# Twistor classification of type D vacuum space-times

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In the framework of the Yang-Mills twistor approach, stationary axisymmetric space-times can be characterized in terms of holomorphic rank 2 bundles over projective twistor space (Ward 1983). If the space-time extends analytically into a neighbourhood of an axis or horizon, then the patching data of the corresponding bundle consist of a single $2 \times 2$ matrix

$$P(z) = \frac{1}{f_0} \begin{pmatrix} 1 & -\psi_0 \\ -\psi_0 & f_0^2 + \psi_0^2 \end{pmatrix}$$

holomorphic in $z$ which is entirely determined by the values of the associated *Ernst potential*

$$E(z,r) = f(z,r) + \mathrm{i}\psi(z,r)\,; \quad f_0(z) \equiv f(z,0)\,, \quad \psi_0(z) \equiv \psi(z,0)$$

on the axis/horizon $r = 0$ (Woodhouse and Mason 1988, Fletcher and Woodhouse 1990). The patching matrices for the ***Weyl solutions*** – i.e. those type D metrics for which the induced metric $J$ on the space of Killing vectors can be diagonalized ($\omega = 0$ in (1) below) – were derived in an earlier note in TN 35. This note is concerned with the ***non-diagonal*** case, for which the metric takes the full stationary axisymmetric form

$$\mathrm{d}s^2 = f(z,r)\big(\mathrm{d}t - \omega(z,r)\mathrm{d}\theta\big)^2 - \frac{r^2}{f(z,r)}\mathrm{d}\theta^2 - \Omega^2(z,r)\left(\mathrm{d}z^2 + \mathrm{d}r^2\right)\,. \tag{1}$$

Here, $f$, $\omega$, and $\Omega$ are real analytic functions on a Riemann surface $\Sigma$ with complex coordinate $w = z + ir$ which is the space of orbits of the Killing vectors, $\partial/\partial t$ (timelike) and $\partial/\partial\theta$ (spacelike). The imaginary part $\psi$ of the Ernst potential is determined (up to an additive constant) by

$$\mathrm{d}\psi = \frac{f^2}{r} * \mathrm{d}\omega\,, \tag{2}$$

$$\mathrm{d}\omega \wedge \mathrm{d}\psi + \mathrm{d}\left(r * \frac{\mathrm{d}f}{f}\right) = 0\,, \tag{3}$$

and

$$\partial_w \log\left(f\Omega^2\right) = \frac{\mathrm{i}r}{f^2}(\partial_w E)(\partial_w \bar{E})\,, \tag{4}$$

where d is the exterior derivative and $*$ the star operator on $\Sigma$. The existence of such a function $\psi$ is a consequence of the vacuum equations for the metric (1).

As before, our approach is based on the fact that all type D vacuum space-times have Killing spinors of valence 2 (Walker and Penrose 1970). Equations (3)–(5) of the earlier note remain valid, but now with

$$J = \begin{pmatrix} f\omega^2 - \frac{r^2}{f} & -f\omega \\ -f\omega & f \end{pmatrix}\,, \quad A = \begin{pmatrix} -\omega\beta - \frac{\mathrm{i}r}{f} * \beta \\ \beta \end{pmatrix}\,,$$

where $\beta = \beta_z\mathrm{d}z + \beta_r\mathrm{d}r$ is a complex[1] one-form on $\Sigma$, and translate into

$$\mathrm{d}\beta + \frac{3}{2}\beta \wedge \frac{\mathrm{d}E}{f} = 0 \tag{5}$$

[1] Unlike the Weyl case, the equations which $\beta$ has to satisfy, i.e. (5)-(7), are not real and thus $\beta_z$ and $\beta_r$ cannot be taken to be real functions.

$$d * \beta - *\beta \wedge \left(\frac{dE}{2f} - 2\frac{dr}{r}\right) = 0 \tag{6}$$

$$\partial_w \log \frac{\beta_z - i\beta_r}{\Omega^2 f} = -\frac{i}{2}\frac{\partial_w \psi}{f} \tag{7a}$$

$$\partial_{\bar{w}} \log \frac{\beta_z + i\beta_r}{\Omega^2 f} = -\frac{i}{2}\frac{\partial_{\bar{w}} \psi}{f} \,. \tag{7b}$$

Expanding all quantities near the axis (or horizon), $r = 0$, and eliminating from (2)-(7) all functions but $E_0(z) = E(z, 0)$, one finds that $E_0$ satisfies the same simple ODE that $f_0(z)$ has to satisfy in the case of the Weyl solutions:

$$3E_0^{(4)} E_0'' - 4\,(E_0''')^2 = 0 \,. \tag{8}$$

Using the freedom $z \mapsto \pm z + \text{const.}$, $\psi \mapsto \psi + \text{const.}$, and $\psi \mapsto \psi + \text{const.}\ z$ (all constants real)[2], the general solution of (8) can be reduced to

$$E_0(z) = \begin{cases} \gamma z^2 + ez + g & \text{if } E_0''' = 0 \\ \alpha(z + ib)^{-1} + c + dz & \text{if } E_0''' \neq 0 \end{cases}$$

where $\alpha, \gamma \in \mathbf{C}$, $b, c, d, e, g \in \mathbf{R}$. As a (positive) real overall factor in $E$ can be incorporated into the metric components by homothetic transformations in the space of Killing vectors, the two sets of parameters, $[\gamma, e, g]$ and $[\alpha, c, d]$, can be regarded as homogeneous coordinates of the solution spaces, and thus there are at most 4 real parameters. This is, of course, the expected number for the general type D vacuum (acceleration, rotation, mass, and NUT parameter).

One of the most prominent examples is *Kerr space-time*. Fletcher and Woodhouse (1990, eqn. (55)) find

$$f_0(z) = \frac{z^2 - m^2 + a^2}{(z+m)^2 + a^2}\,, \quad \psi_0(z) = -\frac{2am}{(z+m)^2 + a^2}$$

where $m$ and $a$ are, respectively, the mass and angular momentum parameter, and $a < m$. Replacing $z + m$ by $z$, one obtains $E_0(z) = 1 - 2m(z - ia)^{-1}$ and thus

$$\alpha = -2m\,, \quad b = -a\,, \quad c = 1\,, \quad d = 0\,.$$

Further examples are under investigation. One of the aims is, of course, to relate the parameters to physical properties of the space-times.

### References

Fletcher, J. & Woodhouse, N.M.J. (1990), in: T. N. Bailey & R. J. Baston (eds.), *Twistors in Mathematics and Physics.* London Mathematical Society Lecture Note Series 156, Cambridge University Press.

Walker, M. & Penrose, R. (1970), *Commun. Math. Phys.* **18**, 265-274.

Ward, R. S. (1983), *Gen. Rel. Grav.* **15**, 105-109.

Woodhouse, N. M. J., & Mason, L. J. (1988), *Nonlinearity* **1**, 73-114.

[2] The freedom to make these changes arises, respectively, from the definitions of $z$ and $\psi$ in terms of their differentials ($z$ is the "harmonic conjugate" of $r$ on $\Sigma$) and from the freedom to make linear transformations in the space of Killing vectors.