Global twistor correspondences in split signature

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Introduction

The purpose of this note is to describe the globalization of twistor correspondences in signature (2,2). In particular I discuss the non-linear graviton construction for anti-self-dual metrics of signature (2,2) on $S^2 \times S^2$ which specializes to give the twistor description of the examples of conformally anti-self-dual metrics described by K.P.Tod in his accompanying article in this issue. Part of the motivation for this construction arose from a desire to understand the globalization of other aspects of the Penrose transform in signature $(2,2)$ and thereby give a twistor description of the Radon transform and the inverse scattering transform. The connection of these ideas with the inverse scattering transform will be discussed elsewhere.

When one considers boundary conditions for solutions of conformally invariant equations on $\mathbb{R}^4$ in signature $(2,2)$, the first condition one might try would be to require that solutions should extend to the conformal compactification of $\mathbb{R}^4$, $S^2 \times S^2/\mathbb{Z}_2$ as the equations are conformally invariant. We will see, however, that these boundary conditions eliminate all solutions in linear theory. This problem is resolved if we go to the double cover, $S^2 \times S^2$ (where one can ask the question as to which fields are invariant under the $\mathbb{Z}_2$ action).

The global geometry

The conformal compactification $\mathbb{M}$ of affine signature $(2,2)$ Minkowski space, obtained by adjoining a 'light cone at infinity', is the projective quadric in $\mathbb{RP}^5$ given by the zero set of a quadratic form Q in $\mathbb{R}^6$ of signature $(3,3)$. The conformal structure is determined by asserting that the light cones of $\mathbb{M}$ are the intersections of $\mathbb{M}$ with the tangent planes of points of $\mathbb{M}$.

To see that the topology of $\mathbb{M}$ is $S^2 \times S^2/\mathbb{Z}_2$ diagonalize Q using a pair of Euclidean 3-vectors $\underline{w}$ and $\underline{y}$ as coordinates on $\mathbb{R}^6$ such that $Q = \underline{w} \cdot \underline{w} - \underline{y} \cdot \underline{y}$. Set the scale by requiring $\underline{w} \cdot \underline{w} = 1$ so that $\underline{y} \cdot \underline{y} = 1$ also on $Q = 0$; this yields $S^2 \times S^2$ in $\mathbb{R}^6$. However, in $\mathbb{RP}^5$, $(\underline{w}, \underline{y}) \sim (-\underline{w}, -\underline{y})$ so the topolgy of $Q = 0$ in $\mathbb{RP}^5$ is $S^2 \times S^2/\mathbb{Z}_2$.

The conformal structure is simply realized on the double cover $\widetilde{\mathbb{M}} = S^2 \times S^2$ by taking the pullback of the round sphere metric $d\Omega^2$ on each factor and taking the difference

$$ds^2 = p_1^* d\Omega^2 - p_2^* d\Omega^2$$

where p_1, p_2 are the projections onto the first and second factors respectively. This is $\mathbb{Z}_2$ invariant and so descends to $\mathbb{M}$.

The global correspondence

We will be interested in the cases where the original region on which the fields are defined are small complexifications of $\mathbb{M}$ and its double cover $\widetilde{\mathbb{M}}$ (which will be denoted by the same symbol). We will now consider the correspondence for these cases (or deformations thereof for ASD metrics).

The twistor space of a space U will be the spaces of connected components of (α-planes) in U . For $Z \in \mathbb{PT}(U)$, we will denote the corresponding α-plane in U by $\hat{Z}$.

The correspondence for $\mathbb{M}$

Just as compactified complexified Minkowski space $\mathbb{CM}$ is the the space of complex lines in $\mathbb{CP}^3$ via the complex Klein correspondence, $\mathbb{M}$ is the space of real lines in $\mathbb{RP}^3$ via the real Klein correspondence. In the context of the complex correspondence, point of $\mathbb{M}$ are complex lines in $\mathbb{CP}^3$ that intersect $\mathbb{RP}^3$ in a real line. Alternatively, they are complex lines in $\mathbb{CP}^3$ that are mapped into themselves by the complex conjugation $Z^\alpha \to \bar{Z}^\beta$ given by standard complex conjugation, component by component.

We have $\mathbb{PT}(\mathbb{M}) = \mathbb{CP}^3$. Given $Z \in \mathbb{CP}^3$, then if $Z = \bar{Z}$, $Z \in \mathbb{RP}^3$ and any real line in $\mathbb{RP}^3$ through Z corresponds to a point in $\mathbb{M}$ on $\hat{Z}$. In this case, $\hat{Z}$ intersects $\mathbb{M}$ in an $\mathbb{RP}^2$. If $Z \neq \bar{Z}$ then the complex line through Z and $\bar{Z}$ is real and corresponds to a point of $\mathbb{M}$. In fact the complex α-plane $\hat{Z}$ intersects $\mathbb{M}$ in the unique point corresponding to this line.

