Introduction

One of the remarkable features of reductions of the self-dual Yang-Mills equations to systems in two dimensions is that the symmetry group of the reduced equations (in the context of space-time symmetries) is much larger than one might have expected, often being infinite dimensional. A priori, one would expect the symmetry group of the reduced equations to be just the projection of those conformal symmetries in 4–dimensions that normalize the invariance group that one is reducing by. In Hitchin (1987) it was observed that the reductions of the self-dual Yang-Mills equations on Euclidean 4-dimensional space by two translations are actually conformally invariant in the infinite dimensional sense in the residual 2-dimensional space (a priori one would only expect the equations to be invariant under the 2-dimensional Euclidean group plus scalings). In Mason & Sparling (1992) it was observed that reductions of self-dual Yang-Mills by 2 translations spanning a 2-plane on which the metric has rank one also has an infinite dimensional symmetry group at least when the gauge group is $SL(2)$—nonlinear analogues of the Galilean group in so called $(1+0)$–dimensions as opposed to just the linear Galilean group in 2-dimensions..

The purpose of this note is to clarify the geometry underlying this result and state it independently of the gauge group. I also discuss two other examples of this phenomena, one being the reduction by symmetries spanning a totally null ASD 2-plane where the symmetry group is the whole diffeomorphism group (rather than just $GL(2)$), and the other being the reduction by two rotations (or a rotation and a translation) in which the symmetry group is the hyperbolic group in 2-dimensions ($SL(2,\mathbb{R})$).

An important corollary of Hitchins result is that it makes it possible to transfer the equations to a general Riemann surface where they considerably enrich the theory of holomorphic vector bundles. The above results give alternate ways of transferring different reductions of the self-dual Yang-Mills equations to 2-dimensional surfaces endowed with different geometric structures.

The Yang-Mills Higgs equation on a Riemann surface

First a brief review of Hitchin's equations. We will use $(z, w, \tilde{z}, \tilde{w})$ as coordinates on $\mathbf{R}^4$ that are independent and real for signature $(2,2)$ or complex with $\tilde{z} = \bar{z}$ etc. for Euclidean signature. We start with the Lax pair formulation of the self-dual Yang-Mills equations.

The self-dual Yang-Mills equations are the compatibility conditions for the pair of operators:

$$L_0 = D_z - \lambda D_{\tilde{w}}, \quad L_1 = D_w + \lambda D_{\tilde{z}}.$$

where $\lambda \in \mathbf{C}$ is an auxilliary complex parameter and D_z is the covariant derivative of some Yang-Mills connection in the direction $\partial/\partial z$.

For Hitchin's equations we start in Euclidean signature and impose symmetries in the $\partial/\partial w$ and $\partial/\partial \bar{w}$ directions. In an invariant gauge (i.e. one in which the gauge potentials are independent of $(w, \bar{w})$, $D_w = \partial/\partial w + \bar{\Phi}'$ and cc. and we can throw away the derivatives with respect to $(w, \bar{w})$ to leave the pair of operators (with a litte rearrangement):

$$L_0 = D_z - \lambda\Phi', \quad L_1 = D_{\bar{z}} + \frac{1}{\lambda}\bar{\Phi}'.$$

We can make this more geometric by multiplying L_0 by dz and L_1 by $d\bar{z}$ and defining $\Phi = \Phi' dz$. We then obtain the form valued operator:

$$L = dz L_0 + d\bar{z} L_1 = D - \lambda\Phi + \frac{1}{\lambda}\bar{\Phi}.$$

The Yang-Mills Higgs equation on a Riemann surface are the consistency conditions for these operators:

$$D^2 = \Phi \wedge \bar{\Phi}, \quad D\Phi = 0, \quad D\bar{\Phi} = 0.$$

Where D is the covariant exterior derivative. These equations are invariant under the conformal group in two dimensions as they only require a bundle with connection and a complex structure to define Φ and $\bar{\Phi}$. One solution will be transformed to another if $z \mapsto z'(z)$ and Φ and D pull back.

Alternatively, these equations depend only on the $*$–operator on 1–forms on the quotient space of the symmetries. The data consists of a connection D on a bundle, E and a section $\Gamma = \Phi + \bar{\Phi}$ of $\Omega^1 \otimes End(E)$. The operator L is

$$L = D + \left(-\lambda\frac{1-i*}{2} + \frac{1+i*}{2\lambda}\right)\Gamma.$$

So the field equations arising from the consistency conditions of this operator are invariant under the diffeomorphisms preserving $*$, i.e. the conformal transformations in 2-dimensions.

The Galilean analogue

If, in (2,2) signature, we impose one null symmetry along $\partial/\partial\tilde{w}$ and one non-null symmetry along $\partial/\partial z - \partial/\partial\tilde{z}$ we obtain the Lax pair:

$$L_0 = D_x - \lambda\Phi, \quad L_1 = D_w + \lambda(D_x + \Psi)$$

where $x = (z + \tilde{z})$ and we have reorganized the covariant derivative in the x direction to include part of the Higgs field associated to the symmetry in the $\partial_z - \partial_{\tilde{z}}$ direction. We can again perform the above trick, multiplying L_0 by dx and L_1 by dw to and adding together to obtain

$$L = dxL_0 + dwL_1 = D + \lambda(\Gamma + dwD_x)$$

where $\Gamma = -\Phi dx + \Psi dw$. To write this more geometrically, we introduce a degenerate $*$–operator that can be thought of as a map from 1–forms to 1–forms:

$$* = dw\frac{\partial}{\partial x}, \quad \alpha \mapsto \alpha(\frac{\partial}{\partial x})dw.$$

The operator L then becomes:

$$L = D + \lambda(*D + \Gamma).$$

The field equations arising from the consistency equations for this system are:

$$D^2 = 0, \quad D\Gamma = 0, \quad D*\Gamma + \Gamma\wedge\Gamma = 0$$

where D above is acting as the covariant exterior derivative so that the equations are all 2-form equations. Geometrically these equations determine a flat connection D on a bundle E, together with a section Γ of $\Omega^1 \otimes End(E)$.

It is clear, now, that the field equations arising from the consistency conditions for this operator will be invariant under diffeomorphisms of $\mathbf{R}^2$ preserving the degenerate $*$–operator, $dw \otimes \partial/\partial x$. These are the nonlinear Galilean transformations referred to previously:

$$(w, x) \mapsto (w', x') = (h(w), (\partial_w h(w))x + g(w))$$

where $h(w)$ and $g(w)$ are free functions except that $\partial_w h \neq 0$.

These equations embed the nonlinear Schrodinger and KdV equations and most of their generalizations to higher rank gauge groups (the Drinfeld Sokolov hierarchies etc.) into a galilean invariant system. At least in the $SL(2)$ case, this coordinate freedom is completely fixed by the reduction to KdV and NLS.

The totally null case

In the case where the symmetries span an anti self-dual null 2–plane we obtain the linear system

$$L = D + \lambda\Gamma$$

where again D is a flat connection on a bundle E and Γ is again a section of $\Omega^1 \otimes End(E)$. The field equations are now:

$$D^2 = 0, \quad D\Gamma = 0, \quad \Gamma \wedge \Gamma = 0.$$

These equations are now invariant under the full 2-dimensional diffeomorphism group (preserving the 'zero' $*$–operator).

