The Bach equations as an exact set of spinor fields

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Recently, L. J. Mason ([1], [2]) has proposed a reformulation of the "light-cone program" originally due to E. T. Newman and C. N. Kozameh (see [3] for a recent review): in an asymptotically flat vacuum spacetime the light-cone cuts of $\mathcal{J}^-$ are taken as the fundamental quantities and one tries to impose the vacuum Einstein equations as one scalar equation for the "cut function" which describes the cuts. In Mason's formulation a prominent role is played by the so called Bach equations. This is for the following reason: it has been shown in [4] that a necessary and sufficient condition for a spacetime to be conformal to an Einstein space is the validity of the following two equations on the Weyl and Ricci curvatures C_{abcd} and R_{ab}:

$$\partial^d C_{abcd} + \omega^d C_{abcd} = 0 \qquad \text{for some } \omega^d \tag{1}$$

$$B_{bc} := \partial^a \partial^d C_{abcd} - \tfrac{1}{2} R^{ad} C_{abcd} = 0 \tag{2}$$

The tensor B_{ab} defined in (2) is called the Bach tensor and has the following properties:

$$B_{ab} = B_{ba}, \qquad \partial^a B_{ab} = 0, \qquad B_a{}^a = 0.$$

In addition, it is conformally invariant. Any spacetime that is conformal to a vacuum spacetime has to satisfy (1) and (2). Mason imposes $B_{ab} = 0$ and studies the implications of this equation on the cut function.

Here, we want to look at the properties of the system of partial differential equations given by the spinorial version of (1) and (2). In particular, we will show that the system is equivalent to an exact set in the sense of Penrose [5].

Expressed in terms of the Weyl and Ricci spinors equation (2) is

$$\partial^A_{A'} \partial^B_{B'} \Psi_{ABCD} + \Phi^{AB}{}_{A'B'} \Psi_{ABCD} = 0. \tag{3}$$

Note, that the first term is automatically symmetric in $(A'B')$. In the following we will use a formalism described in [6] which is based on the isomorphism between totally symmetric spinor fields on spacetime and homogeneous functions on the spin bundle over spacetime:

$$\phi_{A\dots BA'\dots B'}(x) \to \phi(x, \pi, \bar\pi) = \phi_{A\dots BA'\dots B'}(x) \pi^A \dots \pi^B \pi^{A'} \dots \pi^{B'}.$$

π^A may be considered as a coordinate along the fibers of the spin bundle. With $\partial_A = \partial / \partial \pi^A$ we construct the four covariant derivative operators $L = \pi^A \pi^{A'} \partial_{AA'}$, $M = \pi^A \partial^{A'} \partial_{AA'}$, $M' = \partial^A \pi^{A'} \partial_{AA'}$, $N = \partial^A \partial^{A'} \partial_{AA'}$. The commutators between the derivative operators involve the curvature operators $S = \pi^A \pi^B \Box_{AB}$, $T = \pi^A \partial^B \Box_{AB}$, $U = \partial^A \partial^B \Box_{AB}$, the Euler operator $H = \pi^a \partial_A$ and the wave operator $\Box$. Finally, any algebraic operation consisting of outer multiplication and contraction of spinor fields corresponds to a "C-tree", a tree like structure built up from the bilinear products $C_{kk'}(\phi, \psi)$, where k and k' indicate the number of contracted indices, e.g., $C_{21'}(\phi, \psi) \to \phi_{ABA'(C}{}_{C'}\psi^{ABA'}{}_{D')D)}$.

In this formalism, the Bach equations (2) read

$$M'^2\Psi + 12C_{20'}(\Phi, \Psi) = 0. \tag{4}$$

Φ and Ψ are the $(2,2)$- and $(4,0)$-functions corresponding to the Weyl spinor and the Ricci spinor, respectively. In addition to (4) we also have to consider the Bianchi identities

$$\begin{aligned} M'\Psi &= 2M\Phi, \\ N\Phi &= -12L\Lambda. \end{aligned}$$

Our first task is to find an equivalent set of first order equations. We start by introducing a $(3,1)$-function λ by $M'\Psi = 2\lambda$ and obtain the system

$$\begin{aligned} M'\Psi &= 2\lambda, & M'\Phi &= \bar{\lambda}, & M'\lambda &= -6C_{20'}(\Phi, \Psi), \\ N\Psi &= 0, & N\Phi &= -12L\Lambda, & N\lambda &= 0, \\ M\Psi &= 0, & M\Phi &= \lambda, & & \end{aligned}$$

