Indefinite, conformally-ASD metrics on S^2xS^2

There is a natural, conformally-flat, indefinite metric on S^2xS^2, namely

$$ds^2 = \frac{4d\zeta d\bar{\zeta}}{(1+\zeta\bar{\zeta})^2} - \frac{4d\eta d\bar{\eta}}{(1+\eta\bar{\eta})^2} . \qquad 1$$

From the way that this is written, it obviously has an integrable complex structure J. Lowering J with the (indefinite) metric gives a 2-form which therefore must be closed. The scalar-curvature of this metric is zero, being the difference between the scalar curvatures of the summands, so that (1) defines an indefinite, scalar-flat Kähler metric. The Weyl curvature is therefore ASD (in fact this metric is conformally-flat) so it has a twistor-space and we may look for α-planes in the complexification. Free $\bar{\zeta},\bar{\eta}$ from ζ,η respectively and write them as $\tilde{\zeta},\tilde{\eta}$, then the equation

$$\zeta = \frac{a\eta+b}{c\eta+d} ; \qquad \tilde{\zeta} = \frac{d\tilde{\eta}-c}{-b\tilde{\eta}+a} \qquad 2$$

for $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,C)/(I,-I)$ defines an α-plane. As a particular case of (2)

the metric (1) has real α-planes given by (2) for $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SU(2)/(I,-I)$

which is just RP3. Finally, there are some more α-planes missed by (2), namely ones with ζ=const., η=const., and ones with $\tilde{\zeta}$=const.,$\tilde{\eta}$=const.. None of these are real. The way these α-planes fit together to form the twistor-space is described by Lionel Mason elsewhere in this issue.

At a recent seminar in Oxford, (26/1/93; see the accompanying article), Lionel gave a fairly explicit picture of deformations of this twistor space which must lead to deformations of the metric (1) i.e. to real, conformally-ASD metrics with the ultra-hyperbolic signature (++--) globally-defined on S^2xS^2. It is possible to modify the ansatz given by Claude LeBrun in J.Diff.Geom. **34** (1991) 223 to find some of these metrics explicitly, and this is what I want to describe here.

LeBrun's ansatz gives the general, scalar-flat, Kähler metric with S^1-action. Start with the metric in LeBrun's notation and switch some signs to get the indefinite scalar-flat Kähler metric as

$$ds^2 = w(e^u(dx^2+dy^2)-dz^2)-w^{-1}(d\varphi+\omega)^2. \qquad 3$$

With the change of signs, the field equations turn into

$$u_{xx} + u_{yy} - (e^u)_{zz} = 0 \qquad 4$$

and

$$w_{xx} + w_{yy} - (we^u)_{zz} = 0. \qquad 5$$

There is also the equation for ω, which I will postpone giving.

Now take the solution of (4) given by

$$e^u = \frac{4(1-z^2)}{(1+x^2+y^2)^2} \qquad 6$$

and introduce V by $V = w(1-z^2)$. (The quickest route to the solution (6) is to demand that u be separable in the sense that $u = f(x,y) + g(z)$.)

In terms of θ, where $z=\cos\theta$, and $\zeta=x+iy$, the metric (3) becomes

$$ds^2 = \frac{4V\, d\zeta d\bar{\zeta}}{(1+\zeta\bar{\zeta})^2} - Vd\theta^2 - \frac{\sin^2\theta}{V}(d\varphi + \omega)^2 \qquad 7$$

while (5) turns into

$$(1+\zeta\bar{\zeta})^2\frac{\partial^2 V}{\partial\zeta\partial\bar{\zeta}} = (1-z^2)\frac{\partial^2 V}{\partial z^2}. \qquad 8$$

To write the equation for ω, deferred from above, we first introduce angles ψ, χ by $\zeta = \tan(\psi/2)\exp(i\chi)$. Then

$$d\omega = \left(\csc\psi \frac{\partial V}{\partial\chi} d\psi - \sin\psi \frac{\partial V}{\partial\psi} d\chi\right) \wedge \csc\theta\, d\theta + \sin\psi \csc\theta \frac{\partial V}{\partial\theta} d\psi \wedge d\chi \qquad 9$$

The (obvious) solution of (8,9) given by V=1 when plugged in to (3) gives the metric (1) on S^2xS^2 with which we began. For a nice way to write some more, introduce Q by

$$Q = \frac{\partial V}{\partial z} \qquad 10$$

and differentiate (8) with respect to z to get an equation for Q. The equation which results is the 'ultra-hyperbolic wave equation':

$$\nabla_1{}^2 Q = \nabla_2{}^2 Q \qquad 11$$

where the first Laplacian is on the ζ-sphere, the second is on the (θ,φ)-sphere and Q is independent of φ i.e. we get a metric for every axisymmetric (in this sense) solution of (11).

To solve (11), use Legendre polynomials P_ℓ and associated Legendre polynomials $Y_{\ell m}$:

$$Q = \sum a_{\ell m} Y_{\ell m}(\zeta,\bar{\zeta}) P_\ell(z) \;;$$

substitute into (10) and integrate to get V:

$$V = 1 + \sum a_{\ell m} Y_{\ell m}(\zeta,\bar{\zeta}) \int_{-1}^{z} P_\ell(s)ds \;. \qquad 12$$

With these choices, it turns out that V now equals 1 at $z = \pm 1$, which you need in order to avoid conical singularities in the metric (7). There doesn't seem to be a nice formula for ω.

Thanks to Lionel Mason and Claude Lebrun.

Paul Tod