The Orthographic Image of a Regular Tetrahedron

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Theorem Four points $\alpha, \beta, \gamma, \delta \in \mathbf{R}^2$ are the images of the vertices of a regular tetrahedron in $\mathbf{R}^3$ under orthogonal projection $\mathbf{R}^3 \to \mathbf{R}^2$ if and only if

$$(\alpha + \beta + \gamma + \delta)^2 = 4(\alpha^2 + \beta^2 + \gamma^2 + \delta^2) \qquad (1)$$

where $\alpha, \beta, \gamma, \delta$ are regarded as complex numbers.

Proof. It is easy to check that (1) is translation invariant, so we may as well suppose that $\delta = 0$ and that the prospective regular tetrahedron has its corresponding vertex at the origin in $\mathbf{R}^3$. One possibility for the other three vertices is

$$\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}.$$

Taking the projection to be onto the last two coördinates we get, in this case,

$$\alpha = 1, \qquad \beta = i, \qquad \gamma = 1 + i$$

and it is easy to check that

$$(\alpha + \beta + \gamma)^2 = 4(\alpha^2 + \beta^2 + \gamma^2) \qquad (2)$$

as required. The conformal orthogonal group may be double covered by the group of invertible 2×2 complex matrices of the form

$$\Lambda = \begin{pmatrix} a & -b \\ \bar{b} & \bar{a} \end{pmatrix}$$

acting by $X \mapsto \Lambda X \overline{\Lambda}^t$ on the space of Hermitian 2×2 matrices with zero trace (cf. $SU(2) \cong \mathrm{Spin}(3)$). Therefore, the general regular tetrahedron may be obtained by allowing such a Λ to act on the matrices

$$\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \quad \begin{pmatrix} 1 & i \\ -i & -1 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1+i \\ 1-i & 0 \end{pmatrix}$$

and then picking out the top right hand entries. We obtain

$$\Lambda \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \overline{\Lambda}^t = \begin{pmatrix} a\bar{a} - a\bar{b} - b\bar{a} - b\bar{b} & 2ab + a^2 - b^2 \\ 2\bar{a}\bar{b} + \bar{a}^2 - \bar{b}^2 & b\bar{b} + a\bar{b} + b\bar{a} - a\bar{a} \end{pmatrix}$$

$$\Lambda \begin{pmatrix} 1 & i \\ -i & -1 \end{pmatrix} \overline{\Lambda}^t = \begin{pmatrix} a\bar{a} - ia\bar{b} + ib\bar{a} - b\bar{b} & 2ab + ia^2 + ib^2 \\ 2\bar{a}\bar{b} - i\bar{a}^2 - i\bar{b}^2 & b\bar{b} + ia\bar{b} - ib\bar{a} - a\bar{a} \end{pmatrix}$$

$$\Lambda \begin{pmatrix} 0 & 1+i \\ 1-i & 0 \end{pmatrix} \overline{\Lambda}^t = \begin{pmatrix} (i-1)b\bar{a} - (i+1)a\bar{b} & (i+1)a^2 + (i-1)b^2 \\ (1-i)\bar{a}^2 - (1+i)\bar{b}^2 & (1+i)a\bar{b} + (1-i)b\bar{a} \end{pmatrix}$$

so

$$\alpha = 2ab + a^2 - b^2, \quad \beta = 2ab + ia^2 + ib^2, \quad \gamma = (i+1)a^2 + (i-1)b^2$$

and, again, it is easy to check that (2) is satisfied. Conversely (2) is exactly the condition that α, β, γ may be written in this way for some $a, b \in \mathbb{C}$ (since every non-singular conic in $\mathbb{CP}_2$ is rational). □

There are several variations on this theme. Each simplex in $\mathbb{R}^3$ gives rise to a quadratic equation which characterises the orthographic images of similar simplices. For example, if a vertex of a cube is mapped to the origin and its neighbours are sent to $\alpha, \beta, \gamma \in \mathbb{C}$, then $\alpha^2 + \beta^2 + \gamma^2 = 0$ (this is due to Hadwiger [H], though not stated in this form using complex numbers). The corresponding equation for a regular dodecahedron is

$$(\alpha + \beta + \gamma)^2 + (\sqrt{5} - 1)(\alpha^2 + \beta^2 + \gamma^2) = 0.$$

There is a general theory for orthogonal projection $\mathbb{R}^n \to \mathbb{R}^m$ with an interpretation in terms of complex numbers when $m = 2$. Details will appear elsewhere.

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References

[H] H. Hadwiger, Über ausgezeichnete Vectorsterne und reguläre Polytope, Comm. Math. Helv. **13** (1940), 90–108.