These equations are therefore 'topological' and indeed are another way of writing the Wess-Zumino-Witten equations (Strachan 1992). Their reductions include the n–wave equations and those parts of the Drinfeld-Sokolov hierarchies not obtainable from the Galilean reductions. These further reductions require that their exists coordinates and a gauge in which the components of Γ are constant. In the $SL(3)$ case one can fix the coordinate freedom by using these additional conditions.

Stationary axisymmetric systems

In Fletcher & Woodhouse (1990) it was observed that the reduction of SDYM by 2 rotations gave the same field equations as the reduction by a translation and a rotation. This fact alone endows the 2 rotation reduction with one unexpected symmetry, the residual translation symmetry. However, more is true. These equations are invariant under $SL(2,\mathbf{R})$ the group of motions of the residual space preserving a hyperbolic metric. While this was in some sense clear from the reduced twistor correspondence in Woodhouse & Mason (1988), it was difficult to see on space-time.

To see this we impose a rotational invariance with respect to θ in the $w = y\exp(i\theta)$ plane and set $x = z + \tilde{z}$ and impose a symmetry in the $\partial_z - \partial_{\tilde{z}}$ direction. We obtain the linear system:

$$D_x - iA + \lambda e^{i\theta}(D_y + \frac{i}{y}(\partial_\theta - B)), \quad e^{-i\theta}(D_y - \frac{i}{y}(\partial_\theta - B)) - \lambda(D_x + iA).$$

We cannot just throw away the ∂_θ as there is explicit dependence on θ in the operators. This is connected with the fact that the Lie derivative of a spinor and hence λ along ∂_θ is not zero. To work independently of θ and to avoid derivatives with respect to the 'spectral parameter' we must use, instead of λ the parameter

$$\gamma = \frac{ye^{i\theta}\lambda}{2} + x - \frac{y}{2e^{i\theta}\lambda}$$

as this the simplest function on the spin bundle that is both invariant and constant along the twistor distribution.

If we introduce the complex coordinate $\xi = x + iy$, a bit of massage yields the following form for the linear system:

$$2D_\xi + i\sqrt{\frac{\gamma - \bar{\xi}}{\gamma - \xi}}(A + \frac{i}{y}B), \quad 2D_{\bar{\xi}} + i\sqrt{\frac{\gamma - \xi}{\gamma - \bar{\xi}}}(A - \frac{i}{y}B).$$

In order to bring out the invariance properties of this system, we can first of all multiply the first operator by $d\xi$ and the second by $d\bar{\xi}$ and add them together. Then introduce homogeneous coordinates $\gamma_A = (\gamma_0, \gamma_1)$ with $\gamma = \gamma_1/\gamma_0$ and similarly for ξ_A. Define $\bar{\xi}_A$ to be the componentwise complex conjugate of ξ_A and denote the skew product $\gamma_1\xi_0 - \xi_1\gamma_0 = \gamma\cdot\xi$. The linear system then reduces, after some further massage, to:

$$2D + i\sqrt{\frac{\gamma\cdot\bar\xi}{i\xi\cdot\bar\xi\gamma\cdot\xi}}\Phi\xi\cdot d\xi + i\sqrt{\frac{\gamma\cdot\xi}{i\xi\cdot\bar\xi\gamma\cdot\bar\xi}}\bar\Phi\bar\xi\cdot d\bar\xi$$

where we have put $\Phi = (\sqrt{y}A + iB/\sqrt{y})/\xi_0$.

It can now be seen that the linear system is invariant under $SL(2,\mathbf{R})$; the Mobius transformations on ξ_A preserving the reality structure $\xi_A \mapsto \bar\xi_A$ and hence the hyperbolic metric $\xi\cdot d\xi \odot \bar\xi\cdot d\bar\xi/(i\xi\cdot\bar\xi)^2$. The integrability conditions are equations for a connection D on a bundle E and a section $\Phi \in \Gamma(\mathcal{O}(-1)\otimes E)$ that is a dual spinor valued section of $\mathrm{End}(E)$.

The field equations are

$$D^2 = [\Phi,\bar\Phi]\frac{\xi\cdot d\xi\wedge\bar\xi\cdot d\bar\xi}{\xi\cdot\bar\xi},\quad \bar\eth\Phi = \frac{\bar\Phi}{2\xi\cdot\bar\xi},\quad \eth\bar\Phi = -\frac{\Phi}{2\xi\cdot\bar\xi}$$

where $\eth$ and $\bar\eth$ here denote the 'eth' operator and its complex conjugate, the $(0,1)$ and $(1,0)$ parts of the covariant derivative respectively.

So the 'Higgs fields' Φ and $\bar\Phi$ together constitute a Dirac field and satisfy the background coupled massive Dirac equation. Their commutator provides the curvature of the connection.

Remarks

Just as in Hitchin's case, one might hope to be able to transfer the other equations above to a Riemann surface also.

For the Galilean analogue, instead of endowing the Riemann surface with a complex structure, one might endow it with a measured foliation which corresponds to a limit of a complex structure (i.e. the space of measured foliations modulo certain equivalence relations is a good boundary for Teichmuller space). It turns out that this is *not* the same concept as the degenerate $*$-operator introduced above. They both determine a foliation of the Riemann surface, but the degenerate $*$-operator has an affine structure on the leaves, but no structure transverse to the leaves, whereas the measured foliation has a measure transverse to the leaves, but no structure on the leaves. Nevertheless, one might hope that one could prove an equivalence between the two, modulo diffeomorphisms in the global context as a kind of uniformization result. Even if this is feasible, it is still perhaps not clear that one can obtain a good existence theory for solutions of this equation as the linearized analogues of these equations have $*\Gamma$ covariant constant along the leaves of the foliation, a condition that will have no solutions when the leaves are dense.

Further analysis is required for the other cases. The totally null reduction will presumably not give rise to any difficulty as the equations are underdetermined anyway. This leaves the Hyperbolic case for which more analysis is required.

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Integrable Systems and Curved Twistor Spaces

I.A.B.Strachan

One of the ways in which the self-dual Einstein equations may be understood is as a two dimensional chiral model with the gauge fields taking values in the Lie algebra $sdiff(\Sigma^2)$ of volume preserving diffeomorphisms of the 2-surface Σ^2 [1]. Moreover, since $sl(2,\mathbb{C})$ is a subalgebra of $sdiff(\Sigma^2)$, solutions of certain integrable systems associated with $sl(2,\mathbb{C})$ may be encoded within the geometry of the nonlinear graviton [2]. This description breaks down for higher rank algebras, which are not subalgebras of $sdiff(\Sigma^2)$. However, by generalising the algebras such a description may be achieved. Another reason for studying integrable systems with infinite dimensional gauge groups is that the equations often simplify, and in some cases even linearise [3].

Let $\{\ ,\ \}$ be a generalised Poisson bracket acting on some manifold $\mathcal{N}$, satisfying the conditions:

- $\{f,g\} = -\{g,f\}$ (antisymmetry)
- $\{f,gh\} = \{f,g\}h + \{f,h\}g$ (derivation)
- $\{f,\{g,h\}\} + \text{cyclic} = 0$ (Jacobi identity)

With respect to a basis $x^i, i = 1,\ldots,\dim\mathcal{N}$, one may take*

$$\{f,g\} = \sum_{i,j} G^{ij}(x)\frac{\partial f}{\partial x^i}\frac{\partial g}{\partial x^j}, \tag{1}$$

where $G^{ij}(x)$ is constrained by the equations

$$\begin{gathered} G^{ij} + G^{ji} = 0 \\ \sum_{l=1}^{\dim\mathcal{N}} G^{li}\frac{\partial G^{kj}}{\partial x^l} + G^{lj}\frac{\partial G^{ik}}{\partial x^l} + G^{lk}\frac{\partial G^{ji}}{\partial x^l} = 0 \end{gathered} \tag{2}$$

Given such a structure one may define an associated Lie algebra Ham of Hamiltonian vector fields. Let $L_f \in Ham$, where

$$L_f = \sum_{i,j} G^{ij}(x)\frac{\partial f}{\partial x^i}\frac{\partial}{\partial x^j}$$

The Lie bracket for the algebra may be defined in two different, but equivalent, ways:

* Such generalised Poisson structures were first studied by Sophus Lie.

