Orthogonality of General Spin States

E. Majorana (Nuovo Cimento $\underset{\sim}{9}$ (1932) 43–50) described the general spin state, for spin $\frac{n}{2}$, (up to proportionality) in terms of an unordered set of n points on the sphere — perhaps with coincidences between them, but then multiplicities are counted. Although he did not phrase his argument in these terms, his result essentially expresses the canonical decomposition of a general symmetric spinor of valence n:

$$\underbrace{\psi_{AB\cdots N}}_{n} = \alpha_{(A}\beta_{B}\cdots\nu_{N)} .$$

Here $\psi_{AB\ldots N}$ describes the spin state, and its n principal null directions, represented as n points on the Riemann sphere S, provide Majorana's set (cf. Penrose & Rindler, Spinors & Space-Time, Vol. 1 (1984) p. 162; R.P. in 300 years of Gravity (eds. Hawking & Israel) C.U.P. (1987); R.P. in The Emperor's New Mind, O.U.P (1989), Fig. 6.29).

I shall be concerned, here, with the question of the geometrical interpretation of orthogonality between two states of spin $n/2$. In spinor terms this can be written

$$\psi_{AB\ldots N}\, t^{AA'}t^{BB'}\cdots t^{NN'}\,\bar{\phi}_{A'B'\ldots N'} = 0$$

where t^a is the timelike (unit) vector (in 4-space), with respect to which the states are all taken to be stationary. We can write this last relation as

$$\psi_{AB\cdots N}\,\varphi^{AB\ldots N} = 0$$

where

$$\varphi^{AB\cdots N} = t^{AA'}t^{BB'}\cdots t^{NN'}\,\bar{\phi}_{A'B'\ldots N'} .$$

Geometrically, the n Majorana points on S defined by $\varphi_{AB\ldots N}$ are antipodal to the n points defined by

$\phi_{AB\ldots N}$ (see Spinors & Space-Time, Vol 1). The relation between $\varphi_{AB\ldots N}$ and $\psi_{AB\ldots N}$ given by $\psi_{AB\ldots N}\varphi^{AB\ldots N}=0$ is called <u>apolarity</u> (cf. Grace & Young, <u>The Algebra of Invariants</u> C.U.P. (1903)). Apolarity represents a simpler geometrical condition for a pair of sets of n points on S than does orthogonality — not least because apolarity is <u>conformally</u> invariant on S (since there is no dependence on t^a) whereas orthogonality is only rotationally invariant. Then to pass from apolarity to orthogonality, we simply reflect one or other of the sets of points in the centre of S.

Nevertheless, apolarity is itself not an easy thing to express in purely geometrical terms, in the general case. In terms of components, the algebraic condition is

$$\psi_0\varphi_n - n\psi_1\varphi_{n-1} + \frac{n(n-1)}{2!}\psi_2\varphi_{n-2} - \ldots + (-1)^n\psi_n\varphi_0 = 0$$

where we can think of the n Majorana points of $\psi_{AB\ldots N}$ as defined on the Riemann sphere by the roots of

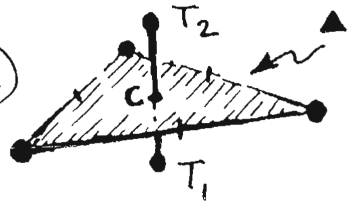
$$\psi_0 + \psi_1 z + \psi_2 z^2 + \ldots + \psi_n z^n$$

($\psi_0 = \psi_{00\ldots0}$, $\psi_1 = \psi_{10\ldots0}$, etc.) whereas orthogonality is given by

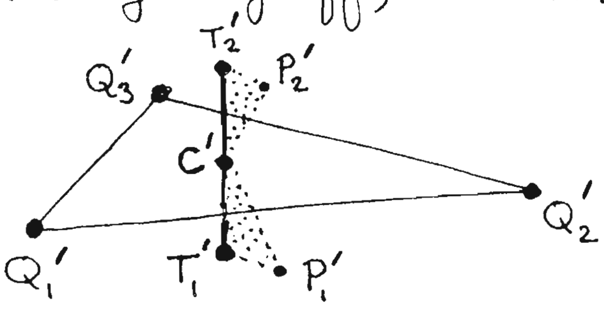
$$\psi_0\bar{\phi}_0 + n\psi_1\bar{\phi}_1 + \frac{n(n-1)}{2!}\psi_2\bar{\phi}_2 + \ldots + \psi_n\bar{\phi}_n = 0.$$

