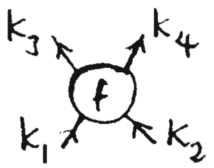
Some new formulas in conformally invariant scattering

The following results were prompted by Franz Muller's work. They fill in some gaps which have been left in the more elementary theory of scatterings which are conformally invariant and free from divergence problems. The common idea, which FM has systematised, is that of thinking of an scattering of zero-mass-fields as an operation on an appropriate space of fields.

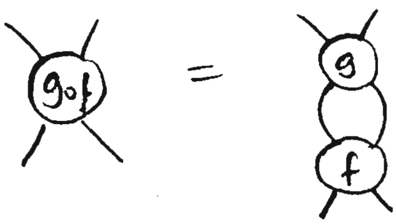
The first question is what happens if we follow one such operation on two fields with another such operation.

Specifically, I restrict to scalar fields and conformally invariant scatterings. I shall use the symbol  to denote the scattering specified on momentum states by



$$= f\left(\frac{k_1.k_3}{k_1.k_2}\right)\delta(k_1+k_2-k_3-k_4)$$

Here f might be any function on [0,1], not necessarily the restriction of an analytic function (and allowing delta-functions, etc.)
Then we have a convolution $g \circ f$ defined by



Clearly it must be possible to give $g \circ f$ directly in terms of f and g. Explicit calculation in momentum space shows the relation can be written as:

$$g \circ f(w) = \int_0^1\int_0^1 du\,dv\; K(u,v,w)\, f(u)\, g(v)$$

where $K(u,v,w) = \begin{cases} \frac{1}{2\pi}(2uv+2vw+2wu-u^2-v^2-w^2-4uvw)^{-\frac{1}{2}} \\ 0 \text{ if this argument } < 0. \end{cases}$

By considering this formula in terms of projections on to the eigenspaces of spin (as used by FM extensively), it can be shown equivalent to the identity

$$\sum_n (2n+1)P_n(x)P_n(y)P_n(z) = \begin{cases} \frac{1}{2\pi}(1-x^2-y^2-z^2+2xyz)^{-\frac{1}{2}} \\ 0 \text{ if this argument } < 0 \end{cases}$$

which must have been well-known long ago (though it wasn't to me.)

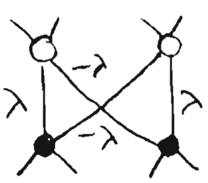
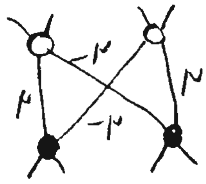
We can also write this as:

$$\sum_n (2n+1)P_n(\cos\theta)P_n(\cos\phi)P_n(\cos\psi) = \begin{cases} \frac{1}{2\pi}\begin{vmatrix}1&\cos\theta&\cos\phi\\ \cos\theta&1&\cos\psi\\ \cos\phi&\cos\psi&1\end{vmatrix}^{-\frac{1}{2}} = \frac{1}{2\pi}[\underline{n}_1,\underline{n}_2,\underline{n}_3]^{-1} \\ 0 \end{cases}$$

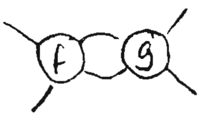
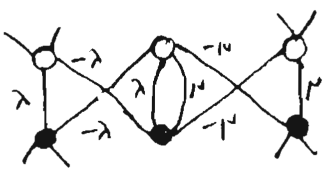
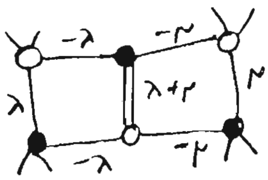
where $\underline{n}_1, \underline{n}_2, \underline{n}_3$ are the vectors joining the centre of a unit sphere to the vertices of a spherical triangle with sides θ, ϕ, ψ.

(the argument of the square root is negative $\Leftrightarrow$ no such spherical triangle exists)

One may use this result (or rather its space-like analogue) to show that in the case where the f and g scatterings are represented by the twistor integrals

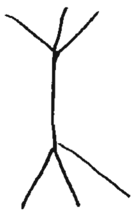
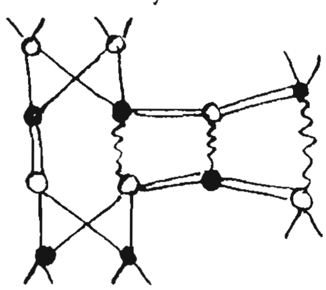
 ,  respectively.

then we find

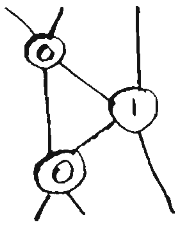
 =  = $\dfrac{\Gamma(\lambda+1)\Gamma(\mu+1)}{\Gamma(\lambda+\mu+2)}$ 

i.e. the convolution f∘g does correspond to "joining the boxes together" in the obvious way. This improves our theory of the "double box" diagram.

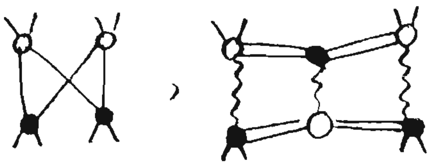
A second topic arose from the Feynman to twistor diagram correspondence

 =  in ϕ^4 scattering.

One can think of this twistor diagram as an operation which projects out the spin-0 part of the product of three z.r.m. scalar fields. FM, using his techniques, was able to interpret its asymmetric form in algebraic terms. This prompted the question of generalisation to n such fields. To do this first note that this twistor diagram can be thought of as composed of three more elementary operations, thus:



where  corresponds to the scattering operation on two fields defined by taking $f(u) = 1$, and  to that with $f(u) = u$, represented by twistor diagrams

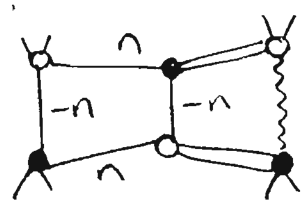


respectively

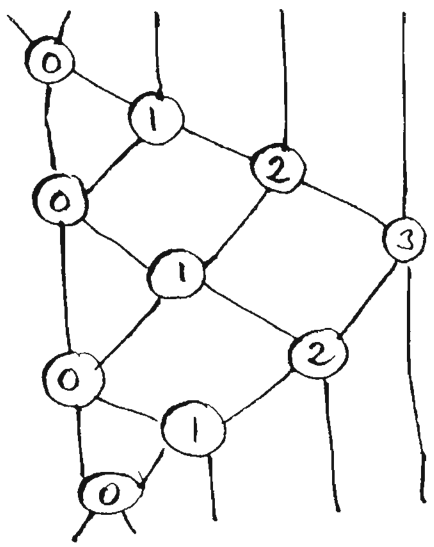
More generally now define

to be the scattering operation defined by $f(u) = u^n$,

which can be represented by twistor diagram



Then it turns out that the projection of the spin-0 part of the product of n z.r.m. scalar fields can be performed (up to a combinatorial factor) by a composite of operations drawn here explicitly in the case n=**5**, and by obvious analogy for general n:



It's remarkable that the complete operation is actually symmetric in the n (ingoing) fields. This formula opens new approaches to finding twistor diagram analogues of certain higher order Feynman tree diagrams.

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