Apolarity between $P_1, \ldots, P_n$ and $Q_1, \ldots, Q_n$ is now simply the condition that $P_1, \ldots, P_{n-1}$ and $R_1, \ldots, R_{n-1}$ be apolar.

suffixes have now become numbers above

Proof Consider $\overset{n}{P}{}^{(A} X^B X^C \ldots X^{N)} \overset{1}{Q}_{(A} \overset{2}{Q}_B \cdots \overset{n}{Q}_{N)} = 0.$

Solutions for X^A give $\overset{1}{R}{}^A, \ldots, \overset{n-1}{R}{}^A$. Hence

$$\overset{n}{P}{}^A \overset{1}{Q}_{(A} \overset{2}{Q}_B \cdots \overset{n}{Q}_{N)} = \overset{1}{R}_{(B} \cdots \overset{n-1}{R}_{N)}$$

taking multiplying factor = 1

Apolarity between $P_1, \ldots, P_{n-1}$ and $R_1, \ldots, R_{n-1}$ is

$$\overset{1}{P}{}^{(B} \ldots \overset{n-1}{P}{}^{N)} \overset{1}{R}_{(B} \cdots \overset{n-1}{R}_{N)} = 0$$

which is the same as the required condition

$$\overset{n}{P}{}^{(A} \overset{1}{P}{}^B \ldots \overset{n-1}{P}{}^{N)} \overset{1}{Q}_{(A} \overset{2}{Q}_B \cdots \overset{n}{Q}_{N)} = 0. \qquad \text{Q.E.D.}$$

A Simple Observation Concerning {22} Vacuums

It is well known that every {22} vacuum ("type D"), with Weyl spinor

$$\Psi_{ABCD} = M r^{-3} \alpha_{(A} \alpha_B \beta_C \beta_{D)}$$

(where M is a constant and $\alpha_A \beta^A = 1$) possesses a Killing spinor — a valence-2 symmetric twistor! —

$$\chi_{AB} = r\, \alpha_{(A} \beta_{B)} \qquad \text{satisfying} \qquad \nabla_{A'(A} \chi_{BC)} = 0.$$

(See Walker & Penrose, Comm. Math. Phys **18** (1970) 265–74; also Spinors & Space-Time Vol. 2, p. 107.)

In the Schwarzschild solution, r is the standard "radial coordinate", but in general r is a complex "radial" quantity.

I am not aware that anyone has pointed out the following simple but striking consequence:

Proposition Along every null geodesic, with parallelly propagated tangent spinor o^A, the null datum

$$\Psi_0 = \Psi_{ABCD}\, o^A o^B o^C o^D = C_{abcd}\, l^a m^b l^c m^d$$

has the precise form

$$\Psi_0 = \frac{K}{r^5}$$

where K is a complex constant depending on the choice of null geodesic (with its choice of scale for o^A).

Proof This is an immediate consequence of the constancy of $\chi_{AB}\, o^A o^B$ along the null geodesic ($o^A \bar{o}^{A'} \nabla_{AA'} (\chi_{BC} o^B o^C) = 0$) and the fact that $\Psi_0 = M r^{-3} (\alpha_A o^A)^2 (\beta_B o^B)^2$ and $\chi_{AB} o^A o^B = r (\alpha_A o^A)(\beta_B o^B)$, whence $\Psi_0 = M / (\chi_{AB} o^A o^B)^2 r^5$. Q.E.D.