Linear theory

The linear problem was completely solved in the case of the wave equation by Fritz-John using the X-ray transform. In twistor notation, the general solution of the hyperbolic wave equation on $\mathbb{R}^4$ satisfying appropriate boundary conditions can be obtained from the integral formula

$$\phi(x^{AA'}) = \oint f(x^{AA'}\pi_{A'}, \pi_{A'})\pi^{A'} d\pi_{A'}.$$

Here f is a freely specifiable smooth section of $\mathcal{O}(-2)$ on $\mathbb{RP}^3$. That ϕ is a solution of the ultrahyperbolic wave equation follows by differentiation under the integral sign.

One might naively think that the function ϕ is naturally a function on the space of lines in $\mathbb{RP}^3$, $\mathbb{M}$. However, ϕ is defined by integrating f along lines and in order to perform the integration, one needs to have an orientation of the line. This means that ϕ is actually defined on the space of *oriented* lines in $\mathbb{RP}^3$. This is $\widetilde{\mathbb{M}}$ the double cover of $\mathbb{M}$. Clearly ϕ changes sign under reversal of orientation of the line and so does not descend to $\mathbb{M}$. Indeed, we will see that there are no solutions of the conformally invariant wave equation on $\mathbb{M}$.

Remark: Actually, there is a possible confusion here owing to the Grgin phenomena—the real point is that these solutions are anti-Grgin. Solutions of the wave equation are sections of $\mathcal{O}[-1]$, the inverse conformal weight bundle. Given just the metric $ds^2 = p_1^* d\Omega^2 - p_2^* d\Omega^2$

on $S^2 \times S^2/\mathbb{Z}_2$ there are two possible choices for $\mathcal{O}[-1]$, the trivial bundle or the Mobius bundle. The correct choice as far as the twistor correspondence is concerned is the Mobius bundle as this is just the restriction of the tautological bundle from $\mathbb{RP}^5$. The solutions above are actually sections of the trivial bundle which is wrong as far as the twistor correspondence is concerned. So one can simply write them down as even *functions* on $S^2 \times S^2$ but as odd sections of $\mathcal{O}[-1]$.

The correspondence for $\widetilde{\mathbb{M}}$

We must therefore study twistor correspondences for $\widetilde{\mathbb{M}}$. We shall abuse notation and denote also by $\widetilde{\mathbb{M}}$ a small complex thickening of $\widetilde{\mathbb{M}}$. We have:

Lemma 0.1 $\mathbb{PT}(\widetilde{\mathbb{M}})$ *is the (non-Hausdorff) space obtained by gluing together two copies of* $\mathbb{CP}^3$, *denoted* $\mathbb{CP}^3_+$ *and* $\mathbb{CP}^3_-$, *together along some small thickening of* $\mathbb{RP}^3$ *using the identity map.*

Proof: Points in the complement of (the thickening of) $\mathbb{RP}^3$ in $\mathbb{CP}^3$ correspond to α-planes that intersect $\mathbb{M}$ in a topologically trivial region and these are necessarily covered by two components in the double cover $\widetilde{\mathbb{M}}$. Whereas, points in the (thickening of) $\mathbb{RP}^3$ correspond to α-planes in (the small thickening of) $\mathbb{M}$ with topology $\mathbb{RP}^2$ so that when one takes the double cover the α-plane has topology S^2.

Thus $\mathbb{PT}(\widetilde{\mathbb{M}})$ double covers $\mathbb{CP}^3$ over the complement of the thickening of $\mathbb{RP}^3$ and the double covering is glued together over the thickening of $\mathbb{RP}^3$.

We reconstruct $\widetilde{\mathbb{M}}$ as the space of complex lines in $\mathbb{PT}(\widetilde{\mathbb{M}})$ that are cut into two pieces by $\mathbb{RP}^3$ with one piece lying in $\mathbb{CP}^3_+$'s and the other in $\mathbb{CP}^3_-$. This yields the space of oriented lines in $\mathbb{RP}^3$; the line is given by the intersection of complex line with $\mathbb{RP}^3$ and the orientation is determined by multiplying the arrow from the intersection with $\mathbb{CP}^3_-$ to that with $\mathbb{CP}^3_+$ by i and thereby rotating it by 90 degrees.

