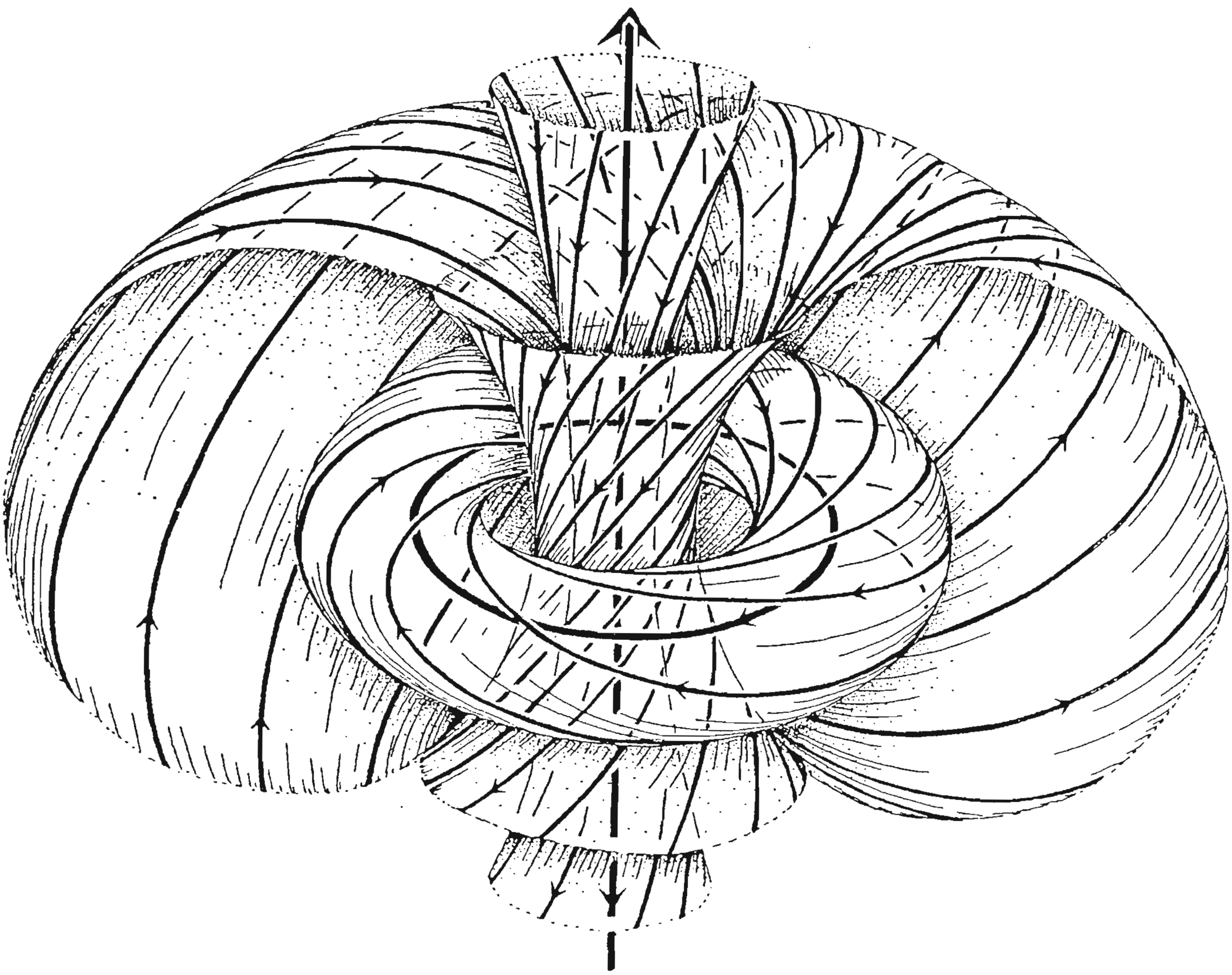
Twistor Newsletter

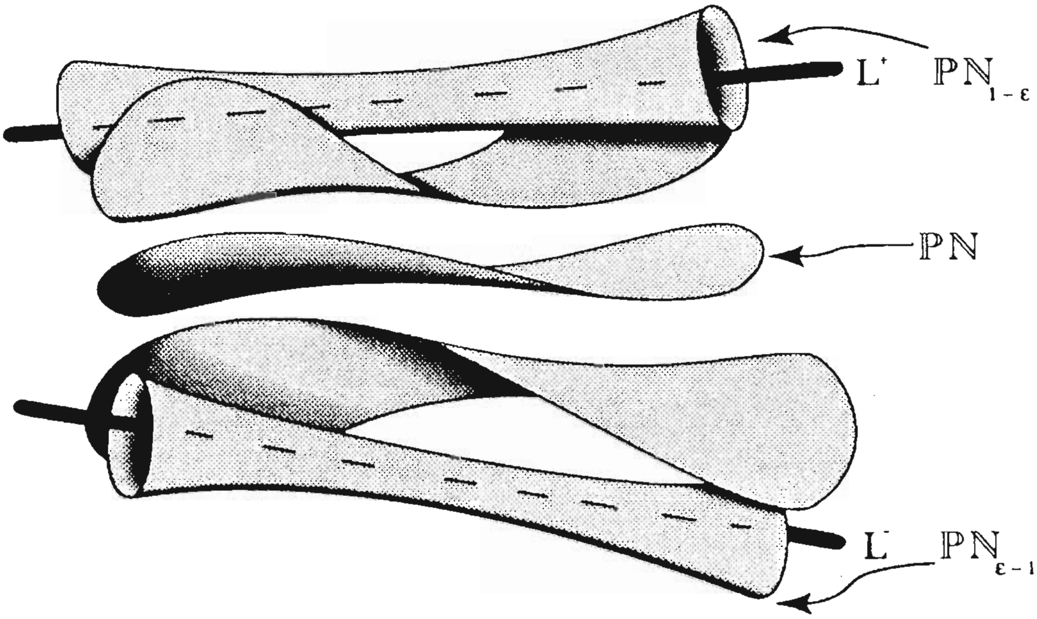
(nº 36: 9 July, 1993)



Mathematical Institute

Oxford, England

TWISTOR THEORY



There will be a conference on Twistor Theory from the 23rd to the 25th August 1993, in Devon.

The principal speakers are:

T N Bailey (Edinburgh)	H Pedersen (Odense)
M G Eastwood (Adelaide)	R Penrose (Oxford)
C LeBrun (SUNY)	K P Tod (Oxford)
S A Merkulov (Odense)	N M J Woodhouse (Oxford)

All are very welcome. There will be a registration fee of the order of £25, and full board for the conference will cost about £90.

For further details, please contact:

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The organisers are grateful to the London Mathematical Society for financial support for this conference.

This is the second announcement of the

TWISTOR THEORY CONFERENCE

23rd - 25th August 1993, in Devon.

Programme:

22nd August	Arrive in time for dinner at 8.00.
23rd August	Roger Penrose "Twistors and the Einstein equations"
	Claude LeBrun "Self-dual metrics on compact 4-manifolds"
	Henrik Pedersen "Self-duality and the connected sums of complex projective planes
	Conference Dinner.
24th August	Mike Eastwood "Twistors in Representation Theory"
	Nick Woodhouse "Twistor theory and isomonodromy"
	Toby Bailey "Conformal Invariants"
25th August	Paul Tod "Self-dual Bianchi type 9 metrics"
	Sergey Merkulov "Relative deformation theory and differential geometry"
	Conference ends after lunch.

Travel information:

The conference is being held at our Faculty of Agriculture, Food, and Land Use at Seale-Hayne, near Newton Abbot, in the South Hams of Devon.

By road, you take the M5 to Exeter and then the A38 and A382 to Newton Abbot, on the outskirts of which you turn right to Ashburton on the A383. Seale-Hayne is then three miles along this road on the right.

By rail, you take the Intercity train to Plymouth or Penzance from London or Edinburgh, alighting at Newton Abbot station. The journey from London (Paddington) takes about three hours, and there are trains arriving at Newton Abbot at 17.20 and 18.19 on Sundays. You would be well advised to book a seat. Trains from Oxford and the "Railair Coach" from Heathrow take you to Reading, where you can join the Penzance or Plymouth train.

By air, you can fly Brymon European from Heathrow to Plymouth (19.45-21.25 on Sunday) and return that way on Wednesday (14.05-15.40 or 17.30-19.10).

Accommodation:

This will be in student single rooms in the main quadrangle. There is a very small number of twin or double rooms.

Conference Dinner:

This will be at Buckland-Tout-Saints, Kingsbridge. The menu will cost £20.

Friends:

Please do not hesitate to bring family or friends to Seale-Hayne if you wish. No special events have been arranged for them, but in fine weather Devon is very beautiful, and in particular Dartmoor is only a few miles away from the conference, so bring walking gear.

Library:

A small collection of our favourite monographs and texts will be available for reference in the Faculty library.

Communications:

The Faculty's fax number is 0626 325605 and its Reception's telephone number is 0626 325800.

Also, it will be possible to use e-mail while at the conference.

Cost:

The registration fee is £25, and full board from Sunday afternoon until Wednesday afternoon is £91.50, all payable to the University of Plymouth. If you are coming to the conference dinner, please deduct £8 from this figure.

The organisers are grateful to the London Mathematical Society for financial support for this conference.

Registration:

The deadline for registration is the 9th of August. Please simply fax, telephone, e-mail, or write to me, saying when and how you will arrive, whether you are coming to the conference dinner, and which menu you would like.

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Kähler-Einstein metrics with SU(2) action.

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Abstract

The aim of this paper is to analyse Kähler-Einstein metrics of real dimension four admitting an isometric action of $SU(2)$ with generically three-dimensional orbits. In the case when the Einstein constant is zero the metrics are hyperkähler and have been classified. We shall take the Einstein constant to be nonzero.

We derive a system of ordinary differential equations whose solutions correspond precisely to such metrics, and we determine which trajectories of the equations give complete metrics. There are two families of complete metrics with negative Einstein constant. One consists of the $U(2)$-invariant metrics previously found by other authors. The other family consists of triaxial metrics.

Orthogonality of General Spin States

E. Majorana (Nuovo Cimento $\underset{\sim}{9}$ (1932) 43–50) described the general spin state, for spin $\frac{n}{2}$, (up to proportionality) in terms of an unordered set of n points on the sphere — perhaps with coincidences between them, but then multiplicities are counted. Although he did not phrase his argument in these terms, his result essentially expresses the canonical decomposition of a general symmetric spinor of valence n:

$$\underbrace{\psi_{AB\cdots N}}_{n} = \alpha_{(A}\beta_{B}\cdots\nu_{N)} \ .$$

Here $\psi_{AB\ldots N}$ describes the spin state, and its n principal null directions, represented as n points on the Riemann sphere S, provide Majorana's set (cf. Penrose & Rindler, Spinors & Space-Time, Vol. 1 (1984) p. 162; R.P. in 300 years of Gravity (eds. Hawking & Israel) C.U.P. (1987); R.P. in The Emperor's New Mind, O.U.P (1989), Fig. 6.29).

