On The Dimension of Elementary States

In [5], Michael Singer proposed a definition for a four dimensional conformal field theory. In this theory the role of compact Riemann surfaces, which occur in standard conformal field theory, is played instead by compact flat twistor spaces. It is therefore tempting to ask questions about these twistor spaces which are, in some way, natural extensions from Riemann surfaces. The properties of compact Riemann surfaces are well known and have been extensively documented and so there is an immensely rich source of possible questions that can be asked about flat twistor spaces.

One such property of compact Riemann surfaces concerns meromorphic functions having poles of prescribed maximum order at given points: Let X be a compact Riemann surface with distinct points $P_1, \ldots, P_k$ on X. If $n_1, \ldots, n_k$ are arbitrary positive integers, how many linearly independent meromorphic functions are there on X, which have poles of order at most n_i at P_i, and no others?

To answer this question one can use the following strategy:

(a) Convert the question to one involving global data.
This is achieved through the introduction of line bundles and divisors. The problem then becomes one of determining the dimension of the cohomology group $H^0(X, \vartheta[D])$ where $[D]$ is the line bundle of the divisor D, which for the above problem is $\sum_{i=1}^{k} n_i P_i$.

(b) Use the Riemann-Roch theorem.
This enables holomorphic data to be calculated in terms of topological data: specifically $\dim H^0(X, \vartheta[D]) - \dim H^1(X, \vartheta[D]) = \deg D + 1 - g$, where g is the genus of X.

(c) Use vanishing theorems for $H^1(X, \mathcal{O}[D])$ in order to eliminate the unwanted term. One such is the Kodaira theorem: if $\deg(D) > 2g - 2$ then $H^1(X\ \mathcal{O}[D]) = 0$.

To extend this problem to flat twistor-spaces we need a good analogue of meromorphic functions with prescribed singularities. Fortunately a candidate for these exist, the elementary states based on a line, but one first has to decide what is meant by a prescribed order of singularity, and then how to extend this notion to a general flat twistor-space.

The first part of this question was answered by Eastwood and Hughston in [1]. If $f(z)$ is homogeneous of degree m in z, and if $f(z)$ is coprime to z_0 and z_1, then $\frac{f(z)}{z_0 z_1}$ is a representative of an elementary state of homogeneity $m - 2$, with a pole of order 1 on the line $z_0 = z_1 = 0$. Similarly, $\frac{f(z)}{z_0 z_1^2} = \frac{z_0 f(z)}{(z_0 z_1)^2}$ has homogeneity $(m + 1) - 4 = m - 3$, and a pole of order 2 on $z_0 = z_1 = 0$.

Thus, at least in the case of $\mathbf{P}^3$, one can formulate the question: Given a fixed line L in $\mathbf{P}^3$, a given homogeneity n, and a positive integer k, how many elementary states based on L are there, with homogeneity n, and with a singularity on L of order at most k?

One way of converting this question, at least in $\mathbf{P}^3$, to one involving global data, was given in [1]. Take the line L, blow-up $\mathbf{P}^3$ along L, to obtain the complex manifold $\tilde{\mathbf{P}}^3$. In this case $\mathbf{P}^3$ is actually a submanifold of $\mathbf{P}^3 \times \mathbf{P}^1$. Now let $a \geq 0$, $b \leq -2$, be integers, and form the bundle $\mathcal{O}(a) \otimes \mathcal{O}(b)$ on $\mathbf{P}^3 \times \mathbf{P}^1$, from $\mathcal{O}(a)$ on $\mathbf{P}^3$ and $\mathcal{O}(b)$ on $\mathbf{P}^1$. Let $\mathcal{O}(a,b)$ be the restriction of this bundle to $\tilde{\mathbf{P}}^3$. Then it is shown in [1], that elements of $H^1(\tilde{\mathbf{P}}^3, \mathcal{O}(a,b))$, when restricted away from the blown-up line, are representatives for elementary states based on L, of homogeneity $a + b$, and order of singularity <u>at most</u> $-b-1$. The question now concerns the dimension of $H^1(\tilde{Z}, \mathcal{O}(a,b))$.

This gives a way of extending the definition of elementary states based on a line, to flat twistor-spaces, since such twistor-spaces have the property that each projective line (fibre) has a neighbourhood which is biholomorphic to $\mathbf{P}_\bullet$.

Given a flat twistor-space Z, and distinguished line L, one can then define the blow-up of Z along L, say $\tilde{Z}$, and the bundle $\vartheta(a,b)$ can be defined on $\tilde{Z}$ in such a way as to preserve the essential properties of $\vartheta(a,b)$ in $\tilde{\mathbf{P}}^3$. Elements of $H^1(\tilde{Z}, \vartheta(a,b))$ then have homogeneity a + b and a 'pole' of order at most -b -1 on L, when restricted away from L, but in a neighbourhood of L.

