Indefinite, conformally-ASD metrics on $S^2 \times S^2$

There is a natural, conformally-flat, indefinite metric on $S^2 \times S^2$, namely

$$ds^2 = \frac{4d\zeta d\bar{\zeta}}{(1+\zeta)^2} - \frac{4d\eta d\bar{\eta}}{(1+\eta)^2}.$$  

From the way that this is written, it obviously has an integrable complex structure $J$. Lowering $J$ with the (indefinite) metric gives a 2-form which therefore must be closed. The scalar-curvature of this metric is zero, being the difference between the scalar curvatures of the summands, so that (1) defines an indefinite, scalar-flat Kähler metric. The Weyl curvature is therefore ASD (in fact this metric is conformally-flat) so it has a twistor-space and we may look for $\alpha$-planes in the complexification. Free $\zeta, \bar{\zeta}$ from $\zeta, \bar{\eta}$ respectively and write them as $\zeta, \bar{\eta}$, then the equation

$$\zeta = \frac{am-bc}{cn+\bar{d}}; \quad \bar{\zeta} = \frac{\bar{d} \eta-c}{-b \bar{\eta} + a}$$

for $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}(2, \mathbb{C})/(1,-1)$ defines an $\alpha$-plane. As a particular case of (2) the metric (1) has real $\alpha$-planes given by (2) for $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SU}(2)/(1,-1)$ which is just RP3. Finally, there are some more $\alpha$-planes missed by (2), namely ones with $\zeta=$const., $\eta=$const., and ones with $\bar{\zeta}=$const., $\bar{\eta}=$const.. None of these are real. The way these $\alpha$-planes fit together to form the twistor-space is described by Lionel Mason elsewhere in this issue.

At a recent seminar in Oxford, (26/1/93; see the accompanying article), Lionel gave a fairly explicit picture of deformations of this twistor space which must lead to deformations of the metric (1) i.e. to real, conformally-ASD metrics with the ultra-hyperbolic signature $(++-)$. It is possible to modify the ansatz given by Claude LeBrun in J.Diff.Geom. 34 (1991) 223 to find some of these metrics explicitly, and this is what I want to describe here.

LeBrun's ansatz gives the general, scalar-flat, Kähler metric with $S^1$-action. Start with the metric in LeBrun's notation and switch some signs to get the indefinite scalar-flat Kähler metric as

$$ds^2 = w(e^u(dx^2 + dy^2) - dz^2) - w^{-1}(d\phi + \omega)^2.$$  

With the change of signs, the field equations turn into

$$u_{\phi\phi} + u_{\phi\phi} - (e^u)_{xx} = 0$$

and

$$w_{\phi\phi} + w_{\phi\phi} - (we^u)_{xx} = 0.$$  

There is also the equation for $\omega$, which I will postpone giving.

Now take the solution of (4) given by
\[ e^\omega = \frac{4(1-z^2)}{(1+x^2+y^2)^2} \]

and introduce \( V \) by \( V = w(1-z^2) \). (The quickest route to the solution (6) is to demand that \( u \) be separable in the sense that \( u = f(x,y) + g(z) \).)

In terms of \( \theta \), where \( z = \cos\theta \), and \( \zeta = x+iy \), the metric (3) becomes
\[ ds^2 = \frac{4V d\zeta^2}{(1+\zeta^2)^2} - V d\theta^2 - \frac{\sin^2\theta}{\sqrt{V}} (d\psi + \omega)^2 \]
while (5) turns into
\[ (1+\zeta^2)\frac{\partial^2 V}{\partial \zeta^2} = (1-z^2)\frac{\partial^2 V}{\partial z^2} \]

To write the equation for \( \omega \), deferred from above, we first introduce angles \( \psi, \chi \) by \( \zeta = \tan(\psi/2) \exp(i\chi) \). Then
\[ d\omega = (\csc \psi \frac{\partial V}{\partial \psi} d\psi - \sin \psi \frac{\partial V}{\partial \chi} d\chi)^2 + \csc \theta d\theta + \sin \psi \csc \psi \frac{\partial V}{\partial \psi} d\psi + d\chi \]

The (obvious) solution of (8, 9) given by \( V=1 \) when plugged in to (3) gives the metric (1) on \( S^2 \times S^2 \) with which we began. For a nice way to write some more, introduce \( Q \) by
\[ Q = \frac{\partial V}{\partial z} \]
and differentiate (8) with respect to \( z \) to get an equation for \( Q \). The equation which results is the 'ultra-hyperbolic wave equation':
\[ V_{z^2} Q = V_{z^2} Q \]
where the first Laplacian is on the \( \zeta \)-sphere, the second is on the \( (\theta, \psi) \)-sphere and \( Q \) is independent of \( \varphi \) i.e. we get a metric for every axisymmetric (in this sense) solution of (11).

To solve (11), use Legendre polynomials \( P_\lambda \) and associated Legendre polynomials \( Y_{\lambda m} \):
\[ Q = \sum a_{\lambda m} Y_{\lambda m}(\zeta, \xi) P_\lambda(z) ; \]
substitute into (10) and integrate to get \( V \):
\[ V = 1 + \sum a_{\lambda m} Y_{\lambda m}(\zeta, \xi) \int P_\lambda(s) ds \]

With these choices, it turns out that \( V \) now equals \( 1 \) at \( z = \pm1 \), which you need in order to avoid conical singularities in the metric (7). There doesn't seem to be a nice formula for \( \omega \).

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