• Regard L_f and L_g as differential operators, and define the Lie bracket for the algebra by $[L_f, L_g] = L_f L_g - L_g L_f$,

• Regard L_f and L_g as vector fields on $\mathcal{N}$ and define the Lie bracket for the algebra be the Lie bracket of vector fields $[L_f, L_g]_{Lie}$.

In both cases $[L_f, L_g] = L_{\{f,g\}}$. The fact that this forms a Lie algebra follows trivially from (1) and (2). The idea now is to study the self-dual Yang-Mills equations with gauge potentials taking values in this infinite dimensional Lie algebra.

Let $y^{AA'}$ be spinor coordinates for $\mathbb{C}^4$ (or perhaps $\mathbf{R}^{2+2}$ etc. depending on a choice of reality condition). The self-dual Yang-Mills equations are the compatibility condition for the otherwise overdetermined linear system:

$$\mathcal{L}_A \Psi = \pi^{A'} \left\{ \frac{\partial}{\partial y^{AA'}} + A_{AA'} \right\} \Psi , \quad A, A' = 0, 1 , \quad \pi^{A'} \in \mathbf{CP}^1 . \tag{3}$$

The $A_{AA'}(y)$ are Lie algebra valued functions known as gauge potentials. In what follows it will be assumed that these take values in the Lie algebra *Ham* constructed above. Thus the $A_{AA'}$'s are represented by vector fields $A_{AA'} \leftrightarrow L_{f_{AA'}}$, where the functions $f_{AA'}$ depend on both the coordinates on $\mathbb{C}^4$ and on $\mathcal{N}$.

With this, the linear operators $\mathcal{L}_A$ are now vector fields on $\mathbb{C}^4 \oplus \mathcal{N}$,

$$\mathcal{L}_A = \pi^{A'} \left\{ \frac{\partial}{\partial y^{AA'}} + \sum_{i,j} G^{ij}(x) \frac{\partial f_{AA'}}{\partial x^i} \frac{\partial}{\partial x^j} \right\} . \tag{4}$$

Owing to the equivalent definition of the Lie bracket, the self-duality equations are a special case of the (Frobenius) integrability conditions for the distribution (4), i.e. $[L_0, L_1]_{Lie} = 0$. The integral surfaces of this distribution may be regarded as curved twistor surfaces, and the space of such surfaces as a curved twistor space, fibred over the Riemann sphere.

The converse construction involves studying an appropriate Riemann-Hilbert problem for the infinite dimensional group. Similiar ideas have been applied to the $SU(\infty)$-Toda equations in [4], which also developes the notion of a τ-function for this system and its associated hierarchy.

As mentioned at the beginning of this article, Mason showed [2] showed that one could give a curved twistor space construction to certain integrable systems associated with $sl(2,\mathbb{C})$ by embedding it in the algebra $sdiff(\Sigma^2)$. The same is true for any finite dimensional Lie algebra g. Let the structure constants for the Lie algebra g, with respect to some basis $e^i, i = 1, \ldots, \dim g$ be $c^{ij}{}_k$, so $[e^i, e^j] = \sum_k c^{ij}{}_k e^k$. From this one may define a generalised Poisson bracket by setting

$$G^{ij}(x) = \sum_k c^{ij}{}_k x^k$$

(the conditions (2) are automatically satisfied due to the properties of the structure functions), and let the associated infinite dimensional Lie algebra of Hamiltonian vector field be denoted by $\tilde{g}$. The original Lie algebra is now a subalgebra of $\tilde{g}$, since

$$[L_{x^i}, L_{x^j}] = \sum_k c^{ij}{}_k L_{x^k}.$$

Thus any solution to the self-dual Yang-Mills equations with a finite dimensional algebra may be encoded within the structure of a curved twistor space by first embedding g in $\tilde{g}$.

Another approach is to use a deformation of $sdiff(\Sigma^2)$ known as the Moyal algebra [5], in which higher order derivatives are present. This leads to some interesting results, but a direct geometrical interpretation of the results is absent.

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Ian Strachan

Endomorphisms $S : \otimes^2 \overline{H^1(PT^+, \mathcal{O}(-2))} \hookleftarrow$

We write H for the Hilbert space completion $\overline{H^1(PT^+,\mathcal{O}(-2))}$ with respect to $<\ |\ >$ of $H^1(\overline{PT^+},\mathcal{O}(-2))$, the space of analytic positive frequency free zero rest-mass (z.r.m.) fields on Minkowski space with finite L^2-norm on the momentum space light cone [1,2]. To a given basis

$$\mathcal{B} = \{\underset{|}{A}, \underset{|}{B}, \underset{|}{C}, \underset{|}{D}\} \subset T^* \cong \mathbf{C}^4 \tag{1}$$

of dual twistor space we can associate functions $e^{\mathbf{i}}_{\mathcal{B}}$ of homogeneity -2 on T as follows:

$$e^{\mathbf{i}}_{\mathcal{B}}(\overset{|}{Z}) = (\overset{C}{\underset{Z}{|}})^c(\overset{D}{\underset{Z}{|}})^d/(\overset{A}{\underset{Z}{|}})^{1+a}(\overset{B}{\underset{Z}{|}})^{1+b} \in H^0(PT - \{\overset{A}{\underset{Z}{|}} = 0\} - \{\overset{B}{\underset{Z}{|}} = 0\}, \mathcal{O}(-2))$$

$$for\ \mathbf{i} = (a,b,c,d) \in \mathbf{I} = \left\{(k,l,m,n) \in \mathbf{N}^4 | k + l - m - n = 0\right\},\ \overset{|}{Z} \in T. \tag{2}$$

If the projective lines $\underset{++}{AB}$, $\underset{++}{CD}$ lie in PT^-, PT^+ resp. these functions give rise to a linearly independent set of states

$$\{|e^{\mathbf{i}}_{\mathcal{B}}>\}_{\mathbf{i}\in\mathbf{I}} \subset H^1(PT - \{\overset{A}{\underset{Z}{|}} = 0\} \cap \{\overset{B}{\underset{Z}{|}} = 0\}, \mathcal{O}(-2)) \tag{3}$$

with dense span in H[2], i.e. they form a Hilbert space basis. An arbitrary element $|f> \in H$ has a unique expansion

$$|f> = \sum_{\mathbf{I}} <e^{\mathcal{B}}_{\mathbf{i}}\,|\,f>\,|e^{\mathbf{i}}_{\mathcal{B}}> \tag{4}$$

where $\{<e^{\mathcal{B}}_{\mathbf{i}}\,|\}_{\mathbf{i}\in\mathbf{I}}$ is the basis dual to (3) which is conveniently defined via the basis in T dual to $\mathcal{B}$ and the twistor transform [4].