I shall be concerned, here, with the question of the geometrical interpretation of orthogonality between two states of spin $n/2$. In spinor terms this can be written

$$\psi_{AB\ldots N}\, t^{AA'} t^{BB'} \cdots t^{NN'}\, \overline{\phi}_{A'B'\ldots N'} = 0$$

where t^a is the timelike (unit) vector (in 4-space), with respect to which the states are all taken to be stationary. We can write this last relation as

$$\psi_{AB\cdots N}\, \varphi^{AB\cdots N} = 0$$

where

$$\varphi^{AB\cdots N} = t^{AA'} t^{BB'} \cdots t^{NN'}\, \overline{\phi}_{A'B'\ldots N'} \ .$$

Geometrically, the n Majorana points on S defined by $\varphi_{AB\ldots N}$ are antipodal to the n points defined by

$\phi_{AB\ldots N}$ (see Spinors & Space-Time, Vol 1). The relation between $\varphi_{AB\ldots N}$ and $\psi_{AB\ldots N}$ given by $\psi_{AB\ldots N}\varphi^{AB\ldots N}=0$ is called <u>apolarity</u> (cf. Grace & Young, <u>The Algebra of Invariants</u> C.U.P. (1903)). Apolarity represents a simpler geometrical condition for a pair of sets of n points on S than does orthogonality – not least because apolarity is <u>conformally</u> invariant on S (since there is no dependence on t^a) whereas orthogonality is only rotationally invariant. Then to pass from apolarity to orthogonality, we simply reflect one or other of the sets of points in the centre of S.

Nevertheless, apolarity is itself not an easy thing to express in purely geometrical terms, in the general case. In terms of components, the algebraic condition is

$$\psi_0\varphi_n - n\psi_1\varphi_{n-1} + \frac{n(n-1)}{2!}\psi_2\varphi_{n-2} - \cdots + (-1)^n\psi_n\varphi_0 = 0$$

where we can think of the n Majorana points of $\psi_{AB\ldots N}$ as defined on the Riemann sphere by the roots of

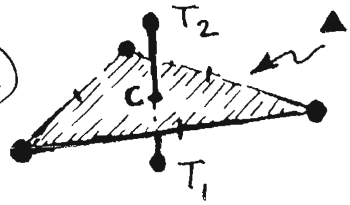
$$\psi_0 + \psi_1 z + \psi_2 z^2 + \ldots + \psi_n z^n$$

($\psi_0 = \psi_{00\ldots0}$, $\psi_1 = \psi_{10\ldots0}$, etc.) whereas orthogonality is given by

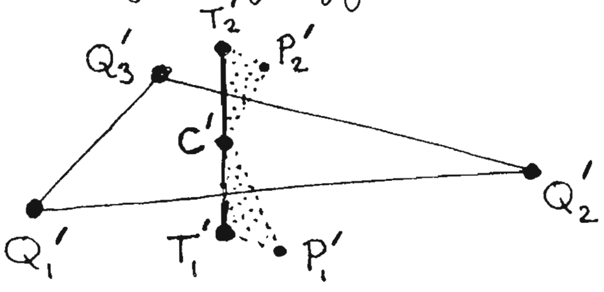
$$\psi_0\bar{\phi}_0 + n\psi_1\bar{\phi}_1 + \frac{n(n-1)}{2!}\psi_2\bar{\phi}_2 + \cdots + \psi_n\bar{\phi}_n = 0.$$

Let us denote by $P_1, \ldots, P_n$ the points on S defined by $\psi_{AB\ldots N}$, and by $Q_1, \ldots, Q_n$ those defined by $\varphi_{AB\ldots N}$. In a recent paper (R.P. "On Bell Nonlocality Without Probabilities: Some Curious Geometry" to be published in a CERN volume in honour of J.S. Bell; cf. also J. Zimba & R.P. "On Bell ... ... More Curious Geometry," to be published: HPS (Cambridge 1993),

$\boxed{2}$ Imagine a cardboard <u>equilateral</u> triangle ▲ with a matchstick through its centre C and perpendicular to its plane. The matchstick extends to points T_1, T_2 on either side of ▲ to a distance equal to its in-radius (i.e. $\frac{1}{2}$ its circumradius).



Now imagine that ▲ is held (in 3-space) so that its orthogonal projection to the plane coincides with △, scaling up or down as necessary (i.e. place ▲ so that, from a long way off, it "looks like" △). See where the points T_1, T_2, C project to; call them T_1', T_2', C'. The required apolarity condition between the P-points and Q-points is: $C'P_2'T_2'$ and $C'T_1'P_1'$ are <u>similar</u>.



<u>Proofs</u>: Exercise for the reader; <u>hint</u>: look at the accompanying article by MGE & RP (also, note that R_1, T_1', R_2, T_2' form a square with centre C').

<u>General n</u> ; <u>inductive argument</u>

Assume we already know a geometrical criterion for apolarity for two pairs of n-1 points. Can we find such a criterion for two pairs $P_1, \cdots, P_n$; $Q_1, \cdots, Q_n$ of sets of n points on S ?

(An) <u>Answer</u> (not very practical!) Fix one of the P-points, say P_n, and all the Q-points, and try to find those <u>special</u> sets of points $P_1, \cdots, P_{n-1}$ with $P_1, \cdots, P_{n-1}, P_n$ apolar to $Q_1, \cdots, Q_n$, for which $P_1 = P_2 = \cdots = P_{n-1}$. There are n-1 such places — call them $R_1, \cdots, R_{n-1}$ — these being the points on S, stereographic from which would yield points P_n' ; $Q_1', \cdots, Q_n'$ with P_n' the centroid of $Q_1', \cdots, Q_n'$ (by (2) above). (Unfortunately I don't of a direct construction of $R_1, \cdots, R_{n-1}$.)

a number of special cases of apolarity are given:

1. If all the P-points coincide then they are apolar to the Q-points iff at least one of the Q-points coincides with the n-fold P-point.

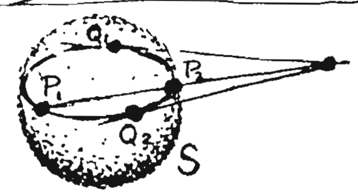
2. If all but one of the P-points coincide, then the P- and Q-sets are apolar if stereographic projection from the multiple P-point sends the remaining P-point to the centroid of the stereographic projection of the Q-points.

3. If n is odd, then any set of P points on S is apolar to itself.

Let us now consider the general case.

$\underline{\underline{n=1}}$ Apolarity holds iff $P_1 = Q_1$

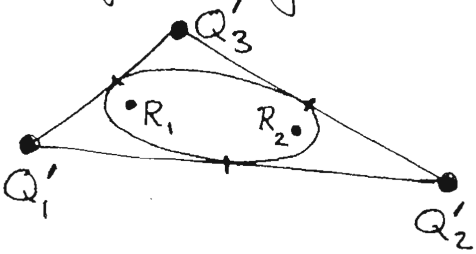
$\underline{\underline{n=2}}$ Apolarity holds iff P_1, P_2 separate Q_1, Q_2 harmonically — i.e. P_1, P_2, Q_1, Q_2 all lie on a circle on S and the line P_1P_2 meets the intersection of the tangents to this circle at Q_1 and at Q_2 (or limiting cases)



$\underline{\underline{n=3}}$ Stereographically project from P_3 (assumed simple — otherwise 2 above); we get points P_1', P_2', Q_1', Q_2', Q_3' as the respective projections of P_1, P_2, Q_1, Q_2, Q_3.

Two ways of seeing the relation of P_1', P_2' to the triangle $Q_1'Q_2'Q_3'$ $(= \Delta)$ that asserts apolarity are as follows.

$\boxed{1}$ R_1, R_2 are the foci of the ellipse touching the sides of Δ at their mid-points. Then the P-points are apolar to the Q-points iff P_1', P_2' separate R_1, R_2 harmonically (in the above sense ($n=2$) — i.e. they separate harmonically on a circle through them).



Apolarity between $P_1,\ldots,P_n$ and $Q_1,\ldots,Q_n$ is now simply the condition that $P_1,\ldots,P_{n-1}$ and $R_1,\ldots,R_{n-1}$ be apolar.

<u>Proof</u> Consider $\overset{n}{P}{}^{(A}X^BX^C\cdots X^{N)}\overset{1}{Q}_{(A}\overset{2}{Q}_B\cdots\overset{n}{Q}_{N)}=0.$ suffixes have now become numbers above

Solutions for X^A give $\overset{1}{R}{}^A,\ldots,\overset{n-1}{R}{}^A$. Hence

$$\overset{n}{P}{}^A\,\overset{1}{Q}_{(A}\overset{2}{Q}_B\cdots\overset{n}{Q}_{N)}=\overset{1}{R}_{(B}\cdots\overset{n-1}{R}_{N)}$$ taking multiplying factor $=1$

Apolarity between $P_1,\ldots,P_{n-1}$ and $R_1,\ldots,R_{n-1}$ is

$$\overset{1}{P}{}^{(B}\cdots\overset{n-1}{P}{}^{N)}\,\overset{1}{R}_{(B}\cdots\overset{n-1}{R}_{N)}=0$$

which is the same as the required condition

$$\overset{n}{P}{}^{(A}\overset{1}{P}{}^{B}\cdots\overset{n-1}{P}{}^{N)}\,\overset{1}{Q}_{(A}\overset{2}{Q}_B\cdots\overset{n}{Q}_{N)}=0.$$

Q.E.D.

<u>A Simple Observation Concerning {22} Vacuums</u>

It is well known that every {22} vacuum ("type D"), with Weyl spinor

$$\Psi_{ABCD}=Mr^{-3}\alpha_{(A}\alpha_B\beta_C\beta_{D)}$$

(where M is a constant and $\alpha_A\beta^A=1$) possesses a <u>Killing spinor</u> — a valence-2 symmetric <u>twistor</u>! —

$$\chi_{AB}=r\,\alpha_{(A}\beta_{B)}\qquad\text{satisfying}\qquad\nabla_{A'(A}\chi_{BC)}=0.$$

(See Walker & Penrose, Comm. Math. Phys <u>18</u> (1970) 265–74; also Spinors & Space-Time Vol. 2, p.107.)
In the Schwarzschild solution, r is the standard "radial coordinate", but in general r is a <u>complex</u> "radial" quantity.

I am not aware that anyone has pointed out the following simple but striking consequence:

<u>Proposition</u> Along every null geodesic, with parallelly propagated tangent spinor o^A, the <u>null datum</u>

$$\Psi_0=\Psi_{ABCD}\,o^Ao^Bo^Co^D=C_{abcd}\,l^am^bl^cm^d$$

has the <u>precise</u> form

$$\Psi_0=\frac{K}{r^5}$$

where K is a complex constant depending on the choice of null geodesic (with its choice of scale for o^A).

<u>Proof</u> This is an immediate consequence of the constancy of $\chi_{AB}o^Ao^B$ along the null geodesic ($o^A\bar{o}^{A'}\nabla_{AA'}(\chi_{BC}o^Bo^C)=0$) and the fact that $\Psi_0=Mr^{-3}(\alpha_Ao^A)^2(\beta_Bo^B)^2$ and $\chi_{AB}o^Ao^B=r(\alpha_Ao^A)(\beta_Bo^B)$, whence $\Psi_0=M/(\chi_{AB}o^Ao^B)^2r^5$. Q.E.D.

The Orthographic Image of a Regular Tetrahedron

Michael Eastwood and Roger Penrose

Theorem Four points $\alpha, \beta, \gamma, \delta \in \mathbf{R}^2$ are the images of the vertices of a regular tetrahedron in $\mathbf{R}^3$ under orthogonal projection $\mathbf{R}^3 \to \mathbf{R}^2$ if and only if

$$(\alpha + \beta + \gamma + \delta)^2 = 4(\alpha^2 + \beta^2 + \gamma^2 + \delta^2) \tag{1}$$

where $\alpha, \beta, \gamma, \delta$ are regarded as complex numbers.