The non-Hausdorffness arises because as one varies an α-plane within $\widetilde{\mathbb{M}}$, it can break into two disconnected parts. The points on the boundary of the glued down region are the ones with the non-Hausdorffly separated partner. This space is actually a deformation retract of that considered in the discussion of sourced fields—see for example the last chapter of Further Advances in Twistor Theory Vol.1.

Complex conjugation

Complex conjugation on the small thickening of $\widetilde{\mathbb{M}}$ sends α-planes to α-planes and hence leads to a conjugation on $\mathbb{PT}(\widetilde{\mathbb{M}})$. This covers the standard complex conjugation on $\mathbb{CP}^3$ that fixes $\mathbb{RP}^3$ and is lifted to $\mathbb{PT}(\widetilde{\mathbb{M}})$ by requiring that it interchange $\mathbb{CP}^3_+$ and $\mathbb{CP}^3_-$. Thus $[Z] \in \mathbb{CP}^3_-$ goes to $[\bar{Z}] \in \mathbb{CP}^3_+$ and the real lines of the conjugation are those described above that correspond to points of $\widetilde{\mathbb{M}}$.

The X-ray transform.

We can now understand the X-ray transform in this context. Solutions of the wave equation on $\widetilde{\mathbb{M}}$ correspond to elements of $H^1(\mathbb{PT}(\widetilde{\mathbb{M}}), \mathcal{O}(-2))$. These can be studied by means of the Meyer Vietoris sequence using the covering of $\mathbb{PT}(\widetilde{\mathbb{M}})$ by $\mathbb{CP}^3_+$ and $\mathbb{CP}^3_-$. Using the fact that $H^1(\mathbb{CP}^3, \mathcal{O}(-2)) = 0 = H^0(\mathbb{CP}^3, \mathcal{O}(-2))$ we find

$$H^1(\mathbb{PT}(\widetilde{\mathbb{M}})) = H^0(\mathbb{CP}^3_+ \cap \mathbb{CP}^3_-, \mathcal{O}(-2)) = H^0(\mathbb{RP}^3, \mathcal{O}(-2))$$

and the formula for the Penrose transform using these representatives is precisely the X-ray transform.

Deformations of $\mathbb{PT}(\widetilde{\mathbb{M}})$

The nonlinear graviton construction implies that (small) ASD deformations of the conformal structure on $S^2 \times S^2$ correspond to (small) deformations of $\mathbb{PT}(\widetilde{\mathbb{M}})$. Since $\mathbb{CP}^3$ is rigid, the only deformable part is the gluing along $\mathbb{RP}^3$. In order to guarantee that the reality structure is preserved, the gluing map P from som open set in $\mathbb{CP}^3_+$ to one in $\mathbb{CP}^3_-$ must be compatible with the conjugation that sends $Z \in \mathbb{CP}^3_+$ to $\bar{Z} \in \mathbb{CP}^3_-$. This yields the condition that $P^{-1} = \bar{P}$ where $\bar{P}$ is the conjugate map. This can be arranged as follows.

Take a small analytic deformation ρ of the standard embedding of $\mathbb{RP}^3$ into $\mathbb{CP}^3$ so that ρ has a small analytic extension to a neighbourhood U of $\mathbb{RP}^3$ in $\mathbb{CP}^3$. Then we also have the conjugate embedding $\bar{\rho}$ of U into $\mathbb{CP}^3$ which is the complexification of the complex conjugate embedding (it is also holomorphic). The deformed gluing from $\mathbb{CP}^3_+$ to $\mathbb{CP}^3_-$ is then done with the map $P = \bar{\rho} \circ \rho^{-1}$.

The complex conjugation map of the deformed glued down twistor space can then be defined by sending the point $Z \in \mathbb{CP}^3_+$ to the point $\bar{Z} \in \mathbb{CP}^3_-$. This conjugation clearly fixes the image of $\mathbb{RP}^3$, is antiholomorphic and acts globally.

The space-time with deformed ASD conformal structure is then reconstructed by constructing the complex lines in the deformed space that are divided into two parts by the glued down region and are half in $\mathbb{CP}^3_+$ and half in $\mathbb{CP}^3_-$.

Thus we have a $1-1$ map from ASD deformations of the conformal structure on $S^2 \times S^2$ and such real gluing maps as above. These can be thought of as the space of (analytically) embedded $\mathbb{RP}^3$'s in $\mathbb{CP}^3$ or $map\{\mathbb{RP}^3 \to \mathbb{CP}^3\}/Diff\{\mathbb{RP}^3\}$ as a diffeomorphism of $\mathbb{RP}^3$ does not affect the final P. This last space can be thought of as the space of complexified diffeomorphism modulo the real ones.

