Diagrams for tensor products of $\mathcal{H}_k$, the discrete series representations of $SU(1,1)$ with lowest weight

One can take a representation theoretical view of free zero rest mass fields on Minkowski space and look at them as vectors in certain "ladder representations" of $SU(2,2)$ [1],[2]. In this context it is natural to look at the $SU(1,1)$ analogue first. There one has the discrete series representations $(\pi_k\,,\,\mathcal{H}_k)\,;\, k\in\mathbb{Z}$ which are generated by a lowest weight vector of weight $|k|+1$, see [3]. In analogy to the twistorial realisation one can realise these representations on spaces of sections of the bundles $\mathcal{O}(-1+k)$ over parts of $\mathbf{P}=\mathbb{C}P^1$. Furthermore these realisations are unitary with respect to $<\,|\,>_k$, the $SU(1,1)$ analogue of the inner product of massless fields, and for $k\neq 0$ the two representations $(\pi_{\pm k}\,,\,\mathcal{H}_{\pm k})$ realised on $\mathcal{O}(-1\pm k)$ are unitarily equivalent. In the notation of [2](§10.3, equ. 26, with misprint) we realise the Hilbert spaces $\mathcal{H}_k$ as

$$(1) \qquad \left[H^0(\overline{\mathbf{P}^+},\mathcal{O}(-1+k))/H^0(\mathbf{P},\mathcal{O}(-1+k))\right]_{<\,|\,>_k},$$

where $[\ldots]_{<\,|\,>_k}$ denotes the Hilbert space completion with respect to $<\,|\,>_k$. $H^0(\mathbf{P},\mathcal{O}(-1+k))$ has dimension $max(0,k)$. ($H^1(\mathbf{PT},\mathcal{O}(-2+k))$, its $SU(2,2)$ analogue, vanishes for all k.) We have Hilbert space bases $\mathcal{B}_k$ for $\mathcal{H}_k$ in terms of elementary states (or K-finite vectors):

$$(2) \qquad \mathcal{B}_k=\begin{cases}\left\{\begin{pmatrix}C\\|\\Z\end{pmatrix}^{n+k}\Big/\begin{pmatrix}A\\|\\Z\end{pmatrix}^{1+n}\right\}_{n=0}^{\infty} & \text{for } k\geq 0,\\[2ex] \left\{\begin{pmatrix}C\\|\\Z\end{pmatrix}^{n}\Big/\begin{pmatrix}A\\|\\Z\end{pmatrix}^{1+n-k}\right\}_{n=0}^{\infty} & \text{for } k\leq 0\end{cases} =: \left\{e_k^n(\underset{|}{A},\underset{|}{C})\right\}_{n=0}^{\infty}$$

where $\underset{|}{A},\underset{|}{C}\in\mathbb{C}^{2*}$ are such that

$$(3) \qquad \begin{aligned} \overset{A}{\underset{Z}{|}} = A_0Z^0+A_1Z^1=0 \Rightarrow \overset{\bar{Z}}{\underset{Z}{|}} &= Z^0\overline{Z^0}-Z^1\overline{Z^1}<0\ , \text{ i.e. } [\overset{|}{Z}]\in\mathbf{P}^-\subset\mathbb{C}P^1\\ \text{and } \overset{C}{\underset{Z}{|}}=0 \Rightarrow \qquad & \qquad [\overset{|}{Z}]\in\mathbf{P}^+\subset\mathbb{C}P^1\,. \end{aligned}$$

It follows that

$$(4) \qquad \overset{A}{\underset{\bar{A}}{|}} = A_0\bar{A}^0+A_1\bar{A}^1 = A_0\overline{A_0}-A_1\overline{A_1}>0 \ \text{ and } \ \overset{C}{\underset{\bar{C}}{|}}<0\,.$$

If $\overset{A}{\underset{\bar{C}}{|}}=0$, the $SU(1,1)$–invariant positive definite inner product $<\,|\,>_k$ on $\mathcal{H}_k$ is given by

$$(5) \quad <e_k^n(\underset{|}{A},\underset{|}{C})\,|\,e_k^m(\underset{|}{A},\underset{|}{C})>_k=\begin{cases}(-1)^{n+k}\,\frac{(n+k)!}{n!}\,\delta_{nm}\begin{pmatrix}C\\|\\\bar{C}\end{pmatrix}^{n+k}\Big/\begin{pmatrix}A\\|\\\bar{A}\end{pmatrix}^{1+n} & \text{for } k\geq 0\\[2ex] (-1)^{n}\,\frac{n!}{(n-k)!}\,\delta_{nm}\begin{pmatrix}C\\|\\\bar{C}\end{pmatrix}^{n}\Big/\begin{pmatrix}A\\|\\\bar{A}\end{pmatrix}^{1+n-k} & \text{for } k\leq 0\end{cases}$$