Proof. It is easy to check that (1) is translation invariant, so we may as well suppose that $\delta = 0$ and that the prospective regular tetrahedron has its corresponding vertex at the origin in $\mathbf{R}^3$. One possibility for the other three vertices is

$$\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}.$$

Taking the projection to be onto the last two coördinates we get, in this case,

$$\alpha = 1, \qquad \beta = i, \qquad \gamma = 1 + i$$

and it is easy to check that

$$(\alpha + \beta + \gamma)^2 = 4(\alpha^2 + \beta^2 + \gamma^2) \tag{2}$$

as required. The conformal orthogonal group may be double covered by the group of invertible 2×2 complex matrices of the form

$$\Lambda = \begin{pmatrix} a & -b \\ \bar{b} & \bar{a} \end{pmatrix}$$

acting by $X \mapsto \Lambda X \overline{\Lambda}^t$ on the space of Hermitian 2×2 matrices with zero trace (cf. $\mathrm{SU}(2) \cong \mathrm{Spin}(3)$). Therefore, the general regular tetrahedron may be obtained by allowing such a Λ to act on the matrices

$$\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \quad \begin{pmatrix} 1 & i \\ -i & -1 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1+i \\ 1-i & 0 \end{pmatrix}$$

and then picking out the top right hand entries. We obtain

$$\Lambda\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}\overline{\Lambda}^t = \begin{pmatrix} a\bar{a}-a\bar{b}-b\bar{a}-b\bar{b} & 2ab+a^2-b^2\\ 2\bar{a}\bar{b}+\bar{a}^2-\bar{b}^2 & b\bar{b}+a\bar{b}+b\bar{a}-a\bar{a}\end{pmatrix}$$

$$\Lambda\begin{pmatrix}1 & i\\ -i & -1\end{pmatrix}\overline{\Lambda}^t = \begin{pmatrix} a\bar{a}-ia\bar{b}+ib\bar{a}-b\bar{b} & 2ab+ia^2+ib^2\\ 2\bar{a}\bar{b}-i\bar{a}^2-i\bar{b}^2 & b\bar{b}+ia\bar{b}-ib\bar{a}-a\bar{a}\end{pmatrix}$$

$$\Lambda\begin{pmatrix}0 & 1+i\\ 1-i & 0\end{pmatrix}\overline{\Lambda}^t = \begin{pmatrix} (i-1)b\bar{a}-(i+1)a\bar{b} & (i+1)a^2+(i-1)b^2\\ (1-i)\bar{a}^2-(1+i)\bar{b}^2 & (1+i)a\bar{b}+(1-i)b\bar{a}\end{pmatrix}$$

so

$$\alpha = 2ab+a^2-b^2, \quad \beta = 2ab+ia^2+ib^2, \quad \gamma = (i+1)a^2+(i-1)b^2$$

and, again, it is easy to check that (2) is satisfied. Conversely (2) is exactly the condition that α, β, γ may be written in this way for some $a, b \in \mathbb{C}$ (since every non-singular conic in $\mathbb{CP}_2$ is rational). □

There are several variations on this theme. Each simplex in $\mathbb{R}^3$ gives rise to a quadratic equation which characterises the orthographic images of similar simplices. For example, if a vertex of a cube is mapped to the origin and its neighbours are sent to $\alpha, \beta, \gamma \in \mathbb{C}$, then $\alpha^2+\beta^2+\gamma^2 = 0$ (this is due to Hadwiger [H], though not stated in this form using complex numbers). The corresponding equation for a regular dodecahedron is

$$(\alpha+\beta+\gamma)^2 + (\sqrt{5}-1)(\alpha^2+\beta^2+\gamma^2) = 0.$$

There is a general theory for orthogonal projection $\mathbb{R}^n \to \mathbb{R}^m$ with an interpretation in terms of complex numbers when $m = 2$. Details will appear elsewhere.

Thanks are due to H.S.M. Coxeter for drawing our attention to Hadwiger's article and to Jane Pitman for useful conversations.

References

[H] H. Hadwiger, Über ausgezeichnete Vectorsterne und reguläre Polytope, Comm. Math. Helv. **13** (1940), 90–108.

Indefinite, conformally-ASD metrics on S^2xS^2

There is a natural, conformally-flat, indefinite metric on S^2xS^2, namely

$$ds^2 = \frac{4d\zeta d\bar{\zeta}}{(1+\zeta\bar{\zeta})^2} - \frac{4d\eta d\bar{\eta}}{(1+\eta\bar{\eta})^2} \quad . \qquad 1$$

From the way that this is written, it obviously has an integrable complex structure J. Lowering J with the (indefinite) metric gives a 2-form which therefore must be closed. The scalar-curvature of this metric is zero, being the difference between the scalar curvatures of the summands, so that (1) defines an indefinite, scalar-flat Kähler metric. The Weyl curvature is therefore ASD (in fact this metric is conformally-flat) so it has a twistor-space and we may look for α-planes in the complexification. Free $\bar{\zeta},\bar{\eta}$ from ζ,η respectively and write them as $\tilde{\zeta},\tilde{\eta}$, then the equation

$$\zeta = \frac{a\eta+b}{c\eta+d} \; ; \qquad \tilde{\zeta} = \frac{d\tilde{\eta}-c}{-b\tilde{\eta}+a} \qquad 2$$

for $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in$ SL(2,C)/(I,-I) defines an α-plane. As a particular case of (2)

the metric (1) has real α-planes given by (2) for $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in$ SU(2)/(I,-I)

which is just RP3. Finally, there are some more α-planes missed by (2), namely ones with ζ=const., η=const., and ones with $\tilde{\zeta}$=const.,$\tilde{\eta}$=const.. None of these are real. The way these α-planes fit together to form the twistor-space is described by Lionel Mason elsewhere in this issue.

At a recent seminar in Oxford, (26/1/93; see the accompanying article), Lionel gave a fairly explicit picture of deformations of this twistor space which must lead to deformations of the metric (1) i.e. to real, conformally-ASD metrics with the ultra-hyperbolic signature (++--) globally-defined on S^2xS^2. It is possible to modify the ansatz given by Claude LeBrun in J.Diff.Geom. **34** (1991) 223 to find some of these metrics explicitly, and this is what I want to describe here.

LeBrun's ansatz gives the general, scalar-flat, Kähler metric with S^1-action. Start with the metric in LeBrun's notation and switch some signs to get the indefinite scalar-flat Kähler metric as

$$ds^2 = w(e^u(dx^2+dy^2)-dz^2)-w^{-1}(d\varphi+\omega)^2. \qquad 3$$

With the change of signs, the field equations turn into

$$u_{xx} + u_{yy} - (e^u)_{zz} = 0 \qquad 4$$

and

$$w_{xx} + w_{yy} - (we^u)_{zz} = 0. \qquad 5$$

There is also the equation for ω, which I will postpone giving.

Now take the solution of (4) given by

$$e^{u} = \frac{4(1-z^2)}{(1+x^2+y^2)^2} \qquad 6$$

and introduce V by $V = w(1-z^2)$. (The quickest route to the solution (6) is to demand that u be separable in the sense that $u = f(x,y) + g(z)$.)

In terms of θ, where $z=\cos\theta$, and $\zeta=x+iy$, the metric (3) becomes

$$ds^2 = \frac{4V\, d\zeta d\bar{\zeta}}{(1+\zeta\bar{\zeta})^2} - V d\theta^2 - \frac{\sin^2\theta}{V}(d\varphi + \omega)^2 \qquad 7$$

while (5) turns into

$$(1+\zeta\bar{\zeta})^2 \frac{\partial^2 V}{\partial\zeta\partial\bar{\zeta}} = (1-z^2)\frac{\partial^2 V}{\partial z^2}. \qquad 8$$

To write the equation for ω, deferred from above, we first introduce angles ψ,χ by $\zeta = \tan(\psi/2)\exp(i\chi)$. Then

$$d\omega = \left(\csc\psi \frac{\partial V}{\partial\chi} d\psi - \sin\psi \frac{\partial V}{\partial\psi} d\chi\right) \wedge \csc\theta\, d\theta + \sin\psi \csc\theta \frac{\partial V}{\partial\theta} d\psi \wedge d\chi \qquad 9$$

The (obvious) solution of (8,9) given by V=1 when plugged in to (3) gives the metric (1) on $S^2 \times S^2$ with which we began. For a nice way to write some more, introduce Q by

$$Q = \frac{\partial V}{\partial z} \qquad 10$$

and differentiate (8) with respect to z to get an equation for Q. The equation which results is the 'ultra-hyperbolic wave equation':

$$\nabla_1{}^2 Q = \nabla_2{}^2 Q \qquad 11$$

where the first Laplacian is on the ζ-sphere, the second is on the (θ,φ)-sphere and Q is independent of φ i.e. we get a metric for every axisymmetric (in this sense) solution of (11).

To solve (11), use Legendre polynomials P_ℓ and associated Legendre polynomials $Y_{\ell m}$:

$$Q = \sum a_{\ell m} Y_{\ell m}(\zeta,\bar{\zeta}) P_\ell(z) \;;$$

substitute into (10) and integrate to get V:

$$V = 1 + \sum a_{\ell m} Y_{\ell m}(\zeta,\bar{\zeta}) \int_{-1}^{z} P_\ell(s)\,ds\,. \qquad 12$$

With these choices, it turns out that V now equals 1 at $z = \pm 1$, which you need in order to avoid conical singularities in the metric (7). There doesn't seem to be a nice formula for ω.

Thanks to Lionel Mason and Claude Lebrun.

Paul Tod

Global twistor correspondences in split signature

L.J. Mason

Introduction

The purpose of this note is to describe the globalization of twistor correspondences in signature (2,2). In particular I discuss the non-linear graviton construction for anti-self-dual metrics of signature (2,2) on $S^2 \times S^2$ which specializes to give the twistor description of the examples of conformally anti-self-dual metrics described by K.P.Tod in his accompanying article in this issue. Part of the motivation for this construction arose from a desire to understand the globalization of other aspects of the Penrose transform in signature $(2,2)$ and thereby give a twistor description of the Radon transform and the inverse scattering transform. The connection of these ideas with the inverse scattering transform will be discussed elsewhere.

When one considers boundary conditions for solutions of conformally invariant equations on $\mathbb{R}^4$ in signature $(2,2)$, the first condition one might try would be to require that solutions should extend to the conformal compactification of $\mathbb{R}^4$, $S^2 \times S^2/\mathbb{Z}_2$ as the equations are conformally invariant. We will see, however, that these boundary conditions eliminate all solutions in linear theory. This problem is resolved if we go to the double cover, $S^2 \times S^2$ (where one can ask the question as to which fields are invariant under the $\mathbb{Z}_2$ action).

The global geometry

The conformal compactification $\mathbb{M}$ of affine signature $(2,2)$ Minkowski space, obtained by adjoining a 'light cone at infinity', is the projective quadric in $\mathbb{RP}^5$ given by the zero set of a quadratic form Q in $\mathbb{R}^6$ of signature $(3,3)$. The conformal structure is determined by asserting that the light cones of $\mathbb{M}$ are the intersections of $\mathbb{M}$ with the tangent planes of points of $\mathbb{M}$.

