Orthogonality of General Spin States

E. Majorana (Nuovo Cimento 9 (1932) 43-50) described the general spin state, for spin \( \frac{n}{2} \), (up to proportionality) in terms of an unordered set of \( n \) points on the sphere — perhaps with coincidences between them, but then multiplicities are counted. Although he did not phrase his argument in these terms, his result essentially expresses the canonical decomposition of a general symmetric spinor of valence \( n \):

\[
\Psi_{\text{AB...N}} = \alpha (A_B^\dagger B_B^\dagger ... N_N^\dagger).
\]

Here \( \Psi_{\text{AB...N}} \) describes the spin state, and its \( n \) principal null directions, represented as \( n \) points on the Riemann sphere \( S \), provide Majorana's set (cf. Penrose & Rindler, Spinors & Space-Time, Vol. 1 (1984) p. 162; RP in 300 Years of Gravitation, eds. Hawking & Israel) C.U.P. (1987); RP in The Emperor's New Mind, O.U.P (1989), Fig. 6.29.

I shall be concerned, here, with the question of the geometrical interpretation of orthogonality between two states of spin \( \frac{n}{2} \). In spinor terms this can be written

\[
\Psi_{\text{AB...N}} \Phi_{\text{AB'...N'}} = 0
\]

where \( \Phi_{\text{AB'...N'}} \) is the timelike (unit) vector (in 4-space) with respect to which the states are all taken to be stationary.

We can write this last relation as

\[
\Psi_{\text{AB...N}} \Phi_{\text{AB'...N'}} = 0
\]

where

\[
\Phi_{\text{AB'...N'}} = t_{AA'}^\dagger t_{BB'}^\dagger ... t_{NN'}^\dagger \Phi_{\text{AB'...N'}}.
\]

Geometrically, the \( n \) Majorana points on \( S \) defined by \( \Phi_{\text{AB...N}} \) are antipodal to the \( n \) points defined by \( \Psi_{\text{AB...N}} \).
\( \Phi_{AB...N} \) (see Spinors & Space-Time, Vol. 1). The relation between 

\( \Phi_{AB...N} \) and \( \Psi_{AB...N} \) given by \( \Psi_{AB...N} \Phi_{AB...N} = 0 \) is 

called apolarity (cf. Grace & Young, The Algebra of Invariants 
C.U.P. (1903)). Apolarity represents a simpler geometrical 
condition for a pair of sets of \( n \) points on \( S \) than does 
orthogonality — not least because apolarity is 
conformally invariant on \( S \) (since there is no dependence 
on \( t^a \)) whereas orthogonality is only rotationally 
invariant. Then to pass from apolarity to orthogonality, 
we simply reflect one or other of the sets of points 
in the centre of \( S \).

Nevertheless, apolarity is itself not an easy 
thing to express in purely geometrical terms, 
in the general case. In terms of components, the 
algebraic condition is 

\[
\Psi_0 \Psi_n - n \Psi_1 \Psi_{n-1} + \frac{n(n-1)}{2!} \Psi_2 \Psi_{n-2} - \ldots + (-1)^n \Psi_n \Psi_0 = 0
\]

where we can think of the \( n \) Majorana points of \( \Psi_{AB...N} \) 
as defined on the Riemann sphere by the roots of 

\[
\Psi_0 + \Psi_1 \bar{z} + \Psi_2 \bar{z}^2 + \ldots + \Psi_n \bar{z}^n
\]

(\( \Psi_0 = \Psi_{10...0}, \Psi_1 = \Psi_{10...0}, \text{etc.} \)) whereas orthogonality is 
given by 

\[
\Psi_0 \bar{\Phi}_0 + n \Psi_1 \bar{\Phi}_1 + \frac{n(n-1)}{2!} \Psi_2 \bar{\Phi}_2 + \ldots + \Psi_n \bar{\Phi}_n = 0.
\]

Let us denote by \( P_1, \ldots, P_n \) the points on \( S \) defined 
by \( \Psi_{AB...N} \), and by \( Q_1, \ldots, Q_n \) those defined by \( \Phi_{AB...N} \). 
In a recent paper (R.P. "On Bell's Non-locality Without Probabilities; 
Some Curious Geometry" to be published in a CERN volume 
in honour of J. S. Bell; cf. also J. Zimbard & R.P. "On Bell... 
... More Curious Geometry," to be published; HPS (Cambridge 1993),
Imagine a cardboard equilateral triangle with a matchstick through its centre and perpendicular to its plane. The matchstick extends to points $T_1, T_2$ on either side of the triangle to a distance equal to its in-radius (i.e. $\frac{1}{2}$ its circumradius).

Now imagine that the triangle is held (in 3-space) so that its orthogonal projection to the plane coincides with the plane, scaling up or down as necessary (i.e. place so that, from a long way off, it "looks like" the triangle). See where the points $T_1', T_2', C'$ project to; call them $T_1', T_2', C'$. The required apolarity condition between the P-points and Q-points is: $C'P_2'T_2'$ and $C'T_1'P_1'$ are similar.

Proofs: Exercise for the reader; hint: look at the accompanying article by MG & RP (also, note that $R_1, T_1', R_2, T_2'$ form a square with centre $C'$).