We then define the elements of this group to be our elementary states based on L, with the prescribed conditions, and the problem we wish to solve is to find the dimension of $H^1(\tilde{Z}, \vartheta(a,b))$. This will give a partial answer to the equivalent problem in Riemann surfaces which formed the motivation for this work.

The strategy for the solution follows closely that for Riemann-surfaces:

(a) The question has already been converted to one involving global data on a compact manifold, though this time it is $\tilde{Z}$, not Z, ie. need to calculate dim $H^1(\tilde{Z}, \vartheta(a,b))$.

(b) The Hirzebruch-Riemann-Roch theorem enables the calculation of the holomorphic Euler characteristic of $\vartheta(a,b)$ on $\tilde{Z}$, using topological data [7].
This states that the holomorphic Euler characteristic $\chi(\tilde{Z}, \vartheta(a,b))$, which is defined by $\chi(\tilde{Z}, \vartheta(a,b)) = \sum_{i=0}^{3}(-1)^i \dim H^i(\tilde{Z}, \vartheta(a,b))$, is given in terms of Chern classes of $\tilde{Z}$, ie

$$\dim H^0(\tilde{Z}, \vartheta(a,b)) - \dim H^1(\tilde{Z}, \vartheta(a,b)) + \dim H^2(\tilde{Z}, \vartheta(a,b)) - \dim H^3(\tilde{Z}, \vartheta(a,b))$$
$$= \left[\mathrm{Ch}(\vartheta(a,b))\,\mathrm{Td}(\tilde{Z})\right]\left[\tilde{Z}\right]$$

where $Ch(\mathcal{O}(a,b))$ is the Chern character of the bundle $\mathcal{O}(a,b)$, and $Td(\tilde{Z})$ is the Todd class of (the holomorphic bundle of) $\tilde{Z}$.

I was able to calculate this when M was a compact, oriented, Riemannian self-dual, 4-manifold, and Z was its twistor space [2]. The method involved using Poincaré duality and intersection of homology classes. I obtained the following results:

$$\chi\left(\tilde{Z},\mathcal{O}(a,b)\right)=\frac{1}{12}(a+b+1)(a+b+2)(a+b+3)\chi$$
$$-\frac{1}{8}(a+b+2)[(a+b+1)(a+b+3)-1]\tau$$
$$-\frac{1}{6}b(b+1)(b+3a+5),$$

where χ is the Euler characteristic, and $\tau = 0$ in this case.

(c) Vanishing theorems.

There is more work to be done in this case since the alternating sum contains 4 terms. In the case where the manifold M has negative scalar curvature, it is easy to show both the H^0 and H^3 terms are zero. This leaves H^2 as the awkward term.

The Serre dual of $H^2(\tilde{Z}, \mathcal{O}(a,b))$ can be calculated, and is $H^1(\tilde{Z}, \mathcal{O}(-a-3,-b-1))$. With the given restrictions on a and b, this means that we have to deal with $\dim H^1\left(\tilde{Z},\mathcal{O}(c,d)\right)$ where $c \leq -3$, $d \geq 1$.

It turns out that, using diagram chasing techniques on two Mayer Vietoris sequences, I was able to prove the following relationship between the cohomologies of the blown-up manifold $\tilde{Z}$, and the flat twistor-space Z: if $H^0(Z, \mathcal{O}(c + d))$ and $H^1(Z, \mathcal{O}(c + d))$ both vanish, then $\dim H^1(\tilde{Z}, \mathcal{O}(c,d)) = \dim H^0(P^+, \mathcal{O}(c,d))$, and this would enable the final calculation to be made [3].

The question now turns on a vanishing theorem for $H^1(Z, \vartheta(m))$. For the case of M having negative scalar curvature, I was then able to prove the following theorem: If M is compact, Riemannian, self-dual, Einstein, with negative scalar curvature, then
$H^1(Z, \vartheta(m)) = 0$ if $m > 0$, [4].

The method of proof used the Penrose transform to identify $H^1(Z, \vartheta(n-2))$ with certain spinor fields in M. This produced a Bochner style vanishing theorem for the spinor fields. This result was brought to my attention by C. LeBrun, from some work of his research student (M. Thornber [6]) on vanishing theorems for quaternionic-Kähler manifolds. His method of approach was completely different, involving a direct attack on the problem in Z, though his proof did not cover the case of 4-dimensional manifolds M, only twistor spaces of 4k-dimensional quaternionic-Kähler M.

This information provides an answer in the following case:
Let M be a compact, Riemannian, conformally-flat, Einstein manifold with negative scalar curvature, and let Z be its twistor-space. If L is a distinguished line in Z and $a \geq 0$, $b \leq -2$, with $a + b < -4$, then the number of elementary states with singularity on L, of order at most $-b-1$ and homogeneity $a + b$, is given by

$$\dim H^0(\mathbf{P}^+, \vartheta(-(a+b)-4)) - \chi(\tilde{Z}, \vartheta(a,b)).$$

We note that the M are precisely the hyperbolic 4-manifolds.

Details will appear anon.

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