1) $\mathcal{B}$-independence

Algebraically, i.e. up to the definition of a topology on $H \otimes H$ we can define endomorphisms $S : H \otimes H \longrightarrow H \otimes H$

$$S\Big(|e^{\mathbf{i}}_{\mathcal{B}}> \otimes |e^{\mathbf{j}}_{\mathcal{B}}>\Big) \quad = \sum_{(\mathbf{k},\mathbf{l})\in\mathbf{I}^2} S^{\mathbf{ij}}_{\mathbf{kl}}(\mathcal{B})\,|e^{\mathbf{k}}_{\mathcal{B}}> \otimes |e^{\mathbf{l}}_{\mathcal{B}}>. \tag{5}$$

Continuity w.r.t. the chosen topology is reflected in appropriate convergence conditions on the coefficients $S^{\mathbf{ij}}_{\mathbf{kl}}(\mathcal{B})$. We assume them to be smooth functions of $\mathcal{B}$ in a neighbourhood of a fixed basis $\mathcal{B}_0$. The (infinitesimal) action of $GL_4(\mathbf{C})$ on T^* then induces an action I on H:

$$g \in GL_4(\mathbf{C}) : \mathcal{B} \mapsto g\mathcal{B} \ \rightsquigarrow\ |e^{\mathbf{i}}_{\mathcal{B}}> \mapsto |e^{\mathbf{i}}_{g\mathcal{B}}> =: I_g\,|e^{\mathbf{i}}_{\mathcal{B}}>. \tag{6}$$

We demand invariance of (5) under the action (6). Let for example $g_\epsilon \in GL_4(\mathbf{C})$ be given by

$$g_\epsilon : \{\underset{|}{A}, \underset{|}{B}, \underset{|}{C}, \underset{|}{D}\} \mapsto \{\underset{|}{A}, \underset{|}{B}, \underset{|}{C} + \epsilon \underset{|}{A}, \underset{|}{D}\} . \tag{7}$$

Applying g_ϵ to both sides of (5) and comparing coefficients of ϵ we find

$$S\left(\overset{A}{\underset{\partial_C}{|}} \mid e^{\mathbf{i}}_{\mathcal{B}} > \otimes \mid e^{\mathbf{j}}_{\mathcal{B}} >\right) = \overset{A}{\underset{\partial_C}{|}} S\left(\mid e^{\mathbf{i}}_{\mathcal{B}} > \otimes \mid e^{\mathbf{j}}_{\mathcal{B}} >\right) \tag{8}$$

and in general we obtain

$$\left[\overset{X}{\underset{\partial_Y}{|}}, S\right] = 0 \ \textit{for}\ \underset{|}{Y} \in \mathcal{B}\ \textit{and arbitrary}\ \underset{|}{X} . \tag{9}$$

The differential operators are understood to act on $H \otimes H$ via their action on the parameters $\mathcal{B}$ of $e^{\mathbf{i}}_{\mathcal{B}}$. This action is the infinitesimal version of (6). It carries over to the cohomology classes (3) — see [3]. (9) places severe consistency restrictions on the $S^{\mathbf{ij}}_{\mathbf{kl}}(\mathcal{B})$. For example, since every $\mid e^{\mathbf{i}}_{\mathcal{B}} > \otimes \mid e^{\mathbf{j}}_{\mathcal{B}} >$ can be obtained as finite linear combination

$$\mid e^{\mathbf{i}}_{\mathcal{B}} > \otimes \mid e^{\mathbf{j}}_{\mathcal{B}} > = \sum_k a_k \left(\prod_{l_k}^{n_k} \overset{X^{l_k}}{\underset{\partial Y^{l_k}}{|}}\right) \mid e^{(m,0,m,0)}_{\mathcal{B}} > \otimes \mid e^{0}_{\mathcal{B}} > \Big|_{\underset{|}{X}^{l_k} \in \mathcal{B}} \tag{10}$$

where $\underset{|}{Y}^{l_k} \in \mathcal{B}$, $a_k \in \mathbf{Q}$ and $m \in \mathbf{N}$ is sufficiently big, we observe that it is enough to give

$$S\left(\mid e^{m}_{\mathcal{B}} > \otimes \mid e^{0}_{\mathcal{B}} >\right) \in H \otimes H \ ; \quad e^m := e^{(m,0,m,0)} \tag{11}$$

for (almost) all $m \in \mathbf{N}$ in order to fix $S : \otimes^2 H \to \otimes^2 H$.

2) Conformal invariance

Looking at the induced action of the subgroup $SU(2,2) \subset GL_n(\mathbf{C})$ (which is a $4:1$ cover of the conformal transformations of space–time) we can investigate the condition on the coefficients in (5) for S to be a 'conformally' invariant map:

$$\begin{aligned} S^{\mathbf{ij}}_{\mathbf{kl}}(\mathcal{B}) &= < e^{\mathcal{B}}_{\mathbf{k}} \mid \otimes < e^{\mathcal{B}}_{\mathbf{l}} \mid S \mid e^{\mathbf{i}}_{\mathcal{B}} > \otimes \mid e^{\mathbf{j}}_{\mathcal{B}} > \\ &= < e^{\mathcal{B}}_{\mathbf{k}} \mid \otimes < e^{\mathcal{B}}_{\mathbf{l}} \mid (I_g)^* S (I_g) \mid e^{\mathbf{i}}_{\mathcal{B}} > \otimes \mid e^{\mathbf{j}}_{\mathcal{B}} > = S^{\mathbf{ij}}_{\mathbf{kl}}(g\mathcal{B}) \end{aligned} \tag{12}$$

for all $g \in SU(2,2)$. Hence we conclude that the $S^{\mathbf{ij}}_{\mathbf{kl}}(\underset{|}{A}, \underset{|}{B}, \underset{|}{C}, \underset{|}{D})$ have to be $SU(2,2)$–invariant, i.e. functions of $\underset{\sqcup\sqcup\sqcup}{ABCD}$. A consideration of homogeneities then shows that in fact they have to be constants. Together with observation (11) this implies that a conformally invariant $S : H \otimes H \hookleftarrow$ is determined by

$$S\left(|e^i_{\mathcal{B}}> \otimes |e^0_{\mathcal{B}}>\right) = \sum_{k+l=i} S^{i\,0}_{k\,l}\,|e^k_{\mathcal{B}}> \otimes |e^l_{\mathcal{B}}>. \tag{13}$$

There are still relations among the $S^{i\,0}_{k\,l}$ due to the condition (9). In fact one gets for example

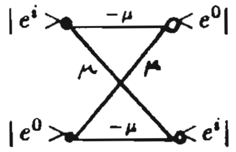
$$S_{i-k\,k}^{\;i\;\;0} = \sum_{l=0}^{k} (-1)^{k+l} \binom{i}{k} \binom{k}{l} S^{i-l\,0}_{i-l\,0} \tag{14}$$

giving S in terms of the (up to convergence conditions) free data $\{S^{i\,0}_{i\,0}\}_{i\in\mathbb{N}}$.