To see that the topology of $\mathbb{M}$ is $S^2 \times S^2/\mathbb{Z}_2$ diagonalize Q using a pair of Euclidean 3-vectors $\underline{w}$ and $\underline{y}$ as coordinates on $\mathbb{R}^6$ such that $Q = \underline{w} \cdot \underline{w} - \underline{y} \cdot \underline{y}$. Set the scale by requiring $\underline{w} \cdot \underline{w} = 1$ so that $\underline{y} \cdot \underline{y} = 1$ also on $Q = 0$; this yields $S^2 \times S^2$ in $\mathbb{R}^6$. However, in $\mathbb{RP}^5$, $(\underline{w}, \underline{y}) \sim (-\underline{w}, -\underline{y})$ so the topolgy of $Q = 0$ in $\mathbb{RP}^5$ is $S^2 \times S^2/\mathbb{Z}_2$.

The conformal structure is simply realized on the double cover $\widetilde{\mathbb{M}} = S^2 \times S^2$ by taking the pullback of the round sphere metric $d\Omega^2$ on each factor and taking the difference

$$ds^2 = p_1^* d\Omega^2 - p_2^* d\Omega^2$$

where p_1, p_2 are the projections onto the first and second factors respectively. This is $\mathbb{Z}_2$ invariant and so descends to $\mathbb{M}$.

The global correspondence

We will be interested in the cases where the original region on which the fields are defined are small complexifications of $\mathbb{M}$ and its double cover $\widetilde{\mathbb{M}}$ (which will be denoted by the same symbol). We will now consider the correspondence for these cases (or deformations thereof for ASD metrics).

The twistor space of a space U will be the spaces of connected components of (α-planes) in U . For $Z \in \mathbb{PT}(U)$, we will denote the corresponding α-plane in U by $\hat{Z}$.

The correspondence for $\mathbb{M}$

Just as compactified complexified Minkowski space $\mathbb{CM}$ is the the space of complex lines in $\mathbb{CP}^3$ via the complex Klein correspondence, $\mathbb{M}$ is the space of real lines in $\mathbb{RP}^3$ via the real Klein correspondence. In the context of the complex correspondence, point of $\mathbb{M}$ are complex lines in $\mathbb{CP}^3$ that intersect $\mathbb{RP}^3$ in a real line. Alternatively, they are complex lines in $\mathbb{CP}^3$ that are mapped into themselves by the complex conjugation $Z^\alpha \to \bar{Z}^\beta$ given by standard complex conjugation, component by component.

We have $\mathbb{PT}(\mathbb{M}) = \mathbb{CP}^3$. Given $Z \in \mathbb{CP}^3$, then if $Z = \bar{Z}$, $Z \in \mathbb{RP}^3$ and any real line in $\mathbb{RP}^3$ through Z corresponds to a point in $\mathbb{M}$ on $\hat{Z}$. In this case, $\hat{Z}$ intersects $\mathbb{M}$ in an $\mathbb{RP}^2$. If $Z \neq \bar{Z}$ then the complex line through Z and $\bar{Z}$ is real and corresponds to a point of $\mathbb{M}$. In fact the complex α-plane $\hat{Z}$ intersects $\mathbb{M}$ in the unique point corresponding to this line.

Linear theory

The linear problem was completely solved in the case of the wave equation by Fritz-John using the X-ray transform. In twistor notation, the general solution of the hyperbolic wave equation on $\mathbb{R}^4$ satisfying appropriate boundary conditions can be obtained from the integral formula

$$\phi(x^{AA'}) = \oint f(x^{AA'}\pi_{A'}, \pi_{A'})\pi^{A'} d\pi_{A'}.$$

Here f is a freely specifiable smooth section of $\mathcal{O}(-2)$ on $\mathbb{RP}^3$. That ϕ is a solution of the ultrahyperbolic wave equation follows by differentiation under the integral sign.

One might naively think that the function ϕ is naturally a function on the space of lines in $\mathbb{RP}^3$, $\mathbb{M}$. However, ϕ is defined by integrating f along lines and in order to perform the integration, one needs to have an orientation of the line. This means that ϕ is actually defined on the space of *oriented* lines in $\mathbb{RP}^3$. This is $\widetilde{\mathbb{M}}$ the double cover of $\mathbb{M}$. Clearly ϕ changes sign under reversal of orientation of the line and so does not descend to $\mathbb{M}$. Indeed, we will see that there are no solutions of the conformally invariant wave equation on $\mathbb{M}$.

Remark: Actually, there is a possible confusion here owing to the Grgin phenomena—the real point is that these solutions are anti-Grgin. Solutions of the wave equation are sections of $\mathcal{O}[-1]$, the inverse conformal weight bundle. Given just the metric $ds^2 = p_1^* d\Omega^2 - p_2^* d\Omega^2$

on $S^2 \times S^2/\mathbb{Z}_2$ there are two possible choices for $\mathcal{O}[-1]$, the trivial bundle or the Mobius bundle. The correct choice as far as the twistor correspondence is concerned is the Mobius bundle as this is just the restriction of the tautological bundle from $\mathbb{RP}^5$. The solutions above are actually sections of the trivial bundle which is wrong as far as the twistor correspondence is concerned. So one can simply write them down as even *functions* on $S^2 \times S^2$ but as odd sections of $\mathcal{O}[-1]$.

The correspondence for $\widetilde{\mathbb{M}}$

We must therefore study twistor correspondences for $\widetilde{\mathbb{M}}$. We shall abuse notation and denote also by $\widetilde{\mathbb{M}}$ a small complex thickening of $\widetilde{\mathbb{M}}$. We have:

Lemma 0.1 $\mathbb{PT}(\widetilde{\mathbb{M}})$ *is the (non-Hausdorff) space obtained by gluing together two copies of* $\mathbb{CP}^3$*, denoted* $\mathbb{CP}^3_+$ *and* $\mathbb{CP}^3_-$*, together along some small thickening of* $\mathbb{RP}^3$ *using the identity map.*

Proof: Points in the complement of (the thickening of) $\mathbb{RP}^3$ in $\mathbb{CP}^3$ correspond to α-planes that intersect $\mathbb{M}$ in a topologically trivial region and these are necessarily covered by two components in the double cover $\widetilde{\mathbb{M}}$. Whereas, points in the (thickening of) $\mathbb{RP}^3$ correspond to α-planes in (the small thickening of) $\mathbb{M}$ with topology $\mathbb{RP}^2$ so that when one takes the double cover the α-plane has topology S^2.

Thus $\mathbb{PT}(\widetilde{\mathbb{M}})$ double covers $\mathbb{CP}^3$ over the complement of the thickening of $\mathbb{RP}^3$ and the double covering is glued together over the thickening of $\mathbb{RP}^3$.

We reconstruct $\widetilde{\mathbb{M}}$ as the space of complex lines in $\mathbb{PT}(\widetilde{\mathbb{M}})$ that are cut into two pieces by $\mathbb{RP}^3$ with one piece lying in $\mathbb{CP}^3_+$'s and the other in $\mathbb{CP}^3_-$. This yields the space of oriented lines in $\mathbb{RP}^3$; the line is given by the intersection of complex line with $\mathbb{RP}^3$ and the orientation is determined by multiplying the arrow from the intersection with $\mathbb{CP}^3_-$ to that with $\mathbb{CP}^3_+$ by i and thereby rotating it by 90 degrees.

The non-Hausdorffness arises because as one varies an α-plane within $\widetilde{\mathbb{M}}$, it can break into two disconnected parts. The points on the boundary of the glued down region are the ones with the non-Hausdorffly separated partner. This space is actually a deformation retract of that considered in the discussion of sourced fields—see for example the last chapter of Further Advances in Twistor Theory Vol.1.

Complex conjugation

Complex conjugation on the small thickening of $\widetilde{\mathbb{M}}$ sends α-planes to α-planes and hence leads to a conjugation on $\mathbb{PT}(\widetilde{\mathbb{M}})$. This covers the standard complex conjugation on $\mathbb{CP}^3$ that fixes $\mathbb{RP}^3$ and is lifted to $\mathbb{PT}(\widetilde{\mathbb{M}})$ by requiring that it interchange $\mathbb{CP}^3_+$ and $\mathbb{CP}^3_-$. Thus $[Z] \in \mathbb{CP}^3_-$ goes to $[\bar{Z}] \in \mathbb{CP}^3_+$ and the real lines of the conjugation are those described above that correspond to points of $\widetilde{\mathbb{M}}$.

The X-ray transform.

We can now understand the X-ray transform in this context. Solutions of the wave equation on $\widetilde{\mathbb{M}}$ correspond to elements of $H^1(\mathbb{PT}(\widetilde{\mathbb{M}}), \mathcal{O}(-2))$. These can be studied by means of the Meyer Vietoris sequence using the covering of $\mathbb{PT}(\widetilde{\mathbb{M}})$ by $\mathbb{CP}^3_+$ and $\mathbb{CP}^3_-$. Using the fact that $H^1(\mathbb{CP}^3, \mathcal{O}(-2)) = 0 = H^0(\mathbb{CP}^3, \mathcal{O}(-2))$ we find

$$H^1(\mathbb{PT}(\widetilde{\mathbb{M}})) = H^0(\mathbb{CP}^3_+ \cap \mathbb{CP}^3_-, \mathcal{O}(-2)) = H^0(\mathbb{RP}^3, \mathcal{O}(-2))$$

and the formula for the Penrose transform using these representatives is precisely the X-ray transform.

Deformations of $\mathbb{PT}(\widetilde{\mathbb{M}})$

The nonlinear graviton construction implies that (small) ASD deformations of the conformal structure on $S^2 \times S^2$ correspond to (small) deformations of $\mathbb{PT}(\widetilde{\mathbb{M}})$. Since $\mathbb{CP}^3$ is rigid, the only deformable part is the gluing along $\mathbb{RP}^3$. In order to guarantee that the reality structure is preserved, the gluing map P from som open set in $\mathbb{CP}^3_+$ to one in $\mathbb{CP}^3_-$ must be compatible with the conjugation that sends $Z \in \mathbb{CP}^3_+$ to $\bar{Z} \in \mathbb{CP}^3_-$. This yields the condition that $P^{-1} = \bar{P}$ where $\bar{P}$ is the conjugate map. This can be arranged as follows.

Take a small analytic deformation ρ of the standard embedding of $\mathbb{RP}^3$ into $\mathbb{CP}^3$ so that ρ has a small analytic extension to a neighbourhood U of $\mathbb{RP}^3$ in $\mathbb{CP}^3$. Then we also have the conjugate embedding $\bar{\rho}$ of U into $\mathbb{CP}^3$ which is the complexification of the complex conjugate embedding (it is also holomorphic). The deformed gluing from $\mathbb{CP}^3_+$ to $\mathbb{CP}^3_-$ is then done with the map $P = \bar{\rho} \circ \rho^{-1}$.

The complex conjugation map of the deformed glued down twistor space can then be defined by sending the point $Z \in \mathbb{CP}^3_+$ to the point $\bar{Z} \in \mathbb{CP}^3_-$. This conjugation clearly fixes the image of $\mathbb{RP}^3$, is antiholomorphic and acts globally.

The space-time with deformed ASD conformal structure is then reconstructed by constructing the complex lines in the deformed space that are divided into two parts by the glued down region and are half in $\mathbb{CP}^3_+$ and half in $\mathbb{CP}^3_-$.

Thus we have a $1-1$ map from ASD deformations of the conformal structure on $S^2 \times S^2$ and such real gluing maps as above. These can be thought of as the space of (analytically) embedded $\mathbb{RP}^3$'s in $\mathbb{CP}^3$ or $map\{\mathbb{RP}^3 \to \mathbb{CP}^3\}/Diff\{\mathbb{RP}^3\}$ as a diffeomorphism of $\mathbb{RP}^3$ does not affect the final P. This last space can be thought of as the space of complexified diffeomorphism modulo the real ones.