Let us denote by $P_1, \ldots, P_n$ the points on S defined by $\psi_{AB\ldots N}$, and by $Q_1, \ldots, Q_n$ those defined by $\varphi_{AB\ldots N}$. In a recent paper (R.P. "On Bell Nonlocality Without Probabilities: Some Curious Geometry" to be published in a CERN volume in honour of J.S. Bell; cf. also J. Zimba & R.P. "On Bell ... ... More Curious Geometry," to be published: HPS (Cambridge 1993),

[2] Imagine a cardboard <u>equilateral</u> triangle ▲ with a matchstick through its centre C and perpendicular to its plane. The matchstick extends to points T_1, T_2 on either side of ▲ to a distance equal to its in-radius (i.e. $\frac{1}{2}$ its circumradius).



Now imagine that ▲ is held (in 3-space) so that its orthogonal projection to the plane coincides with △, scaling up or down as necessary (i.e. place ▲ so that, from a long way off, it "looks like" △). See where the points T_1, T_2, C project to; call them T_1', T_2', C'. The required apolarity condition between the P-points and Q-points is: $C'P_2'T_2'$ and $C'T_1'P_1'$ are <u>similar</u>.



<u>Proofs</u>: Exercise for the reader; <u>hint</u>: look at the accompanying article by MGE & RP (also, note that R_1, T_1', R_2, T_2' form a square with centre C').

<u>General n</u> ; <u>inductive argument</u>

Assume we already know a geometrical criterion for apolarity for two pairs of $n-1$ points. Can we find such a criterion for two pairs $P_1, \ldots, P_n$; $Q_1, \ldots, Q_n$ of sets of n points on S ?

(An) <u>Answer</u> (not very practical!) Fix one of the P-points, say P_n, and all the Q-points, and try to find those <u>special</u> sets of points $P_1, \cdots, P_{n-1}$ with $P_1, \cdots, P_{n-1}, P_n$ apolar to $Q_1, \ldots, Q_n$, for which $P_1 = P_2 = \cdots = P_{n-1}$. There are $n-1$ such places — call them $R_1, \ldots, R_{n-1}$ — these being the points on S, stereographic from which would yield points P_n' ; $Q_1', \ldots, Q_n'$ with P_n' the centroid of $Q_1', \ldots, Q_n'$ (by (2) above). (Unfortunately I don't of a <u>direct</u> construction of $R_1, \cdots, R_{n-1}$.)

a number of special cases of apolarity are given:

① If all the P-points coincide then they are apolar to the Q-points iff at least one of the Q-points coincides with the n-fold P-point.

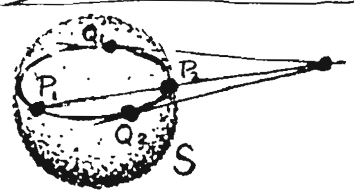
② If all but one of the P-points coincide, then the P- and Q-sets are apolar if stereographic projection from the multiple P-point sends the remaining P-point to the centroid of the stereographic projection of the Q-points.

③ If n is odd, then any set of P points on S is apolar to <u>itself</u>.

Let us now consider the <u>general</u> case.

<u>$n=1$</u> Apolarity holds iff $P_1 = Q_1$

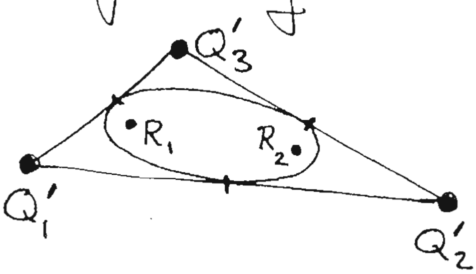
<u>$n=2$</u> Apolarity holds iff P_1, P_2 separate Q_1, Q_2 harmonically — i.e. P_1, P_2, Q_1, Q_2 all lie on a circle on S and the line P_1P_2 meets the intersection of the tangents to this circle at Q_1 and at Q_2 (or limiting cases)



<u>$n=3$</u> Stereographically project from P_3 (assumed simple — otherwise ② above); we get points $P_1', P_2', Q_1', Q_2', Q_3'$ as the respective projections of P_1, P_2, Q_1, Q_2, Q_3.