Applications

3) Conformally invariant twistor diagrams

For the single and double (and presumably also for higher order) box diagrams it is quite easy to compute the coefficients $S^{i\,0}_{0\,i}$ which, in an analogous way to (14), determine the respective maps completely. For the single box one finds

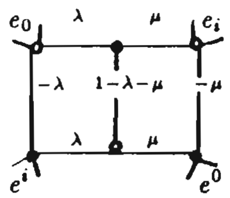


$$= \quad (-1)^{\mu}\frac{\Gamma(1-\mu)\Gamma(1+i+\mu)}{\Gamma(2+i)} =: (s_\mu)^{i\,0}_{0\,i} \tag{15}$$

from which one can compute

$$\begin{aligned} s_\mu\left(|e^i> \otimes |e^0>\right) &= \sum_{k+l=i} (s_\mu)^{i\,0}_{k\,l}\,|e^k> \otimes |e^l> \\ &= \frac{(-1)^\mu}{1+i} \sum_{k+l=i} \frac{\Gamma(1+k-\mu)\Gamma(1+l+\mu)}{\Gamma(1+k)\Gamma(1+l)}\,|e^k> \otimes |e^l>. \end{aligned} \tag{16}$$

For the double box we get



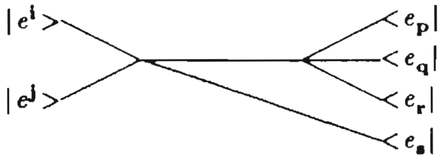
$$= \quad \frac{\Gamma(1+\lambda+i)\Gamma(1+\mu+i)}{\Gamma(2+i)\Gamma(1+\lambda+\mu+i)} =: (s_{\lambda,\mu})^{i\,0}_{0\,i}. \tag{17}$$

Certain properties of the double box, such as derived for example in §2.4 of [5] are immediate from this expression.

4) Factorization of Feynman Diagrams

Extending the algebraic approach we try to use compositions of linear maps

$S : \otimes^m H \longrightarrow \otimes^n H$ (and their analytic continuations) to build up complex Feynman diagrams from simpler ones. Consider for example the conformally non-invariant diagram of ϕ^4-theory:



(18)

One finds (see (23) - (26)) that the corresponding map factorizes

$$S : \otimes^2 H \longrightarrow \otimes^4 H$$

$$S(|e^i> \otimes |e^j>) \;=\; (S_2 \otimes 1)(|\infty> \otimes S_1(|e^i> \otimes |e^j>)) \;. \qquad (19)$$

Here $S_1 : \otimes^2 H \longrightarrow \otimes^3 H$ and $S_2 : \otimes^3 H \longrightarrow \otimes^3 H$ are determined by

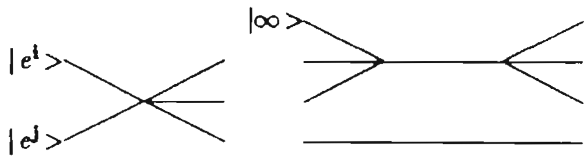
$$S_1(|e^a> \otimes |e^0>) \;=\; \frac{1}{1+a}\Bigg(\sum_{a=b+c+d} \underset{\sqcup}{AB} |e^b> \otimes |e^c> \otimes |e^d> \qquad (20)$$

$$+ \sum_{a-1=b+c+d} \underset{\sqcup}{CB} |e^b> \otimes |e^c> \otimes |e^d> \Bigg)$$

$$S_2(|e^{(a,0,a,0)}> \otimes |e^{(b,0,0,b)}> \otimes |e^0>) \;= \qquad (21)$$

$$\frac{a!\,b!}{(a+b)!} \sum_{\substack{\sum a_i = a \\ \sum b_i = b}} \bigotimes_{i=1}^{3} \frac{(a_i+b_i)!}{a_i!\,b_i!} |e^{(a_i+b_i,0,a_i,b_i)}>$$

where one uses the appropriate version of (11) in (21). S_2 is conformally invariant and has an analytic continuation to $|\infty> \otimes H \otimes H$. The state $|\infty> \notin H$ corresponds to the constant field $\phi(x) \equiv 1$, i.e. it can be represented by a twistor function $\left(\begin{smallmatrix} A & B \\ | & | \\ Z & Z \end{smallmatrix}\right)^{-1}$ in the limit $\underset{++}{AB} \to \sqcap$, the line at infinity. In diagrammatic notation we can represent this factorization by



(22)

To arrive at this one uses two facts. First one assumes that in the space–time integral corresponding to (18)

$$\int \phi^{\mathbf{i}}(x)\phi^{\mathbf{j}}(x)\phi_{\mathbf{s}}(x)\Delta_F(x-y)\phi_{\mathbf{p}}(y)\phi_{\mathbf{q}}(y)\phi_{\mathbf{r}}(y)\, d^4x d^4y$$

$$= \int \phi^{\mathbf{i}}(x)\phi^{\mathbf{j}}(x)\phi_{\mathbf{s}}(x)\Box_x^{-1}\left(\phi_{\mathbf{p}}(x)\phi_{\mathbf{q}}(x)\phi_{\mathbf{r}}(x)\right) d^4x \tag{23}$$

one can write

$$\Box_x^{-1}\left(\phi_{\mathbf{p}}(x)\phi_{\mathbf{q}}(x)\phi_{\mathbf{r}}(x)\right) = \sum_{(\mathbf{k},\mathbf{l})\in\Gamma^2} a_{\mathbf{pqr}}^{\mathbf{kl}}\, \psi_{\mathbf{k}}(x)\psi_{\mathbf{l}}(x) \tag{24}$$

i.e. that products of two free negative frequency z.r.m fields are dense in the space of general (L^2) negative frequency fields. Secondly, one observes that

$$(s_0 \otimes 1)\circ S_1 = S_1 \tag{25}$$

where $s_0 = s_{\mu=0}$ from (16) corresponds to the integral of four (two + and two – frequency) z.r.m. fields. This enables one to write

$$\sum_{(\mathbf{k},\mathbf{l})} a_{\mathbf{pqr}}^{\mathbf{kl}}\ (<e_{\mathbf{k}}| \otimes <e_{\mathbf{l}}| \otimes <e_{\mathbf{s}}|)\ S_1\ (|e^{\mathbf{i}}> \otimes |e^{\mathbf{j}}>)$$

$$= \sum_{(\mathbf{m},\mathbf{n},\mathbf{t})} (S_1)_{\mathbf{mnt}}^{\mathbf{ij}} \int \phi^{\mathbf{m}}(x)\phi^{\mathbf{n}}(x)\Box_x^{-1}\left(\phi_{\mathbf{p}}(x)\phi_{\mathbf{q}}(x)\phi_{\mathbf{r}}(x)\right) d^4x <e_{\mathbf{s}}|e^{\mathbf{t}}> . \tag{26}$$

It remains to be seen how such factorizations can be extended to general Feynman diagrams and whether they are reflected on the twistor diagram level. A rigorous treatment requires careful consideration of the various topologies involved.

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Franz Müller

Twistor description for Weyl's class of type D vacuum space-times

Thomas von Schroeter

In his article in TN 34, Nick Woodhouse derived the patching matrix for the Ward transform of the general anti-self-dual type D vacuum metric, the essential component of which turns out to be rational of degree 2 in a single complex variable w. As we know from other previously calculated examples, a simple form of the patching matrix in terms of rational components of low degree is also a characteristic feature of the *real* type D vacuum solutions.

In this note, I shall describe how, for the *Weyl solutions* among the type D vacuum metrics i.e. the ones which can be given in the form

$$ds^2 = f(z,r)dt^2 - \frac{r^2 d\theta^2}{f(z,r)} - \Omega^2(z,r)\left(dz^2 + dr^2\right) \tag{1}$$

our observation can be shown to arise from the existence of a Killing spinor of valence 2 - that is, a spinor field X_{AB} such that

$$\nabla_{A'(A}X_{BC)} = 0. \tag{2}$$

As this is true for all type D vacuum space-times (Walker & Penrose 1970), we expect that we will be able to generalize the argument given here.