Examples: The examples that Paul Tod writes down in his article in this issue are obtained from a split signature analogue of LeBrun's hyperbolic Gibbons-Hawking ansatz, LeBrun 1991. The basic idea is to take a global holomorphic vector field on $\mathbb{CP}^3$ that is real on $\mathbb{RP}^3$ and to drag the standard gluing some fixed amount along the imaginary part of the vector field. This is a global version of the construction of Jones & Tod, (1985).

Use 2×2 matrices as homogeneous coordinates on twistor space with columns (λ_A, μ_A). The real slice $\mathbb{RP}^3$ sits inside as PSU(2) with λ_A the SU(2) complex conjugate of μ_A. The vector field $V = i\lambda_A\partial/\partial\lambda_A - i\mu_A\partial/\partial\mu_A$ corresponds to right multiplication by diagonal SU(2) matrices. The quotient by the complexified vector field is the quadric Q coordinatized by $([\lambda_A],[\mu_A])$ with real slice S^2. On $\widetilde{\mathbb{M}}$ the symmetry can be represented so that it rotates just one of the S^2 factors and leaves the other invariant.

Choose a real analytic function on S^2, $f([\lambda_A],[\mu_A])$ defined for $\lambda_A = \hat{\mu}_A$ and continue it to some small thickening of the real slice. Then we identify $(\exp(-f)\lambda_A, \exp(f)\mu_A)$ in $\mathbb{CP}^3_+$ with $(\exp(f)\lambda_A, \exp(-f)\mu_A)$ in $\mathbb{CP}^3_-$ where $[\lambda_A]$ and $[\mu_A]$ are close to being conjugate. Thus, if we take out the lines $\lambda_A = 0$ and $\mu_A = 0$, the twistor space PT is a complex line bundle over the space $\tilde{Q}$ obtained by gluing one copy of the quadric Q_+ to another Q_- over some thickening of the real slice.

The space-time metric is given in standard form in Jones & Tod (1985) as

$$ds^2 = d\Sigma_3^2 + (d\phi + \omega)^2/V^2$$

where here $d\Sigma_3^2$ is the Einstein-Weyl space corresponding to the quadric which is just Lorentzian hyperbolic space and ω and V are the parts of an invariant ASD $U(1)$ connection that are respectively orthogonal and tangent to the symmetry direction and thus satisfy $d\omega = *_3 dV$ where $*_3$ is the hodge dual with respect to $d\Sigma_3^2$.

It is a straightforward, but slightly tedious exercise to show that the metrics in Tod's article can be put into this form after a conformal rescaling (the main non trivial part is to show that the 3-metric $(d\theta^2 - 4d\zeta d\bar{\zeta}/(1+|\zeta|^2)^2)/\sin^2\theta$ is the Lorentzian hyperbolic metric).

Discussion:

1) It is possible to write down metrics that are Ricci flat with conformal structures that extend over $S^2 \times S^2$ but whose null infinity cuts the space in half. This uses the same construction but with a translation symmetry generated by $\pi^A\partial/\partial\omega^A$ where the real slice is now given by real values for the components of (ω^A, π^A). The gluing identifies $\omega^A - if\pi^A$ on $\mathbb{CP}^3_+$ with $\omega^A + if\pi^A$ on $\mathbb{CP}^3_-$ where $f := f(\omega^A\pi_A, \pi_A)$ has homogeneity zero and is rapidly decreasing as $\omega^A\pi_A/(\pi_0^2 + \pi_1^2) \to \infty$.

2) The analogous (general) construction for ASD Yang-Mills fields gives a correspondence between ASDYM connections on $\widetilde{\mathbb{M}}$ and pairs consisting of a holomorphic vector bundle E on $\mathbb{CP}^3$ and a map $P : E \to \bar{E}$ on the thickening of $\mathbb{RP}^3$. This gives a paradigm for the inverse scattering transform with the bundle E corresponding to the solitonic part of the data and the map P corresponding to the 'scattering data'. When c_2 of the bundle on space-time is 0, E is trivial and the data is precisely a matric function P on $\mathbb{RP}^3$ satisfying $\bar{P} = P^{-1}$, or alternatively P is a map from $\mathbb{RP}^3$ to $G_{\mathbb{C}}/G$.

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On The Dimension of Elementary States

In [5], Michael Singer proposed a definition for a four dimensional conformal field theory. In this theory the role of compact Riemann surfaces, which occur in standard conformal field theory, is played instead by compact flat twistor spaces. It is therefore tempting to ask questions about these twistor spaces which are, in some way, natural extensions from Riemann surfaces. The properties of compact Riemann surfaces are well known and have been extensively documented and so there is an immensely rich source of possible questions that can be asked about flat twistor spaces.

One such property of compact Riemann surfaces concerns meromorphic functions having poles of prescribed maximum order at given points: Let X be a compact Riemann surface with distinct points $P_1, \ldots, P_k$ on X. If $n_1, \ldots, n_k$ are arbitrary positive integers, how many linearly independent meromorphic functions are there on X, which have poles of order at most n_i at P_i, and no others?

To answer this question one can use the following strategy:

(a) Convert the question to one involving global data.
This is achieved through the introduction of line bundles and divisors. The problem then becomes one of determining the dimension of the cohomology group $H^0(X, \vartheta[D])$ where $[D]$ is the line bundle of the divisor D, which for the above problem is $\sum_{i=1}^{k} n_i P_i$.

(b) Use the Riemann-Roch theorem.
This enables holomorphic data to be calculated in terms of topological data: specifically $\dim H^0(X, \vartheta[D]) - \dim H^1(X, \vartheta[D]) = \deg D + 1 - g$, where g is the genus of X.

(c) Use vanishing theorems for $H^1(X, \mathcal{O}[D])$ in order to eliminate the unwanted term. One such is the Kodaira theorem: if $\deg(D) > 2g - 2$ then $H^1(X\ \mathcal{O}[D]) = 0$.

To extend this problem to flat twistor-spaces we need a good analogue of meromorphic functions with prescribed singularities. Fortunately a candidate for these exist, the elementary states based on a line, but one first has to decide what is meant by a prescribed order of singularity, and then how to extend this notion to a general flat twistor-space.

The first part of this question was answered by Eastwood and Hughston in [1]. If f(z) is homogeneous of degree m in z, and if f(z) is coprime to z_0 and z_1, then $\frac{f(z)}{z_0 z_1}$ is a representative of an elementary state of homogeneity m − 2, with a pole of order 1 on the line $z_0 = z_1 = 0$. Similarly, $\frac{f(z)}{z_0 z_1^2} = \frac{z_0 f(z)}{(z_0 z_1)^2}$ has homogeneity $(m + 1) - 4 = m - 3$, and a pole of order 2 on $z_0 = z_1 = 0$.

Thus, at least in the case of $\mathbf{P}^3$, one can formulate the question: Given a fixed line L in $\mathbf{P}^3$, a given homogeneity n, and a positive integer k, how many elementary states based on L are there, with homogeneity n, and with a singularity on L of order at most k?

One way of converting this question, at least in $\mathbf{P}^3$, to one involving global data, was given in [1]. Take the line L, blow-up $\mathbf{P}^3$ along L, to obtain the complex manifold $\tilde{\mathbf{P}}^3$. In this case $\mathbf{P}^3$ is actually a submanifold of $\mathbf{P}^3 \times \mathbf{P}^1$. Now let $a \geq 0$, $b \leq -2$, be integers, and form the bundle $\mathcal{O}(a) \otimes \mathcal{O}(b)$ on $\mathbf{P}^3 \times \mathbf{P}^1$, from $\mathcal{O}(a)$ on $\mathbf{P}^3$ and $\mathcal{O}(b)$ on $\mathbf{P}^1$. Let $\mathcal{O}(a,b)$ be the restriction of this bundle to $\tilde{\mathbf{P}}^3$. Then it is shown in [1], that elements of $H^1(\tilde{\mathbf{P}}^3, \mathcal{O}(a,b))$, when restricted away from the blown-up line, are representatives for elementary states based on L, of homogeneity a + b, and order of singularity <u>at most</u> −b −1. The question now concerns the dimension of $H^1(\tilde{Z}, \mathcal{O}(a,b))$.

This gives a way of extending the definition of elementary states based on a line, to flat twistor-spaces, since such twistor-spaces have the property that each projective line (fibre) has a neighbourhood which is biholomorphic to $\mathbf{P}_{\cdot}$.

Given a flat twistor-space Z, and distinguished line L, one can then define the blow-up of Z along L, say $\tilde{Z}$, and the bundle $\vartheta(a,b)$ can be defined on $\tilde{Z}$ in such a way as to preserve the essential properties of $\vartheta(a,b)$ in $\tilde{\mathbf{P}}^3$. Elements of $H^1(\tilde{Z}, \vartheta(a,b))$ then have homogeneity a + b and a 'pole' of order at most -b -1 on L, when restricted away from L, but in a neighbourhood of L.

We then define the elements of this group to be our elementary states based on L, with the prescribed conditions, and the problem we wish to solve is to find the dimension of $H^1(\tilde{Z}, \vartheta(a,b))$. This will give a partial answer to the equivalent problem in Riemann surfaces which formed the motivation for this work.

The strategy for the solution follows closely that for Riemann-surfaces:

(a) The question has already been converted to one involving global data on a compact manifold, though this time it is $\tilde{Z}$, not Z, ie. need to calculate dim $H^1(\tilde{Z}, \vartheta(a,b))$.

(b) The Hirzebruch-Riemann-Roch theorem enables the calculation of the holomorphic Euler characteristic of $\vartheta(a,b)$ on $\tilde{Z}$, using topological data [7].
This states that the holomorphic Euler characteristic $\chi(\tilde{Z}, \vartheta(a,b))$, which is defined by $\chi(\tilde{Z}, \vartheta(a,b)) = \sum_{i=0}^{3}(-1)^i \dim H^i(\tilde{Z}, \vartheta(a,b))$, is given in terms of Chern classes of $\tilde{Z}$, ie

$$\dim H^0(\tilde{Z}, \vartheta(a,b)) - \dim H^1(\tilde{Z}, \vartheta(a,b)) + \dim H^2(\tilde{Z}, \vartheta(a,b)) - \dim H^3(\tilde{Z}, \vartheta(a,b))$$
$$= \left[\mathrm{Ch}(\vartheta(a,b))\,\mathrm{Td}(\tilde{Z})\right]\left[\tilde{Z}\right]$$

where $Ch(\vartheta(a,b))$ is the Chern character of the bundle $\vartheta(a,b)$, and $Td(\tilde{Z})$ is the Todd class of (the holomorphic bundle of) $\tilde{Z}$.

I was able to calculate this when M was a compact, oriented, Riemannian self-dual, 4-manifold, and Z was its twistor space [2]. The method involved using Poincaré duality and intersection of homology classes. I obtained the following results:

$$\chi\left(\tilde{Z},\vartheta(a,b)\right)=\frac{1}{12}(a+b+1)(a+b+2)(a+b+3)\chi$$
$$-\frac{1}{8}(a+b+2)\left[(a+b+1)(a+b+3)-1\right]\tau$$
$$-\frac{1}{6}b(b+1)(b+3a+5),$$

where χ is the Euler characteristic, and $\tau = 0$ in this case.

(c) Vanishing theorems.

There is more work to be done in this case since the alternating sum contains 4 terms. In the case where the manifold M has negative scalar curvature, it is easy to show both the H^0 and H^3 terms are zero. This leaves H^2 as the awkward term.