and the bases $\mathcal{B}_k$ consist of orthogonal vectors. Let

(6) $$X = \overset{C}{\underset{\partial A}{|}} + \overset{A}{\underset{\partial C}{|}} \, , \; Y = i \overset{C}{\underset{\partial A}{|}} - i \overset{A}{\underset{\partial C}{|}} \, , \; Z = i \overset{C}{\underset{\partial C}{|}} - i \overset{A}{\underset{\partial A}{|}}$$

be generators of the $SU(1,1)$ action π_k on $\mathcal{H}_k$ with commutation relations for X, Y, Z corresponding to $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$, $\left(\begin{smallmatrix} 0 & i \\ -i & 0 \end{smallmatrix}\right)$, $\left(\begin{smallmatrix} i & 0 \\ 0 & -i \end{smallmatrix}\right) \in su(1,1)$. Then $\mathcal{H}_k$ is generated by repeated application of $E_+ = X - iY = 2 \overset{C}{\underset{\partial A}{|}}$ to the lowest weight vector e_k^0. One has

(7) $$Ze_k^0 = i(1+|k|)e_k^0 \, , \; Ze_k^n \sim Z(E_+)^n e_k^0 = i(1+|k|+2n)(E_+)^n e_k^0$$

because $[Z \, , \, E_+] = 2iE_+$.

Tensor poducts and projection operators

In a way completely analogous to finite dimensional representations one finds with straightforward algebra that under the action $\pi_k \otimes \pi_l$ the Hilbert space $\mathcal{H}_k \otimes \mathcal{H}_l$ decomposes into an orthogonal sum of invariant subspaces $(\mathcal{H}_k \otimes \mathcal{H}_l)_n$, $n = 0, 1, \ldots$ on which $\pi_k \otimes \pi_l$ is unitarily equaivalent to $(\pi_{\alpha_n} \, , \, H_{\alpha_n})$, $\alpha_n = 1 + |k| + |l| + 2n$. In abbreviated notation:

(8) $$\pi_k \otimes \pi_l = \bigoplus_{n=0}^{\infty} \pi_{1+|k|+|l|+2n} \, .$$

For example, for k, $l \leq 0$, a lowest weight vector χ_n^0 with weight $2 - k - l + 2n$ is given by

(9) $$\chi_n^0 = \sum_{i=0}^{n} (-1)^i \binom{n}{i} e_k^i \otimes e_l^{n-i} \, .$$

One quickly establishes that $E_- \chi_n^0 = (X + iY)\chi_n^0 = 0$ and $< E_+^m \, \chi_i^0 \, | E_+^n \, \chi_j^0 >_{k \otimes l} = 0$ for $i \neq j$ or $m \neq n$.

Having twistor diagrams in mind it is then natural to ask:

1. Are there diagrams which, given the above realisation of $(\pi_k \, , \, \mathcal{H}_k)$, have the effect of projection operators

   (10) $$P_n^{k,l} : \mathcal{H}_k \otimes \mathcal{H}_l \longrightarrow (\mathcal{H}_k \otimes \mathcal{H}_l)_n \; ?$$

   In other words, are there diagrams with in states $|\phi_k^1 >$, $|\phi_l^2 >$ and out states $< \psi_k^1 |$, $< \psi_l^2 |$ attached which integrate to

   (11) $$< \psi_k^1 \otimes \psi_l^2 \, | P_n^{k,l} | \, \phi_k^1 \otimes \phi_l^2 > \; ?$$

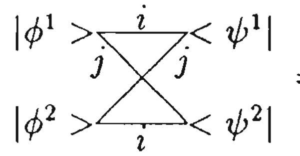
2. Can one compose such diagrams to construct projection operators

   (12) $$P_{n_1,\ldots,n_i}^{k_1,\ldots,k_{i+1}} : \mathcal{H}_{k_1} \otimes \cdots \otimes \mathcal{H}_{k_{i+1}} \longrightarrow (\cdots(\mathcal{H}_{k_1} \otimes \mathcal{H}_{k_2})_{n_1} \otimes \cdots \otimes \mathcal{H}_{k_{i+1}})_{n_i}$$

   onto spaces of irreducible subrepresentations ?