General $n$ inductive argument

Assume we already know a geometrical criterion for apolarity for two pairs of $n-1$ points. Can we find such a criterion for two pairs $P_1, \ldots, P_n; Q_1, \ldots, Q_n$ of sets of $n$ points on $S$?

(An) Answer (not very practical!) Fix one of the P-points, say $P_n$, and all the Q-points, and try to find those special sets of points $P_1, \ldots, P_{n-1}$ with $P_1, \ldots, P_{n-1}, P_n$ apolar to $Q_1, \ldots, Q_n$, for which $P_1 = P_2 = \ldots = P_{n-1}$. There are $n-1$ such places — call them $R_1, \ldots, R_{n-1}$ — these being the points on $S$, stereographic from which would yield points $P_n'; Q_1', \ldots, Q_n'$ with $P_n'$ the centroid of $Q_1', \ldots, Q_n'$ (by 2 above). (Unfortunately I don't of a direct construction of $R_1, \ldots, R_{n-1}$.)
a number of special cases of apolarity are given:

1) If all the P-points coincide then they are apolar to the Q-points iff at least one of the Q-points coincides with the n-fold P-point.

2) If all but one of the P-points coincide, then the P- and Q-sets are apolar if stereographic projection from the multiple P-point sends the remaining P-point to the centroid of the stereographic projection of the Q-points.

3) If n is odd, then any set of P-points on S is apolar to itself.

Let us now consider the general case.

\[ n = 1 \] Apolarity holds iff \( P_1 = Q_1 \)

\[ n = 2 \] Apolarity holds iff \( P_1, P_2 \) separate \( Q_1, Q_2 \) - i.e. \( P_1, P_2, Q_1, Q_2 \) all lie on a circle on S and the line \( P_1 P_2 \) meets the intersection of the tangents to this circle at \( Q_1 \) and at \( Q_2 \) (or limiting cases)

\[ n = 3 \] Stereographically project from \( P_3 \) (assumed simple – otherwise 2 above); we get points \( P'_1, P'_2, Q'_1, Q'_2, Q'_3 \) as the respective projections of \( P_1, P_2, Q_1, Q_2, Q_3 \).

Two ways of seeing the relation of \( P'_1, P'_2 \) to the triangle \( Q'_1 Q'_2 Q'_3 \) (= \( \Delta \)) that asserts apolarity are as follows.

1) \( R_1, R_2 \) are the foci of the ellipse touching the sides of \( \Delta \) at their mid-points. Then the P-points are apolar to the Q-points iff \( P'_1, P'_2 \) separate \( R_1, R_2 \) harmonically (in the above sense \( n = 2 \)) - i.e. they separate harmonically on a circle through them.)
A polarity between \( P_1, ..., P_n \) and \( Q_1, ..., Q_n \) is now simply the condition that \( P_1, ..., P_{n-1} \) and \( R_1, ..., R_{n-1} \) be apolar.

Proof: Consider \( \tilde{p}^{(A} x^B x^C ... x^N) q_{(A} q_{B} ... q_{N)} = 0 \).

Solutions for \( x^A \) give \( R^A, ..., \tilde{R}^A \). Hence
\[
\tilde{p}^A q_{(A} q_{B} ... q_{N)} = R^B \tilde{R}^N
\]
which is the same as the required condition
\[
\tilde{p}^{(A} p^B ... p^N) q_{(A} q_{B} ... q_{N)} = 0.
\]
A.E.D.

A Simple Observation Concerning \( \{2,2,3 \} \) Vacuums

It is well known that every \( \{2,2,3 \} \) vacuum ("type D"), with Weyl spinor
\[
\Psi_{ABCD} = M r^{-3} \epsilon^{A} \alpha_{B} \beta_{C} \beta_{D}
\]
(where \( M \) is a constant and \( \alpha_{A} \beta^{A} = 1 \)) possesses a Killing spinor—a valence-2 symmetric twistor,
\[
X_{AB} = r \alpha_{(A} \beta_{B)} \text{ satisfying } \nabla_{A'} (A' X_{B'C}) = 0.
\]

In the Schwarzschild solution, \( r \) is the standard "radial coordinate," but in general \( r \) is a complex "radial" quantity.

I am not aware that anyone has pointed out the following simple but striking consequence:

Proposition. Along every null geodesic, with parallelly propagated tangent spinor \( O^A \), the null datum
\[
\Psi_{0} = \Psi_{ABCD} O^{A} O^{B} O^{C} O^{D} = C a b c d \epsilon^{a} m^{b} \epsilon^{c} n^{d}
\]
has the precise form
\[
\Psi_{0} = \frac{K}{r^5}
\]
where \( K \) is a complex constant depending on the choice of null geodesic (with its choice of scale for \( O^A \)).

Proof. This is an immediate consequence of the constancy of \( X_{AB} O^A O^B \) along the null geodesic \( (O^A \nabla^A \nabla_A (X_{BC} O^B O^C) = 0) \) and the fact that \( \Psi_{0} = M r^{-3} (\alpha_{A} O^{A} )^2 (\beta_{B} O^{B} )^2 \) and \( X_{AB} O^A O^B = r (\alpha_{A} O^{A} ) (\beta_{B} O^{B} ) \), whence \( \Psi_{0} = M / (X_{AB} O^A O^B )^2 r^{-5} \). Q.E.D.