The Serre dual of $H^2(\tilde{Z}, \vartheta(a,b))$ can be calculated, and is $H^1(\tilde{Z}, \vartheta(-a-3,-b-1))$. With the given restrictions on a and b, this means that we have to deal with $\dim H^1\left(\tilde{Z},\vartheta(c,d)\right)$ where $c \leq -3$, $d \geq 1$.

It turns out that, using diagram chasing techniques on two Mayer Vietoris sequences, I was able to prove the following relationship between the cohomologies of the blown-up manifold $\tilde{Z}$, and the flat twistor-space Z: if $H^0(Z, \vartheta(c + d))$ and $H^1(Z, \vartheta(c + d))$ both vanish, then $\dim H^1(\tilde{Z}, \vartheta(c,d)) = \dim H^0(P^+, \vartheta(c,d))$, and this would enable the final calculation to be made [3].

The question now turns on a vanishing theorem for $H^1(Z, \mathcal{O}(m))$. For the case of M having negative scalar curvature, I was then able to prove the following theorem: If M is compact, Riemannian, self-dual, Einstein, with negative scalar curvature, then
$H^1(Z, \mathcal{O}(m)) = 0$ if $m > 0$, [4].

The method of proof used the Penrose transform to identify $H^1(Z, \mathcal{O}(n - 2))$ with certain spinor fields in M. This produced a Bochner style vanishing theorem for the spinor fields. This result was brought to my attention by C. LeBrun, from some work of his research student (M. Thornber [6]) on vanishing theorems for quaternionic-Kähler manifolds. His method of approach was completely different, involving a direct attack on the problem in Z, though his proof did not cover the case of 4-dimensional manifolds M, only twistor spaces of 4k-dimensional quaternionic-Kähler M.

This information provides an answer in the following case:
Let M be a compact, Riemannian, conformally-flat, Einstein manifold with negative scalar curvature, and let Z be its twistor-space. If L is a distinguished line in Z and $a \geq 0$, $b \leq -2$, with $a + b < -4$, then the number of elementary states with singularity on L, of order at most $-b -1$ and homogeneity $a + b$, is given by

$$\dim H^0(\mathbf{P}^+, \mathcal{O}(-(a + b) - 4)) - \chi(\tilde{Z}, \mathcal{O}(a,b)).$$

We note that the M are precisely the hyperbolic 4-manifolds.

Details will appear anon.

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Diagrams for tensor products of $\mathcal{H}_k$, the discrete series representations of $SU(1,1)$ with lowest weight

One can take a representation theoretical view of free zero rest mass fields on Minkowski space and look at them as vectors in certain "ladder representations" of $SU(2,2)$ [1],[2]. In this context it is natural to look at the $SU(1,1)$ analogue first. There one has the discrete series representations $(\pi_k\,,\,\mathcal{H}_k)\,;k\in\mathbb{Z}$ which are generated by a lowest weight vector of weight $|k|+1$, see [3]. In analogy to the twistorial realisation one can realise these representations on spaces of sections of the bundles $\mathcal{O}(-1+k)$ over parts of $\mathbf{P}=\mathbb{C}P^1$. Furthermore these realisations are unitary with respect to $<\,|\,>_k$, the $SU(1,1)$ analogue of the inner product of massless fields, and for $k\neq 0$ the two representations $(\pi_{\pm k}\,,\,\mathcal{H}_{\pm k})$ realised on $\mathcal{O}(-1\pm k)$ are unitarily equivalent. In the notation of [2](§10.3, equ. 26, with misprint) we realise the Hilbert spaces $\mathcal{H}_k$ as

$$(1)\qquad \Big[H^0(\overline{\mathbf{P}^+},\mathcal{O}(-1+k))/H^0(\mathbf{P},\mathcal{O}(-1+k))\Big]_{<\,|\,>_k}\,,$$

where $[\ldots]_{<\,|\,>_k}$ denotes the Hilbert space completion with respect to $<\,|\,>_k$. $H^0(\mathbf{P},\mathcal{O}(-1+k))$ has dimension $max(0,k)$. ($H^1(\mathbf{PT},\mathcal{O}(-2+k))$, its $SU(2,2)$ analogue, vanishes for all k.) We have Hilbert space bases $\mathcal{B}_k$ for $\mathcal{H}_k$ in terms of elementary states (or K-finite vectors):

$$(2)\qquad \mathcal{B}_k=\begin{cases}\{\begin{pmatrix}C\\|\\Z\end{pmatrix}^{n+k}\Big/\begin{pmatrix}A\\|\\Z\end{pmatrix}^{1+n}\}_{n=0}^{\infty} & \text{for } k\geq 0,\\[2mm] \{\begin{pmatrix}C\\|\\Z\end{pmatrix}^{n}\Big/\begin{pmatrix}A\\|\\Z\end{pmatrix}^{1+n-k}\}_{n=0}^{\infty} & \text{for } k\leq 0\end{cases}=:\Big\{e_k^n(\underset{|}{A},\underset{|}{C})\Big\}_{n=0}^{\infty}$$

where $\underset{|}{A},\underset{|}{C}\in\mathbb{C}^{2*}$ are such that

$$(3)\qquad \begin{aligned}\overset{A}{\underset{Z}{|}}=A_0Z^0+A_1Z^1=0\Rightarrow \overset{\bar Z}{\underset{Z}{|}}=Z^0\overline{Z^0}-Z^1\overline{Z^1}<0\ ,\ \text{i.e.}\quad &[\overset{|}{Z}]\in\mathbf{P}^-\subset\mathbb{C}P^1\\ \text{and}\ \overset{C}{\underset{Z}{|}}=0\Rightarrow\qquad\qquad\qquad\qquad &[\overset{|}{Z}]\in\mathbf{P}^+\subset\mathbb{C}P^1\,.\end{aligned}$$

It follows that

$$(4)\qquad \overset{A}{\underset{\bar A}{|}}=A_0\bar A^0+A_1\bar A^1=A_0\overline{A_0}-A_1\overline{A_1}>0\ \text{ and }\ \overset{C}{\underset{\bar C}{|}}<0\,.$$

If $\overset{A}{\underset{\bar C}{|}}=0$, the $SU(1,1)$–invariant positive definite inner product $<\,|\,>_k$ on $\mathcal{H}_k$ is given by

$$(5)\quad <e_k^n(\underset{|}{A},\underset{|}{C})\,|\,e_k^m(\underset{|}{A},\underset{|}{C})>_k=\begin{cases}(-1)^{n+k}\,\frac{(n+k)!}{n!}\,\delta_{nm}\begin{pmatrix}C\\|\\\bar C\end{pmatrix}^{n+k}\Big/\begin{pmatrix}A\\|\\\bar A\end{pmatrix}^{1+n} & \text{for } k\geq 0\\[2mm] (-1)^{n}\,\frac{n!}{(n-k)!}\,\delta_{nm}\begin{pmatrix}C\\|\\\bar C\end{pmatrix}^{n}\Big/\begin{pmatrix}A\\|\\\bar A\end{pmatrix}^{1+n-k} & \text{for } k\leq 0\end{cases}$$

and the bases $\mathcal{B}_k$ consist of orthogonal vectors. Let

$$(6)\qquad X = \overset{C}{\underset{\partial A}{|}} + \overset{A}{\underset{\partial C}{|}}\ ,\ Y = i\overset{C}{\underset{\partial A}{|}} - i\overset{A}{\underset{\partial C}{|}}\ ,\ Z = i\overset{C}{\underset{\partial C}{|}} - i\overset{A}{\underset{\partial A}{|}}$$

be generators of the $SU(1,1)$ action π_k on $\mathcal{H}_k$ with commutation relations for X, Y, Z corresponding to $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$, $\left(\begin{smallmatrix} 0 & i \\ -i & 0 \end{smallmatrix}\right)$, $\left(\begin{smallmatrix} i & 0 \\ 0 & -i \end{smallmatrix}\right) \in su(1,1)$. Then $\mathcal{H}_k$ is generated by repeated application of $E_+ = X - iY = 2\overset{C}{\underset{\partial A}{|}}$ to the lowest weight vector e_k^0. One has

$$(7)\qquad Ze_k^0 = i(1+|k|)e_k^0\ ,\ Ze_k^n \sim Z(E_+)^n e_k^0 = i(1+|k|+2n)(E_+)^n e_k^0$$

because $[Z\ ,\ E_+] = 2iE_+$.

Tensor poducts and projection operators

In a way completely analogous to finite dimensional representations one finds with straightforward algebra that under the action $\pi_k \otimes \pi_l$ the Hilbert space $\mathcal{H}_k \otimes \mathcal{H}_l$ decomposes into an orthogonal sum of invariant subspaces $(\mathcal{H}_k \otimes \mathcal{H}_l)_n\ ,\ n = 0, 1, \ldots$ on which $\pi_k \otimes \pi_l$ is unitarily equaivalent to $(\pi_{\alpha_n}\ ,\ H_{\alpha_n})$, $\alpha_n = 1 + |k| + |l| + 2n$. In abbreviated notation:

$$(8)\qquad \pi_k \otimes \pi_l = \bigoplus_{n=0}^{\infty} \pi_{1+|k|+|l|+2n}\ .$$

For example, for k, $l \leq 0$, a lowest weight vector χ_n^0 with weight $2 - k - l + 2n$ is given by

$$(9)\qquad \chi_n^0 = \sum_{i=0}^{n} (-1)^i \binom{n}{i} e_k^i \otimes e_l^{n-i}\ .$$

One quickly establishes that $E_- \chi_n^0 = (X + iY)\chi_n^0 = 0$ and $< E_+^m \chi_i^0 | E_+^n \chi_j^0 >_{k\otimes l} = 0$ for $i \neq j$ or $m \neq n$.

Having twistor diagrams in mind it is then natural to ask:

1. Are there diagrams which, given the above realisation of $(\pi_k\ ,\ \mathcal{H}_k)$, have the effect of projection operators

   $$(10)\qquad P_n^{k,l} : \mathcal{H}_k \otimes \mathcal{H}_l \longrightarrow (\mathcal{H}_k \otimes \mathcal{H}_l)_n\ ?$$

   In other words, are there diagrams with in states $|\phi_k^1>$, $|\phi_l^2>$ and out states $<\psi_k^1|$, $<\psi_l^2|$ attached which integrate to

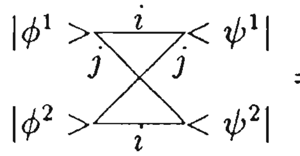
   $$(11)\qquad < \psi_k^1 \otimes \psi_l^2 | P_n^{k,l} | \phi_k^1 \otimes \phi_l^2 > ?$$

2. Can one compose such diagrams to construct projection operators

   $$(12)\ P_{n_1,\ldots,n_i}^{k_1,\ldots,k_{i+1}} : \mathcal{H}_{k_1} \otimes \cdots \otimes \mathcal{H}_{k_{i+1}} \longrightarrow (\cdots(\mathcal{H}_{k_1} \otimes \mathcal{H}_{k_2})_{n_1} \otimes \cdots \otimes \mathcal{H}_{k_{i+1}})_{n_i}$$

   onto spaces of irreducible subrepresentations ?