One could then think of such compositions as some kind of Young diagrams for the representations $(\pi_k\,,\,\mathcal{H}_k)$ where the operators $P_n^{k,l}$ replace (anti)symmetrisation operations for finite dimensional representations, see (18),(19).
Of course there are several other questions one might address in this context. For example, for $k,\,l \geq 1$ equation (8) implies that one has $SU(1,1)$–module homomorphisms between $\mathcal{H}_k \otimes \mathcal{H}_l$ and $\mathcal{H}_{k-1} \otimes \mathcal{H}_{l+1}$. Therefore one might want to construct diagrams for these and for more general homomorphisms. Here we just want to assert that $SU(1,1)$ analogues of the higher dimensional twistor diagrams introduced in [4] are of sufficient scope to allow realisations of $P_n^{k,l}$ for all k,l,n.
For example let $k = l = 0$ and set

(13)



$$=:< \psi^1 \otimes \psi^2 |\, \mathcal{D}_{ij}\, | \phi^1 \otimes \phi^2 >$$

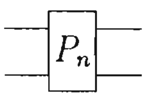
where $\mathcal{D}_{ij}$ stands for both, the interior of the diagram or, alternatively, for the operator corresponding to its integration over a contour in $(\mathbb{C}^{3+i+j})^2 \times (\mathbb{C}^{*(3+i+j)})^2$ with in and out states ϕ^1, ϕ^2 and ψ^1, $\psi^2 \in \mathcal{H}_0$. Writing P_n for $P_n^{0,0}$, one finds

(14)
$$\begin{aligned}
\mathcal{D}_{i0} &\sim \sum_{n=0}^{i} \frac{1}{(i-n)!(i+1+n)!}\, P_n \;, \\
\mathcal{D}_{i1} &\sim \sum_{n=o}^{i+1} \frac{(i+1-n(n+1))}{(i+1-n)!(i+2+n)!}\, P_n \;, \\
\mathcal{D}_{i2} &\sim \cdots
\end{aligned}$$

and, conversely, diagrams for the projections $P_0, P_1, \ldots, P_n$ can be constructed as linear combinations of $\{\mathcal{D}_{ij}\}_{i+j=m}$ for $m \geq n$. In fact for the linear spans one has

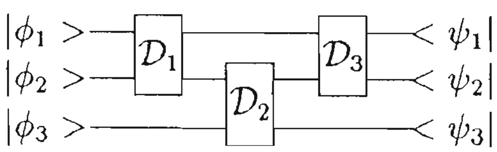
(15)
$$< \{P_k\}_{k=o}^{n} > \;=\; < \{\mathcal{D}_{ij}\}_{i+j=n} > \;.$$

We write



for a linear combination of diagrams $\mathcal{D}_{ij}$ with the effect of $P_n^{0,0}$. With regard to the second question, one can look at compositions of diagrams (13) and show that there are standard contours for which integration yields results corresponding to the composition of operators. For example, there is a contour for

(16)

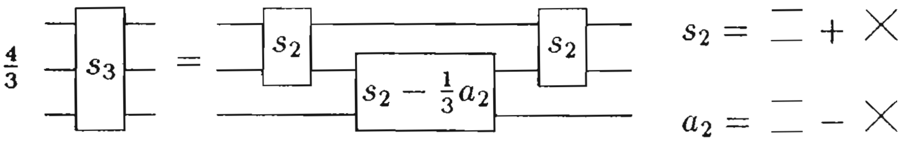


which gives

$$<\psi^1 \otimes \psi^2 \otimes \psi^3 | (\mathcal{D}_3 \otimes \mathbb{I}) \circ (\mathbb{I} \otimes \mathcal{D}_2) \circ (\mathcal{D}_1 \otimes \mathbb{I}) | \phi^1 \otimes \phi^2 \otimes \phi^3 > \tag{17}$$

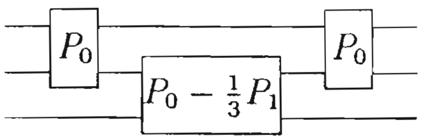
where ϕ^1, ϕ^2, ϕ^3 and ψ^1, ψ^2, $\psi^3 \in \mathcal{H}_0$ are arbitrary in and out states and $\mathcal{D}_1$, $\mathcal{D}_2$, $\mathcal{D}_3$ are linear combinations of $\mathcal{D}_{ij}$'s of a fixed dimension ($i + j = n$, fixed). Given this structure, we are left with a purely algebraic question: Can we fill in the boxes in (16) to obtain projections $P^{(0,0)}_{n_1,n_2}$?
Starting with a finite dimensional analogue one can write symmetrisation s_3 of three indices in an obvious notation as