The equations on λ express equation (4) and the symmetry in the primed indices in (3). We still need an equation for $M\lambda$. Inspection of the $[M, M']$ commutator gives a relation between $M\lambda$ and $\Box\lambda$. A similar such relation can be obtained from the operator identity

$$LN - MM' = -(H'+1)T + \tfrac{1}{2}H(H'+1)\Box \tag{5}$$

acting on Ψ. However, in the present case, these two relations are exactly the same. Therefore, we need to introduce another $(4,0)$-function χ by $M\lambda = \chi$ and derive equations for χ. By homogeneity we have $M\chi = 0$ and $N\chi = 0$. Now the $[M, M']$ commutator and the above identity acting on λ give independent relations between $M'\chi$ and $\Box\lambda$, which can be used to derive an equation for $M'\chi$. So we end up with the system

$$\begin{aligned} N\Phi &= -12L\Lambda, & N\lambda &= 0, \\ M\Phi &= \lambda, & M\lambda &= \chi, \\ M'\Phi &= \bar{\lambda}, & M'\lambda &= -6C_{20'}(\Phi, \Psi), \\[6pt] N\Psi &= 0, & N\chi &= 0, \\ M\Psi &= 0, & M\chi &= 0, \\ M'\Psi &= 2\lambda, & M'\chi &= 8C_{21'}(\Phi, L\Psi) - 16C_{10'}(L\Lambda, \Psi) + 8\Lambda\lambda \\ & & &\quad + \tfrac{48}{5}C_{11'}(\Phi, \lambda) - 12C_{20'}(\lambda, \Psi). \end{aligned}$$

This system is consistent, which can be seen by applying each of the commutators $[M, M']$, $[N, M]$ and $[N, M']$ to the four functions. This results either in expressions for the wave operator acting on the functions or in identities. Obviously, any solution of the system gives rise to a solution of (4) and vice versa.

It is clear from the conformal invariance of the Bach equations that we do not get any equation for the scalar curvature Λ. We can handle this situation in two ways: either we enlarge the system by adding Λ to the variables and postulate an evolution equation like $\Box\Lambda = 0$, or we consider Λ as a given function on spacetime. Both ways lead to the result that the system constitutes an exact set. The first case leads to an exact set which is also invariant [5], while we can not expect to obtain an invariant exact set in the second case because Λ and all its derivatives will enter in the expression for the unsymmetrized derivatives of the fields in terms of the totally symmetric derivatives. Nevertheless, we will treat Λ (and all its symmetrized derivatives $L^k\Lambda$) as given.

The proof that our system is in fact an exact set consists of verifying two conditions:

(i) all powers of L acting on the functions are algebraically independent,

(ii) arbitrary products of operators L, M, M' and N acting on the functions can be expressed in terms of the powers L^k acting on the functions.

The proof of condition (ii) is a reprise of the proof that the vacuum Bianchi identity on the Weyl spinor gives rise to an exact set ([6]). It rests on the fact that commuting the derivative operators only introduces the powers $L^k\Lambda$, the unknowns and the wave operator whose action on the functions can be expressed as a C-tree containing only powers. In addition, the right hand sides of the equations are C-trees in the variables so that when we encounter any derivative operator other than L acting on a function we can replace it with a C-tree.

As for condition (i) again the same argument as in the case of the vacuum Bianchi identities holds: the field equations impose conditions on all expressions of the form $s_n O\phi$, where s_n is any string of length n in the derivative operators, O is any of the operators M, M' N and ϕ stands for any of the unknowns. There are no restrictions on the expressions of the form $s_n L\phi$. Also, the commutator relations and relation (5) above only link expressions $s_n\phi$ for which the string does not consist entirely of L's. Therefore, in all the relations generated by the commutators, identity (5) and the field equations there can never appear a power and, hence, the powers are all independent. This proves exactness of the set of fields consisting of the four functions Ψ, Φ, λ and χ.

In this formal setting the characteristic initial value problem for the Bach equations is well posed. The initial data are all the powers $L^k\Psi$, $L^k\Phi$, $L^k\lambda$ and $L^k\chi$ corresponding to the totally symmetric spinor derivatives of the spinor fields at the vertex of the initial light cone.

Let us now assume that Λ and all its derivatives vanish. Additionally, we will assume that we have existence and uniqueness of solutions to equations which give rise to an exact set. This is certainly true in the formal sense. It is still not known whether the characteristic initial value problem is well posed for any reasonable function space. Suppose we give Ψ and all its symmetrized derivatives at the vertex of the initial light cone. Then evolution with the vacuum Bianchi identity $M'\Psi = 0$ (which is an exact set) produces a vacuum spacetime which necessarily provides a solution to the Bach equations. On the other hand, evolution of the same initial data together with $L^k\Phi = 0$, $L^k\chi = 0$ and $L^k\lambda = 0$ with the Bach equations gives a spacetime which is necessarily the same vacuum spacetime because of the uniqueness of the solution. This argument shows that one obtains solutions to the vacuum equations by appropriately restricting the initial data for the Bach equations.

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Hierarchy of Conservation Laws for Self-Dual Gravity

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Abstract

An infinite hierarchy of non-local conservation laws is constructed for the self-dual vacuum equations. Further, it is shown that the construction of such conserved currents has a natural description in terms of Penrose's non-linear graviton construction of such self-dual vacuum metrics.

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