Our strategy is to use the Yang-Mills twistor description of stationary axisymmetric vacuum space-times (Fletcher & Woodhouse 1990), in which the space-time M splits into a product of the orbits of the two Killing vectors $\partial/\partial t$ and $\partial/\partial\theta$ and a two-dimensional manifold Σ with co-ordinates (z,r), the space of orbits. For the metric (1), the patching matrix is simply $P(w) = \operatorname{diag}(1/f(w,0), f(w,0))$ and thus all we need to determine is the restriction of f to the axis (or horizon), $\{r=0\}$. If we write $J = \operatorname{diag}(-r^2/f, f)$ for the induced metric on the space of Killing vectors (i.e. $T\Sigma^\perp$), equation (2) translates into

$$\begin{aligned} F &= \mathrm{i} *F, & (3)\\ \mathrm{d}A &= -\tfrac{3}{2}A\wedge J^{-1}\mathrm{d}J, & (4)\\ \mathrm{D}_{(\alpha}A_{\beta)} &= \tfrac{1}{2}A_{(\alpha}J^{-1}\partial_{\beta)}J - \Omega^2\xi\,\delta_{\alpha\beta} & (5)\end{aligned}$$

where $F_{ab} = \epsilon_{A'B'}X_{AB}$ and $A = A_{\alpha j}dx^\alpha$ is the one-form on Σ with values in the dual of the space of Killing vectors that corresponds to X_{AB} via

$$(F_{ab}) = \begin{pmatrix} 0 & A_{\alpha j} \\ -A_{\alpha j} & 0 \end{pmatrix}.$$

Here, the matrix decomposition corresponds to the splitting $TM = T\Sigma^\perp \oplus T\Sigma$, $*F$ is the 4-space dual of F, d and D are, respectively, the operators of exterior and covariant differentiation on Σ, the indices α and β label elements of $T\Sigma$, j those of $T\Sigma^\perp$ and ξ is a complicated expression in J, A and DA with no further relevance for our purpose. (Note that (3)-(5) remain true also in the general case where J is no longer diagonal.)

If we satisfy (3) by putting

$$A = \left(-\frac{\mathrm{i}r}{f} *\beta,\ \beta\right)$$

with $\beta =: \Omega^2\sqrt{f}\,(u\mathrm{d}z + v\mathrm{d}r)$ a one-form on Σ and $*\beta$ now denoting its *two*-space dual on Σ, then (5) implies that $h(w) := u + \mathrm{i}v$ is holomorphic in $w = z + \mathrm{i}r$, and (4) is equivalent to

$$\mathrm{d}\left(f^{-3/2}\beta\right) = 0 \text{ and } \mathrm{d}\left(r^{-2}\sqrt{f} *\beta\right) = 0.$$

As these equations are real, the real and the imaginary part of β will satisfy them too, and thus u and v can be taken to be real functions. As a consequence of the vacuum field equations, the conformal factor is related to f by

$$-\mathrm{i}\partial_w \log\left(\Omega^2 f\right) = r\left(\partial_w \log f\right)^2,$$

and since J is a solution of Yang's form of the ASDYM equations, $\lambda = \log f$ has to satisfy

$$\lambda_r + r\left(\lambda_{zz} + \lambda_{rr}\right) = 0$$

(Fletcher & Woodhouse 1990).

Finally, eliminating Ω and h and expanding f near the axis (or horizon), one obtains a remarkably simple ODE for $f_0(z) := f(z, 0)$, namely

$$3f_0^{(4)}f_0'' - 4\left(f_0'''\right)^2 = 0$$

of which the general solution can be reduced to

$$f_0(z) = \begin{cases} az^{-1} + b + cz & \text{if } f_0''' \neq 0 \\ dz^2 + e & \text{if } f_0''' = 0 \end{cases}$$

by using the freedom $z \mapsto z + \text{const.}$ (here, a, b, c, d and e are real constants). As a constant overall factor in f can be absorbed into $\mathrm{d}t$ and $\mathrm{d}\theta$, one should think of both sets of parameters as homogeneous co-ordinates labelling a projective space of solutions, which is two-dimensional in the case $f_0''' \neq 0$ and one-dimensional in the case $f_0''' = 0$.

Examples:

- Flat space with time translation and rotation:

$$\mathrm{d}s^2 = \mathrm{d}t^2 - r^2\mathrm{d}\theta^2 - \mathrm{d}r^2 - \mathrm{d}z^2$$

  and hence $f \equiv 1$, $\Omega \equiv 1$, thus $a = c = 0$, $b = 1$.

- Schwarzschild space-time with time translation and rotation:

$$\mathrm{d}s^2 = \left(1 - \frac{2m}{R}\right)\mathrm{d}t^2 - \left(1 - \frac{2m}{R}\right)^{-1}\mathrm{d}R^2 - R^2\left(\mathrm{d}\psi^2 + \sin^2\psi\,\mathrm{d}\theta^2\right).$$

  The Weyl co-ordinates are $z = (R - m)\cos\psi$ and $r = \sqrt{R^2 - 2mR}\sin\psi$ and one finds $a = -2m$, $b = 1$, $c = 0$.

- Kinnersley's metric IV A (Kinnersley 1969) can be transformed to

$$ds^2 = (x^2 + a^2)\left[Cv^2dt^2 + 2dtdv\right] - \tfrac{1}{2}\Delta^{-1}dx^2 - 2\Delta d\theta^2$$

where

$$\Delta(x) = \frac{2mx + C(a^2 - x^2)}{2(x^2 + a^2)}.$$

As $r = v\sqrt{2C\Delta(x^2 + a^2)}$ and $z = v(Cx - m)$, $v = 0$ is just a single point on Σ and, in order to evaluate f on $\{r = 0\}$, we have to put $\Delta = 0$. We obtain

$$d = \frac{2}{C}\left(1 + \frac{m}{\sqrt{m^2 + C^2a^2}}\right) \quad \text{and} \quad e = 0.$$

- The vacuum C metric (Ehlers & Kundt 1962). Kinnersley & Walker (1970) give it in the form

$$ds^2 = A^{-2}(x + y)^{-2}\left(Fdt^2 - F^{-1}dy^2 - G^{-1}dx^2 - Gd\theta^2\right)$$

where

$$F(y) = -1 + y^2 - 2mAy^3 \quad \text{and} \quad G(x) = 1 - x^2 - 2mAx^3.$$

One finds $r = A^{-2}(x+y)^{-2}\sqrt{FG}$ and $z = A^{-2}(x+y)^{-2}[mAxy(y-x) - xy - 1]$. The patching matrix can be adapted to different parts of $\{r = 0\}$. For the physically relevant one, Fletcher (1990) found

$$a = 2, \quad b = -\frac{2m}{A}(\beta_1 + \beta_2), \quad \text{and} \quad c = \frac{m^2}{A^2}\beta_1\beta_2,$$

where β_1 and β_2 are roots of F and G (both polynomials have the same roots and all three of them are real provided $m^2A^2 < 1/27$).

It is not yet clear to me what the most general case with $f_0''' = 0$ (i.e. d and e arbitrary) corresponds to.

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Puzzle Page (as in TN 34)

Who wrote this? When was it published?
↓ What does it mean?