(18)



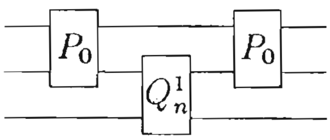
Similarly, one finds that $P_{0,0}$ can be obtained as

(19)



and, more generally, $P_{0,n}$ is obtained as $c_n \times$

(20)



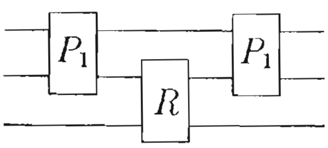
where

$$\boxed{Q_i^k = \sum_{t=0}^{k} (-1)^t \frac{(2i+t)!}{t!(k-t)!(2i+k+t+1)!} P_{i+t}}\,. \tag{21}$$

With $c_n = 2(n+1)^3$ we obtain $c_n Q_n^1 = (n+1)^2(\frac{1}{2n+1} P_n - \frac{1}{2n+3} P_{n+1})$ and one verifies

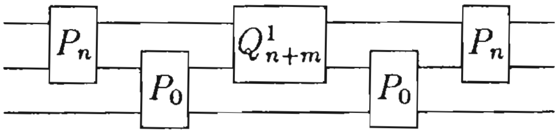
$$\sum_{n=0}^{\infty} P_{0,n} = \sum_{n=0}^{\infty} c_n (P_0 \otimes \mathbb{I}) \circ (\mathbb{I} \otimes Q_n^1) \circ (P_0 \otimes \mathbb{I}) = P_0 \otimes \mathbb{I} \tag{22}$$

because $\sum_{n=0}^{\infty} P_n = \mathbb{I} \otimes \mathbb{I}$.
How do we get $P_{i,n}$ for $i \neq 0$? It turns out, for example, that there is no finite linear combination $R = \sum_{n=0}^{N} c_n P_n$ such that



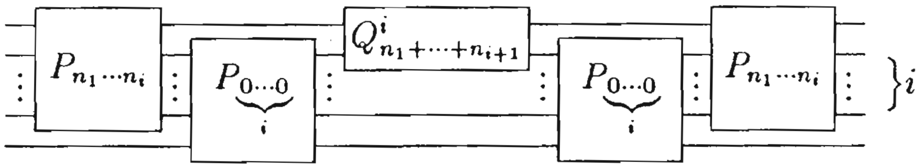
realises $P_{1,0}$. However, we can realise $P_{n,m}$ up to a factor as

(23)



the factor being $(n+m+1)^2/(2n+1)$. Finally, we can realise $P_{n_1,\dots,n_{i+1}}$ recursively as (some factor times)

(24)



This establishes the fact that combinations of (13) are sufficient to realise projections onto irreducible subrepresentations of $\otimes^n \mathcal{H}_0$. I believe this will extend (with appropriate $\mathcal{D}^{k,l}_{ij}$) to arbitrary tensor products $\otimes^n_{i=0}\mathcal{H}_{k_i}$.
Moreover, although the general combinatorics is much more involved for $SU(2,2)$ (having rank 3), for the ladder representation on

$$(25) \qquad {}^{SU(2,2)}\mathcal{H}_0 =: \tilde{\mathcal{H}}_0 = \left[H^1(\overline{PT^+}, \mathcal{O}(-2))\right]_{<\ |\ >_0}$$

we still have a decomposition $\tilde{\mathcal{H}}_0 \otimes \tilde{\mathcal{H}}_0 = \oplus^{\infty}_{n=0}(\tilde{\mathcal{H}}_0 \otimes \tilde{\mathcal{H}}_0)_n$. Substituting the corresponding projections $\tilde{P}_n$ into the above construction (24) we again get projections from $\otimes^n\tilde{\mathcal{H}}_0$ onto irreducible subspaces, although no longer all of them. (See [5] for the special case of $\tilde{P}^{0\dots 0}_{0\dots 0}$.) This is analogous to the fact that Young diagrams for $SU(2)$ are also Young diagrams for $SU(n)$, $n > 2$. The genral question therefore arises:
Can all projection operators for tensor products of the ladder representations of $SU(2,2)$ be built up as combinations of diagrams of the type (13)? Is there a Young diagram like algorithm ?
Much more work can be done ... [6].

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