Two ways of seeing the relation of P_1', P_2' to the triangle $Q_1'Q_2'Q_3'$ $(=\triangle)$ that asserts apolarity are as follows.

$\boxed{1}$ R_1, R_2 are the foci of the ellipse touching the sides of $\triangle$ at their mid-points. Then the P-points are apolar to the Q-points iff P_1', P_2' separate R_1, R_2 harmonically (in the above sense ($n=2$) — i.e. they separate harmonically on a circle through them).



Apolarity between $P_1, \dots, P_n$ and $Q_1, \dots, Q_n$ is now simply the condition that $P_1, \dots, P_{n-1}$ and $R_1, \dots, R_{n-1}$ be apolar.

Proof Consider $\overset{n}{P}{}^{(A} X^B X^C \cdots X^{N)} \overset{1}{Q}_{(A} \overset{2}{Q}_B \cdots \overset{n}{Q}_{N)} = 0$. suffixes have now become numbers above

Solutions for X^A give $\overset{1}{R}{}^A, \dots, \overset{n-1}{R}{}^A$. Hence

$$\overset{n}{P}{}^A \overset{1}{Q}_{(A} \overset{2}{Q}_B \cdots \overset{n}{Q}_{N)} = \overset{1}{R}_{(B} \cdots \overset{n-1}{R}_{N)}$$

taking multiplying factor = 1

Apolarity between $P_1, \dots, P_{n-1}$ and $R_1, \dots, R_{n-1}$ is

$$\overset{1}{P}{}^{(B} \cdots \overset{n-1}{P}{}^{N)} \overset{1}{R}_{(B} \cdots \overset{n-1}{R}_{N)} = 0$$

which is the same as the required condition

$$\overset{n}{P}{}^{(A} \overset{1}{P}{}^{B} \cdots \overset{n-1}{P}{}^{N)} \overset{1}{Q}_{(A} \overset{2}{Q}_B \cdots \overset{n}{Q}_{N)} = 0.$$

Q.E.D.

A Simple Observation Concerning {22} Vacuums

It is well known that every {22} vacuum ("type D"), with Weyl spinor

$$\Psi_{ABCD} = M r^{-3} \alpha_{(A} \alpha_B \beta_C \beta_{D)}$$

(where M is a constant and $\alpha_A \beta^A = 1$) possesses a Killing spinor — a valence-2 symmetric twistor! —

$$\chi_{AB} = r\, \alpha_{(A} \beta_{B)} \quad \text{satisfying} \quad \nabla_{A'(A} \chi_{BC)} = 0.$$

(See Walker & Penrose, Comm. Math. Phys **18** (1970) 265–74; also Spinors & Space-Time Vol. 2, p. 107.)

In the Schwarzschild solution, r is the standard "radial coordinate", but in general r is a complex "radial" quantity.

I am not aware that anyone has pointed out the following simple but striking consequence:

Proposition Along every null geodesic, with parallelly propagated tangent spinor o^A, the null datum

$$\Psi_0 = \Psi_{ABCD}\, o^A o^B o^C o^D = C_{abcd}\, l^a m^b l^c m^d$$

has the precise form

$$\Psi_0 = \frac{K}{r^5}$$

where K is a complex constant depending on the choice of null geodesic (with its choice of scale for o^A).

Proof This is an immediate consequence of the constancy of $\chi_{AB}\, o^A o^B$ along the null geodesic ($o^A \bar{o}^{A'} \nabla_{AA'} (\chi_{BC}\, o^B o^C) = 0$) and the fact that $\Psi_0 = M r^{-3} (\alpha_A o^A)^2 (\beta_B o^B)^2$ and $\chi_{AB}\, o^A o^B = r (\alpha_A o^A)(\beta_B o^B)$, whence $\Psi_0 = M / (\chi_{AB}\, o^A o^B)^2 r^5$. Q.E.D.