One could then think of such compositions as some kind of Young diagrams for the representations $(\pi_k\,,\,\mathcal{H}_k)$ where the operators $P_n^{k,l}$ replace (anti)symmetrisation operations for finite dimensional representations, see (18),(19).
Of course there are several other questions one might address in this context. For example, for $k,\ l\geq 1$ equation (8) implies that one has $SU(1,1)$–module homomorphisms between $\mathcal{H}_k\otimes\mathcal{H}_l$ and $\mathcal{H}_{k-1}\otimes\mathcal{H}_{l+1}$. Therefore one might want to construct diagrams for these and for more general homomorphisms. Here we just want to assert that $SU(1,1)$ analogues of the higher dimensional twistor diagrams introduced in [4] are of sufficient scope to allow realisations of $P_n^{k,l}$ for all k,l,n.
For example let $k=l=0$ and set

(13)  $=:<\psi^1\otimes\psi^2|\,\mathcal{D}_{ij}\,|\phi^1\otimes\phi^2>$

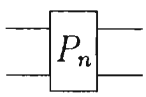
where $\mathcal{D}_{ij}$ stands for both, the interior of the diagram or, alternatively, for the operator corresponding to its integration over a contour in $(\mathbb{C}^{3+i+j})^2\times(\mathbb{C}^{*(3+i+j)})^2$ with in and out states ϕ^1, ϕ^2 and ψ^1, $\psi^2\in\mathcal{H}_0$. Writing P_n for $P_n^{0,0}$, one finds

(14)
$$\begin{aligned}
\mathcal{D}_{i0} &\sim \sum_{n=0}^{i}\frac{1}{(i-n)!(i+1+n)!}\,P_n\ ,\\
\mathcal{D}_{i1} &\sim \sum_{n=o}^{i+1}\frac{(i+1-n(n+1))}{(i+1-n)!(i+2+n)!}\,P_n\ ,\\
\mathcal{D}_{i2} &\sim \cdots
\end{aligned}$$

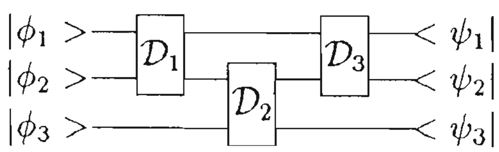
and, conversely, diagrams for the projections $P_0,\,P_1,\ldots,P_n$ can be constructed as linear combinations of $\{\mathcal{D}_{ij}\}_{i+j=m}$ for $m\geq n$. In fact for the linear spans one has

(15)
$$<\{P_k\}_{k=o}^{n}>\;=\;<\{\mathcal{D}_{ij}\}_{i+j=n}>\ .$$

We write



for a linear combination of diagrams $\mathcal{D}_{ij}$ with the effect of $P_n^{0,0}$. With regard to the second question, one can look at compositions of diagrams (13) and show that there are standard contours for which integration yields results corresponding to the composition of operators. For example, there is a contour for

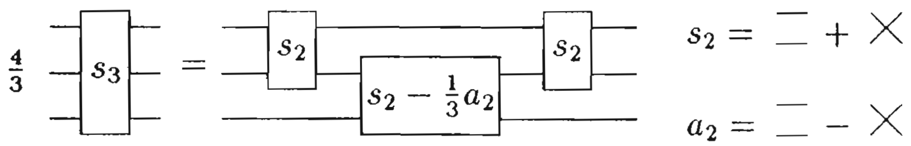
(16) 

which gives

$$(17) \qquad < \psi^1 \otimes \psi^2 \otimes \psi^3 | (\mathcal{D}_3 \otimes \mathbb{I}) \circ (\mathbb{I} \otimes \mathcal{D}_2) \circ (\mathcal{D}_1 \otimes \mathbb{I}) | \phi^1 \otimes \phi^2 \otimes \phi^3 >$$

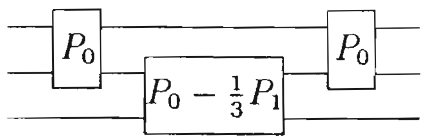
where ϕ^1, ϕ^2, ϕ^3 and ψ^1, ψ^2, $\psi^3 \in \mathcal{H}_0$ are arbitrary in and out states and $\mathcal{D}_1$, $\mathcal{D}_2$, $\mathcal{D}_3$ are linear combinations of $\mathcal{D}_{ij}$'s of a fixed dimension ($i + j = n$, fixed). Given this structure, we are left with a purely algebraic question: Can we fill in the boxes in (16) to obtain projections $P^{(0,0)}_{n_1,n_2}$?
Starting with a finite dimensional analogue one can write symmetrisation s_3 of three indices in an obvious notation as

(18)



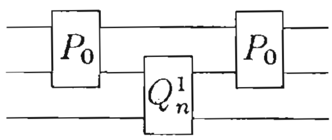
Similarly, one finds that $P_{0,0}$ can be obtained as

(19)



and, more generally, $P_{0,n}$ is obtained as $c_n \times$

(20)



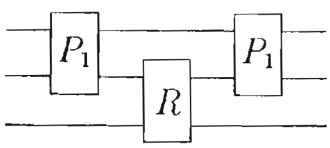
where

$$(21) \qquad \boxed{Q^k_i = \sum_{t=0}^{k} (-1)^t \frac{(2i+t)!}{t!(k-t)!(2i+k+t+1)!} P_{i+t}} \, .$$

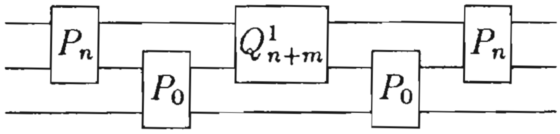
With $c_n = 2(n+1)^3$ we obtain $c_n Q^1_n = (n+1)^2 (\frac{1}{2n+1} P_n - \frac{1}{2n+3} P_{n+1})$ and one verifies

$$(22) \qquad \sum_{n=0}^{\infty} P_{0,n} = \sum_{n=0}^{\infty} c_n (P_0 \otimes \mathbb{I}) \circ (\mathbb{I} \otimes Q^1_n) \circ (P_0 \otimes \mathbb{I}) = P_0 \otimes \mathbb{I}$$

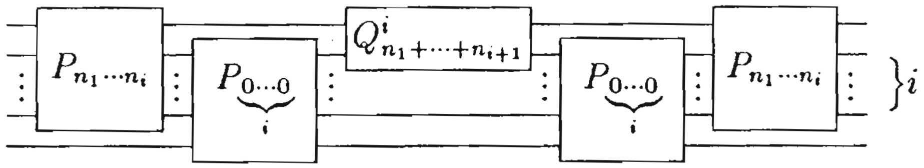
because $\sum_{n=0}^{\infty} P_n = \mathbb{I} \otimes \mathbb{I}$.
How do we get $P_{i,n}$ for $i \neq 0$? It turns out, for example, that there is no finite linear combination $R = \sum_{n=0}^{N} c_n P_n$ such that



realises $P_{1,0}$. However, we can realise $P_{n,m}$ up to a factor as

(23) 

the factor being $(n+m+1)^2/(2n+1)$. Finally, we can realise $P_{n_1,\dots,n_{i+1}}$ recursively as (some factor times)

(24) 

This establishes the fact that combinations of (13) are sufficient to realise projections onto irreducible subrepresentations of $\otimes^n \mathcal{H}_0$. I believe this will extend (with appropriate $\mathcal{D}^{k,l}_{ij}$) to arbitrary tensor products $\otimes^n_{i=0}\mathcal{H}_{k_i}$.
Moreover, although the general combinatorics is much more involved for $SU(2,2)$ (having rank 3), for the ladder representation on

(25) $$^{SU(2,2)}\mathcal{H}_0 =: \tilde{\mathcal{H}}_0 = \left[H^1(\overline{PT^+}, \mathcal{O}(-2))\right]_{<\ |\ >_0}$$

we still have a decomposition $\tilde{\mathcal{H}}_0 \otimes \tilde{\mathcal{H}}_0 = \oplus^{\infty}_{n=0}(\tilde{\mathcal{H}}_0 \otimes \tilde{\mathcal{H}}_0)_n$. Substituting the corresponding projections $\tilde{P}_n$ into the above construction (24) we again get projections from $\otimes^n\tilde{\mathcal{H}}_0$ onto irreducible subspaces, although no longer all of them. (See [5] for the special case of $\tilde{P}^{0\dots0}_{0\dots0}$.) This is analogous to the fact that Young diagrams for $SU(2)$ are also Young diagrams for $SU(n)$, $n > 2$. The genral question therefore arises:
Can all projection operators for tensor products of the ladder representations of $SU(2,2)$ be built up as combinations of diagrams of the type (13)? Is there a Young diagram like algorithm ?
Much more work can be done ... [6].

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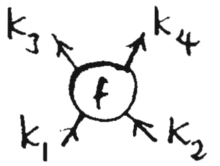
Franz Müller

Some new formulas in conformally invariant scattering

The following results were prompted by Franz Muller's work. They fill in some gaps which have been left in the more elementary theory of scatterings which are conformally invariant and free from divergence problems. The common idea, which FM has systematised, is that of thinking of an scattering of zero-mass-fields as an operation on an appropriate space of fields.

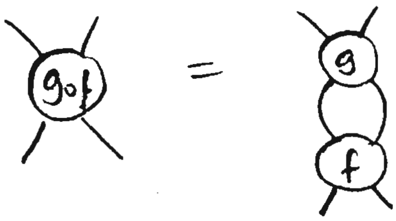
The first question is what happens if we follow one such operation on two fields with another such operation.

Specifically, I restrict to scalar fields and conformally invariant scatterings. I shall use the symbol  to denote the scattering specified on momentum states by



$$= f\left(\frac{k_1 . k_3}{k_1 . k_2}\right) \delta(k_1 + k_2 - k_3 - k_4)$$

Here f might be any function on [0,1], not necessarily the restriction of an analytic function (and allowing delta-functions, etc.)
Then we have a convolution $g \circ f$ defined by



Clearly it must be possible to give $g \circ f$ directly in terms of f and g. Explicit calculation in momentum space shows the relation can be written as:

$$g \circ f(w) = \int_0^1 \int_0^1 du\, dv\; K(u,v,w)\, f(u)\, g(v)$$

where $K(u,v,w) = \begin{cases} \frac{1}{2\pi}(2uv + 2vw + 2wu - u^2 - v^2 - w^2 - 4uvw)^{-\frac{1}{2}} \\ 0 \text{ if this argument } < 0. \end{cases}$

By considering this formula in terms of projections on to the eigenspaces of spin (as used by FM extensively), it can be shown equivalent to the identity

$$\sum_n (2n+1) P_n(x) P_n(y) P_n(z) = \begin{cases} \frac{1}{2\pi}(1 - x^2 - y^2 - z^2 + 2xyz)^{-\frac{1}{2}} \\ 0 \text{ if this argument } < 0 \end{cases}$$

which must have been well-known long ago (though it wasn't to me.)

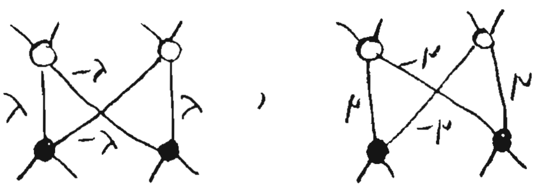
We can also write this as:

$$\sum_n (2n+1)P_n(\cos\theta)P_n(\cos\phi)P_n(\cos\psi) = \begin{cases} \frac{1}{2\pi}\begin{vmatrix} 1 & \cos\theta & \cos\phi \\ \cos\theta & 1 & \cos\psi \\ \cos\phi & \cos\psi & 1 \end{vmatrix}^{-\frac{1}{2}} = \frac{1}{2\pi}[\underline{n}_1,\underline{n}_2,\underline{n}_3]^{-1} \\ 0 \end{cases}$$

where $\underline{n}_1, \underline{n}_2, \underline{n}_3$ are the vectors joining the centre of a unit sphere to the vertices of a spherical triangle with sides θ, ϕ, ψ.