Examples: The examples that Paul Tod writes down in his article in this issue are obtained from a split signature analogue of LeBrun's hyperbolic Gibbons-Hawking ansatz, LeBrun 1991. The basic idea is to take a global holomorphic vector field on $\mathbb{CP}^3$ that is real on $\mathbb{RP}^3$ and to drag the standard gluing some fixed amount along the imaginary part of the vector field. This is a global version of the construction of Jones & Tod, (1985).

Use 2×2 matrices as homogeneous coordinates on twistor space with columns (λ_A, μ_A). The real slice $\mathbb{RP}^3$ sits inside as PSU(2) with λ_A the SU(2) complex conjugate of μ_A. The vector field $V = i\lambda_A \partial/\partial\lambda_A - i\mu_A\partial/\partial\mu_A$ corresponds to right multiplication by diagonal SU(2) matrices. The quotient by the complexified vector field is the quadric Q coordinatized by $([\lambda_A], [\mu_A])$ with real slice S^2. On $\widetilde{\mathbb{M}}$ the symmetry can be represented so that it rotates just one of the S^2 factors and leaves the other invariant.

Choose a real analytic function on S^2, $f([\lambda_A], [\mu_A])$ defined for $\lambda_A = \bar{\mu}_A$ and continue it to some small thickening of the real slice. Then we identify $(\exp(-f)\lambda_A, \exp(f)\mu_A)$ in $\mathbb{CP}^3_+$ with $(\exp(f)\lambda_A, \exp(-f)\mu_A)$ in $\mathbb{CP}^3_-$ where $[\lambda_A]$ and $[\mu_A]$ are close to being conjugate. Thus, if we take out the lines $\lambda_A = 0$ and $\mu_A = 0$, the twistor space PT is a complex line bundle over the space $\tilde{Q}$ obtained by gluing one copy of the quadric Q_+ to another Q_- over some thickening of the real slice.

The space-time metric is given in standard form in Jones & Tod (1985) as

$$ds^2 = d\Sigma_3^2 + (d\phi + \omega)^2/V^2$$

where here $d\Sigma_3^2$ is the Einstein-Weyl space corresponding to the quadric which is just Lorentzian hyperbolic space and ω and V are the parts of an invariant ASD $U(1)$ connection that are respectively orthogonal and tangent to the symmetry direction and thus satisfy $d\omega = *_3 dV$ where $*_3$ is the hodge dual with respect to $d\Sigma_3^2$.

It is a straightforward, but slightly tedious exercise to show that the metrics in Tod's article can be put into this form after a conformal rescaling (the main non trivial part is to show that the 3-metric $(d\theta^2 - 4d\zeta d\bar{\zeta}/(1+|\zeta|^2)^2)/\sin^2\theta$ is the Lorentzian hyperbolic metric).

Discussion:

1) It is possible to write down metrics that are Ricci flat with conformal structures that extend over $S^2 \times S^2$ but whose null infinity cuts the space in half. This uses the same construction but with a translation symmetry generated by $\pi^A\partial/\partial\omega^A$ where the real slice is now given by real values for the components of (ω^A, π^A). The gluing identifies $\omega^A - if\pi^A$ on $\mathbb{CP}^3_+$ with $\omega^A + if\pi^A$ on $\mathbb{CP}^3_-$ where $f := f(\omega^A\pi_A, \pi_A)$ has homogeneity zero and is rapidly decreasing as $\omega^A\pi_A/(\pi_0^2 + \pi_1^2) \to \infty$.

2) The analogous (general) construction for ASD Yang-Mills fields gives a correspondence between ASDYM connections on $\widetilde{\mathbb{M}}$ and pairs consisting of a holomorphic vector bundle E on $\mathbb{CP}^3$ and a map $P : E \to \bar{E}$ on the thickening of $\mathbb{RP}^3$. This gives a paradigm for the inverse scattering transform with the bundle E corresponding to the solitonic part of the data and the map P corresponding to the 'scattering data'. When c_2 of the bundle on space-time is 0, E is trivial and the data is precisely a matric function P on $\mathbb{RP}^3$ satisfying $\bar{P} = P^{-1}$, or alternatively P is a map from $\mathbb{RP}^3$ to $G_{\mathbb{C}}/G$.

References

Jones, P. and Tod, K.P. (1985) Minitwistor spaces and Einstein Weyl spaces, Class. & Quant. Grav. **2** 565-577.

LeBrun, C. (1991) Explicit self-dual metrics on $\mathbb{CP}^2\#\ldots\#\mathbb{CP}^2$, J. Differential Geometry, **34** (1991) 223-253.