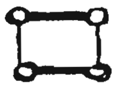
$f =$ 

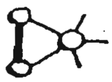
$g =$ 

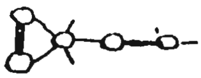
~~[illegible]~~

$i =$ 

~~[illegible]~~

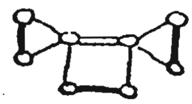
$A =$ 

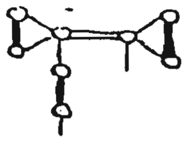
$j =$ 

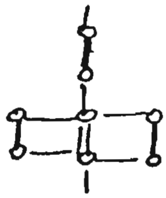
$k =$ 

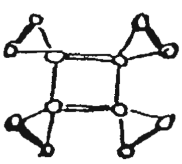
$\tau =$ 

$\Delta_k = A\tau - Bi = \upsilon$

$B =$ 

$\vartheta =$ 

$\vartheta =$ 

$C =$ 

$D_{A\tau - Bi} = A^2C - AB^2 + AB^2$
$= A^2C$

$\Theta = \kappa^2 + \frac{A}{2}\lambda^2$

~~[illegible]~~ $J = \kappa j + \lambda k$

$f_F = F = \kappa f + \lambda g$

$g_F = \frac{1}{2}\Theta . F\Theta$

Answers on the next page

Solution to Puzzle in TN 34

The "twistor diagrams" were not drawn by Andrew Hodges (or by me) but by William Kingdon Clifford, the great 19th century twistor theorist (1845–1879), discoverer of Clifford algebras (on which the theory of n-dimensional spinors – and twistors – is based) and of Clifford parallels on S^3 (whose stereographic projection to Euclidean 3-space provides the twistor picture that now adorns all TN covers), and who also anticipated Einstein in suggesting that the presence of matter is to be identified with the curving of space (a profound insight that twistor theory still strives doggedly to come to terms with). After he died, at an early age (34), several of Clifford's unpublished papers were collected together posthumously and published in 1881 under the title "Mathematical Fragments, being Facsimilies of his Unfinished Papers Relating to the Theory of Graphs" — which suggests to me that those who collected the papers together did not really understand what they were about!

What *are* they about then? It should be borne in mind that one of the prevailing interests among algebraists of the late 19th century was the theory of invariants and, in particular, the construction of "complete sets of invariants" for binary forms. A *binary form* is a homogeneous polynomial (normally with complex coefficients) in two variables ξ, η. (Three or more variables was considered to be too hard, at that time.) An *invariant* of a binary form (or collection of such forms) would be a polynomial in the coefficients of the form(s) which is invariant under (complex) unimodular linear transformations of ξ and η. A *covariant* would be a corresponding thing, but which is a homogeneous polynomial in ξ, η. Nowadays, we would use tensor algebra to construct such things, but in the late 19th and early 20th century, the "symbolic calculus" was widely used. This calculus employed a strange notation that can be described roughly as follows, using a comparison with the spinor notation that is more familiar to us now.

Spinor notn.	Symbolic notn.
Take $x^0 = \xi$, $x^1 = \eta$ All spinors symmetric	All factors commute, but $(\alpha\beta) = -(\beta\alpha)$

Examples	Spinor notn.	Symbolic notn.
forms	$\alpha_A x^A$	$\alpha_x = \beta_x = \ldots$
	$\alpha_{AB}\, x^A x^B$	α_x^2 or $\alpha_x \alpha_x$ or β_x^2 etc.
	$\alpha_{\underbrace{AB\ldots L}_{n}}\, x^A x^B \ldots x^L$	$\alpha_x^n = \beta_x^n = \gamma_x^n = \ldots$
invariants	$\alpha_{AB}\,\alpha^{AB} = \alpha_{AB}\alpha_{CD}\varepsilon^{AC}\varepsilon^{BD}$	$(\alpha\beta)^2 = (\alpha\beta)(\alpha\beta)$
	$\alpha_{ABC}\alpha^{ABD}\alpha_{EFD}\,\alpha^{EFC}$	$(\alpha\beta)^2(\alpha\gamma)(\beta\delta)(\gamma\delta)^2$
covariants	$\alpha_{AB}\,\alpha_{CD}\, x^A x^B x^C x^D$	$\alpha_x^2\, \beta_x^2$
	$\alpha_{AB}\alpha_{CD}\alpha_{EF}\, x^A x^F \varepsilon^{BC}\varepsilon^{DE}$	$\alpha_x(\alpha\beta)(\beta\gamma)\gamma_x$

A complete set of invariants [covariants] is a set from which all others can be constructed as polynomials in members of the set.

Now Clifford, with characteristic insight, not only effectively employs tensor (or spinor) ideas, but also uses a diagrammatic notation (similar to that described in the appendix to vol. 1 of Spinors and Space-Time). On the Puzzle Page, TN 34, we have (all expressions up to sign — I'm not sure how Clifford dealt with signs)

f standing for, say α_{ABCDE} (i.e. for $\alpha_{\ldots\ldots} x^{\cdot}x^{\cdot}x^{\cdot}x^{\cdot}$, i.e. α_x^5)

i " " $\alpha_{ABCD(E}\alpha_{F)}{}^{ABCD}$ (i.e. $\alpha_{ABCDE}\alpha_F{}^{ABCD} x^E x^F$, i.e. $(\alpha\beta)^4\alpha_x\beta_x$)

(I'm not sure abt. symmetry bracket.)

g " " $\alpha_{ABCD(E}\,\alpha^{ABCDF}\,\alpha_{GHJK)F}$ (i.e. $(\alpha\beta)^4(\gamma\beta)\alpha_x\gamma_x^4$)

etc. The thick lines seem to be 4-fold lines.

At the bottom of p. xxii Clifford seems to be working out a (complete?) set of covariants for a binary cubic (black spots) together with a binary quartic (white spots). Note the times at the left-hand side, evidently indicating how long it took him to work things out, using his notation.

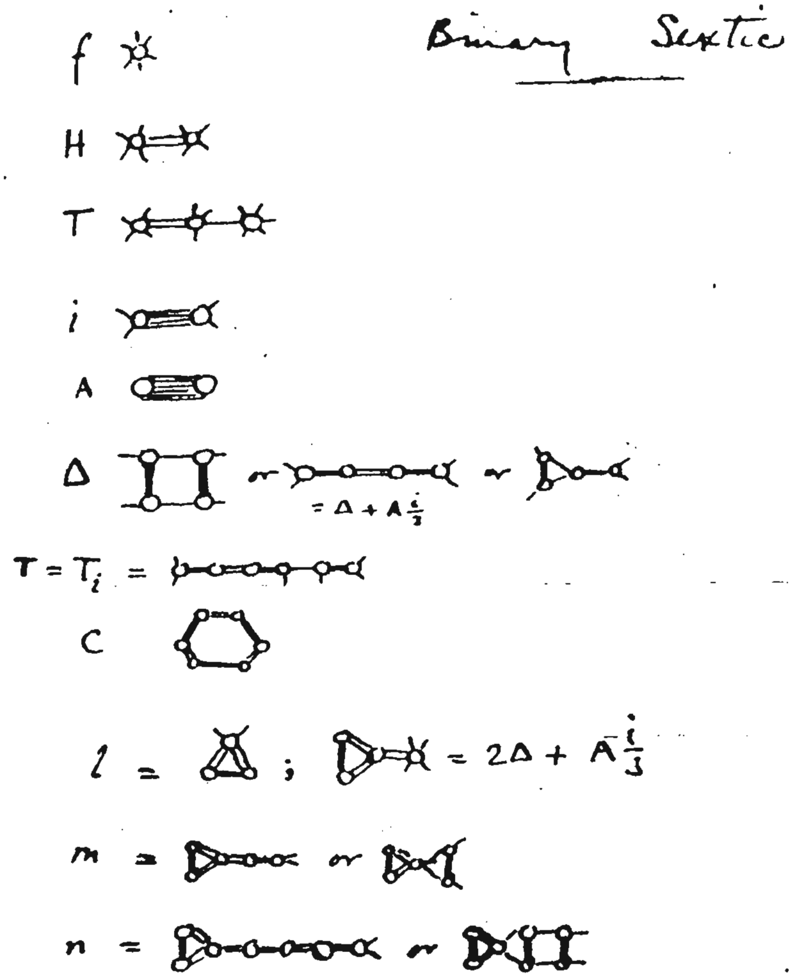
All credit to K.P.T. for spotting this publication among the books being cleared out of St Johns library.

Roger Penrose

Reference

Grace, J.H. & Young, A. (1903) The Algebra of Invariants (Camb. Univ. Press)

Binary Sextic



MATHEMATICAL FRAGMENTS

BEING

FACSIMILES OF HIS UNFINISHED PAPERS

RELATING TO THE

THEORY OF GRAPHS

BY THE LATE

W. K. CLIFFORD

London

MACMILLAN AND CO.

1881

these are really equivalent because i is symmetrical

Before of invariants of ~~cubic &~~ quadratic & cubic, f_2, φ_3

, 0 $D = (ab)^2$

, 4 $R = (\Delta\Delta')^2$

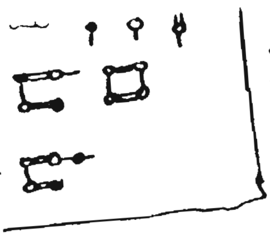
, 2 $E = (a\Delta)^2$

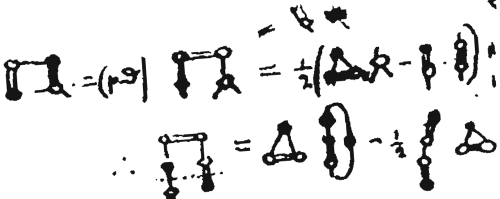
, 2 $F = (ap)^2 =$

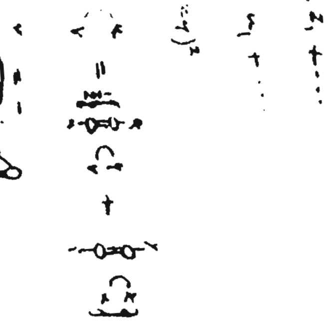
$M = (ap)(ar)$

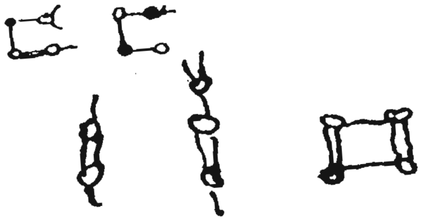
here is a series of invariants of a quadratic and any other binary form analogous to the resultant: viz. $(x\alpha . x\beta . y\alpha . y\beta)^k \left(\overline{x\alpha' y\beta}^{\,n-k} + \overline{x\beta . y\alpha}^{\,n-k}\right)$ where x, y are factors of the quadratic and $\alpha_x^n = \beta_x^n$ is the form of the n^{th} order. Their evanescence is the condition that one element of the quadric should be the k^{th} polar of the other.

...me in the case of a conic C_2 and a curve K_n ~~we me~~ there is a curve of







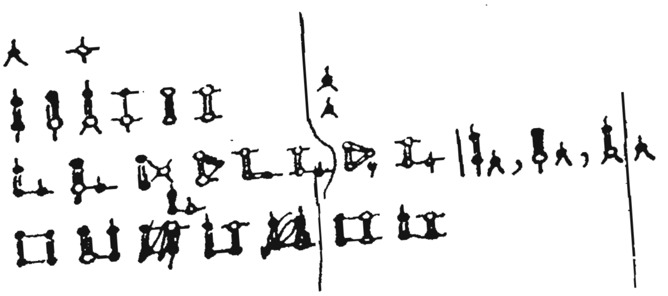


$$a(xu).b(yu) + a(yu).b(xu) = a(yu).b(yu)..|xy|$$

$$a(xu).b(yu).c(yz) + a(yu).b(xu).c(yz) = -a(yu).b(yu)..c(xz)$$

$$a(yu).b(zu).c(yx) + a(yu).b(xu).c(yz) = a(yu).b(zu)c(yx)..|xz|$$

$$\therefore a(xu).b(yu).c(yz) + c(xy).a(uy).b(uz)$$



1

$\frac{1}{2}$ hr

10 min

ABSTRACTS

The Stützfunktion and the cut-function

K.P.Tod

A convex body B in R^3 is defined by its *Stützfunktion* or support function; the boundary of the future of B meets future-null-infinity in a cut determined by a cut-function; these turn out to be proportional. I review parts of the theory of convex bodies and show how they can be generalised to cover future- and past-convex space-like 2-surfaces in Minkowski space.

(in 'Recent Advances in General Relativity' eds A.I.Janis and J.R.Porter, Birkhauser:Boston 1992)

The Hoop conjecture and the Gibbons-Penrose construction of trapped surfaces

K.P.Tod

The Hoop conjecture in the form that a marginally-trapped surface has its maximum 'circumference' less than about 4π times the mass it contains is studied for marginally-trapped surfaces produced by the construction of Gibbons and Penrose from shells of matter falling in at the speed of light in flat space. Some forms of the hoop conjecture are proved as new geometric inequalities on convex bodies; other forms of the conjecture are shown to be false. It is also shown how, despite a widespread belief to the contrary, marginally-trapped surfaces can be formed in the collapse of cylindrical or extremely prolate bodies.

(Class.Quant.Grav. 9 (1992) 1581-1591)

An algebraic treatment of certain classes of spinor equations with an application to General Relativity

JÖRG FRAUENDIENER
AND
GEORGE A. J. SPARLING

ABSTRACT. A new formulation for treating spinor equations on a spacetime is introduced and applied to the spin-2 equation for the Weyl spinor in vacuum General Relativity. The power of the formalism rests on the fact that it is index free, describing structures in terms of algebraic relations which makes it well adapted for use in algebraic manipulation programs. The starting point is the fact that connections in bundles can be viewed as derivations on the algebra of sections of the bundle. In the case of the spin bundle there is a canonical operator basis that allows one to take components in a canonical way so that one can express everything in terms of scalar operators. Thus, essentially, this is a non-commutative Newman-Penrose formalism. In an application to General Relativity we present an algorithm that recursively produces the terms of a Taylor series expansion of the Weyl spinor around the apex of a light cone from characteristic data given on that cone.

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COMPLEX STRUCTURES ON QUATERNIONIC MANIFOLDS

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September 7, 1992

ABSTRACT. In the first part of this work we consider compact riemannian manifolds M with holonomy in $Sp(n)Sp(1)$. We show that M admits a compatible complex structure if and only if the holonomy is in $Sp(n)$, up to finite coverings. We also show that the sign of the Ricci curvature completely determines the algebraic dimension of the twistor space.

In the second part, by way of contrast, we give two geometric constructions of simply-connected quaternionic manifolds with a compatible complex structure which is not hypercomplex. The first examples are non-compact and symmetric. The second one is compact and follows from general reults of Joyce [J].

TWISTOR NEWSLETTER No. 35

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Short contributions for TN 36 should be sent to

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