(the argument of the square root is negative $\Longleftrightarrow$ no such spherical triangle exists)

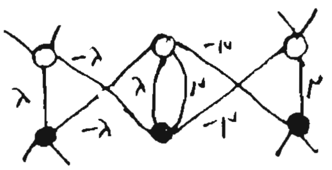
One may use this result (or rather its space-like analogue) to show that in the case where the f and g scatterings are represented by the twistor integrals



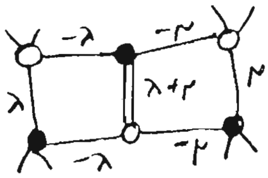
respectively.

then we find

$=$

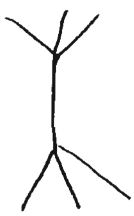


$= \dfrac{\Gamma(\lambda+1)\Gamma(\mu+1)}{\Gamma(\lambda+\mu+2)}$

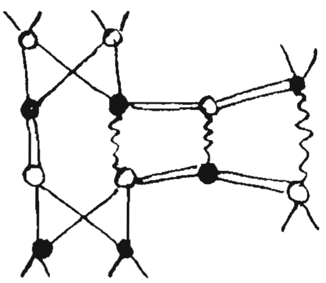


i.e. the convolution f∘g does correspond to "joining the boxes together" in the obvious way. This improves our theory of the "double box" diagram.

A second topic arose from the Feynman to twistor diagram correspondence

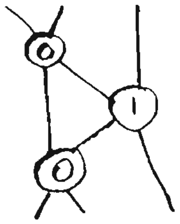


$=$

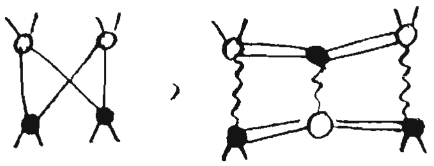


in ϕ^4 scattering.

One can think of this twistor diagram as an operation which projects out the spin-0 part of the product of three z.r.m. scalar fields. FM, using his techniques, was able to interpret its asymmetric form in algebraic terms. This prompted the question of generalisation to n such fields. To do this first note that this twistor diagram can be thought of as composed of three more elementary operations, thus:



where  corresponds to the scattering operation on two fields defined by taking $f(u) = 1$, and  to that with $f(u) = u$, represented by twistor diagrams

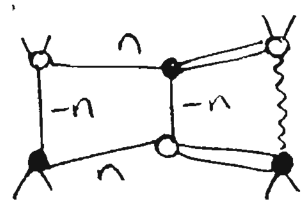


respectively

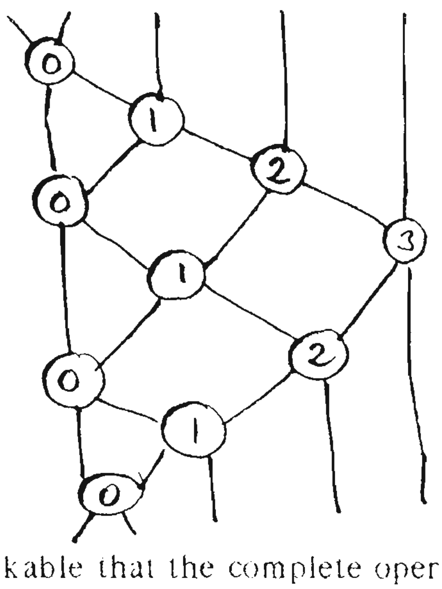
More generally now define

to be the scattering operation defined by $f(u) = u^n$,

which can be represented by twistor diagram



Then it turns out that the projection of the spin-0 part of the product of n z.r.m. scalar fields can be performed (up to a combinatorial factor) by a composite of operations drawn here explicitly in the case n=**5**, and by obvious analogy for general n:



It's remarkable that the complete operation is actually symmetric in the n (ingoing) fields. This formula opens new approaches to finding twistor diagram analogues of certain higher order Feynman tree diagrams.

Andrew Hodges

The Bach equations as an exact set of spinor fields

J. Frauendiener

Recently, L. J. Mason ([1], [2]) has proposed a reformulation of the "light-cone program" originally due to E. T. Newman and C. N. Kozameh (see [3] for a recent review): in an asymptotically flat vacuum spacetime the light-cone cuts of $\mathcal{J}^-$ are taken as the fundamental quantities and one tries to impose the vacuum Einstein equations as one scalar equation for the "cut function" which describes the cuts. In Mason's formulation a prominent role is played by the so called Bach equations. This is for the following reason: it has been shown in [4] that a necessary and sufficient condition for a spacetime to be conformal to an Einstein space is the validity of the following two equations on the Weyl and Ricci curvatures C_{abcd} and R_{ab}:

$$\partial^d C_{abcd} + \omega^d C_{abcd} = 0 \qquad \text{for some } \omega^d \tag{1}$$
$$B_{bc} := \partial^a \partial^d C_{abcd} - \tfrac{1}{2} R^{ad} C_{abcd} = 0 \tag{2}$$

The tensor B_{ab} defined in (2) is called the Bach tensor and has the following properties:

$$B_{ab} = B_{ba}, \qquad \partial^a B_{ab} = 0, \qquad B_a{}^a = 0.$$

In addition, it is conformally invariant. Any spacetime that is conformal to a vacuum spacetime has to satisfy (1) and (2). Mason imposes $B_{ab} = 0$ and studies the implications of this equation on the cut function.

Here, we want to look at the properties of the system of partial differential equations given by the spinorial version of (1) and (2). In particular, we will show that the system is equivalent to an exact set in the sense of Penrose [5].

Expressed in terms of the Weyl and Ricci spinors equation (2) is

$$\partial^A_{A'} \partial^B_{B'} \Psi_{ABCD} + \Phi^{AB}{}_{A'B'} \Psi_{ABCD} = 0. \tag{3}$$

Note, that the first term is automatically symmetric in $(A'B')$. In the following we will use a formalism described in [6] which is based on the isomorphism between totally symmetric spinor fields on spacetime and homogeneous functions on the spin bundle over spacetime:

$$\phi_{A\ldots BA'\ldots B'}(x) \to \phi(x, \pi, \bar\pi) = \phi_{A\ldots BA'\ldots B'}(x) \pi^A \ldots \pi^B \pi^{A'} \ldots \pi^{B'}.$$

π^A may be considered as a coordinate along the fibers of the spin bundle. With $\partial_A = \partial / \partial \pi^A$ we construct the four covariant derivative operators $L = \pi^A \pi^{A'} \partial_{AA'}$, $M = \pi^A \partial^{A'} \partial_{AA'}$, $M' = \partial^A \pi^{A'} \partial_{AA'}$, $N = \partial^A \partial^{A'} \partial_{AA'}$. The commutators between the derivative operators involve the curvature operators $S = \pi^A \pi^B \Box_{AB}$, $T = \pi^A \partial^B \Box_{AB}$, $U = \partial^A \partial^B \Box_{AB}$, the Euler operator $H = \pi^a \partial_A$ and the wave operator $\Box$. Finally, any algebraic operation consisting of outer multiplication and contraction of spinor fields corresponds to a "C-tree", a tree like structure built up from the bilinear products $C_{kk'}(\phi, \psi)$, where k and k' indicate the number of contracted indices, e.g., $C_{21'}(\phi, \psi) \to \phi_{ABA'(C} \psi^{ABA'}{}_{D')D)}$.

In this formalism, the Bach equations (2) read

$$M'^2\Psi + 12C_{20'}(\Phi,\Psi) = 0. \tag{4}$$

Φ and Ψ are the (2,2)- and (4,0)-functions corresponding to the Weyl spinor and the Ricci spinor, respectively. In addition to (4) we also have to consider the Bianchi identities

$$\begin{aligned} M'\Psi &= 2M\Phi, \\ N\Phi &= -12L\Lambda. \end{aligned}$$

Our first task is to find an equivalent set of first order equations. We start by introducing a (3,1)-function λ by $M'\Psi = 2\lambda$ and obtain the system

$$\begin{aligned} M'\Psi &= 2\lambda, & M'\Phi &= \bar{\lambda}, & M'\lambda &= -6C_{20'}(\Phi,\Psi), \\ N\Psi &= 0, & N\Phi &= -12L\Lambda, & N\lambda &= 0, \\ M\Psi &= 0, & M\Phi &= \lambda, & & \end{aligned}$$

The equations on λ express equation (4) and the symmetry in the primed indices in (3). We still need an equation for $M\lambda$. Inspection of the $[M,M']$ commutator gives a relation between $M\lambda$ and $\Box\lambda$. A similar such relation can be obtained from the operator identity

$$LN - MM' = -(H'+1)T + \tfrac{1}{2}H(H'+1)\Box \tag{5}$$

acting on Ψ. However, in the present case, these two relations are exactly the same. Therefore, we need to introduce another (4,0)-function χ by $M\lambda = \chi$ and derive equations for χ. By homogeneity we have $M\chi = 0$ and $N\chi = 0$. Now the $[M,M']$ commutator and the above identity acting on λ give independent relations between $M'\chi$ and $\Box\lambda$, which can be used to derive an equation for $M'\chi$. So we end up with the system

$$\begin{aligned} N\Phi &= -12L\Lambda, & N\lambda &= 0, \\ M\Phi &= \lambda, & M\lambda &= \chi, \\ M'\Phi &= \bar{\lambda}, & M'\lambda &= -6C_{20'}(\Phi,\Psi), \\ N\Psi &= 0, & N\chi &= 0, \\ M\Psi &= 0, & M\chi &= 0, \\ M'\Psi &= 2\lambda, & M'\chi &= 8C_{21'}(\Phi, L\Psi) - 16C_{10'}(L\Lambda,\Psi) + 8\Lambda\lambda \\ & & &\quad + \tfrac{48}{5}C_{11'}(\Phi,\lambda) - 12C_{20'}(\lambda,\Psi). \end{aligned}$$

This system is consistent, which can be seen by applying each of the commutators $[M,M']$, $[N,M]$ and $[N,M']$ to the four functions. This results either in expressions for the wave operator acting on the functions or in identities. Obviously, any solution of the system gives rise to a solution of (4) and vice versa.

It is clear from the conformal invariance of the Bach equations that we do not get any equation for the scalar curvature Λ. We can handle this situation in two ways: either we enlarge the system by adding Λ to the variables and postulate an evolution equation like $\Box\Lambda = 0$, or we consider Λ as a given function on spacetime. Both ways lead to the result that the system constitutes an exact set. The first case leads to an exact set which is also invariant [5], while we can not expect to obtain an invariant exact set in the second case because Λ and all its derivatives will enter in the expression for the unsymmetrized derivatives of the fields in terms of the totally symmetric derivatives. Nevertheless, we will treat Λ (and all its symmetrized derivatives $L^k\Lambda$) as given.

The proof that our system is in fact an exact set consists of verifying two conditions:

(i) all powers of L acting on the functions are algebraically independent,

(ii) arbitrary products of operators L, M, M' and N acting on the functions can be expressed in terms of the powers L^k acting on the functions.

The proof of condition (ii) is a reprise of the proof that the vacuum Bianchi identity on the Weyl spinor gives rise to an exact set ([6]). It rests on the fact that commuting the derivative operators only introduces the powers $L^k\Lambda$, the unknowns and the wave operator whose action on the functions can be expressed as a C-tree containing only powers. In addition, the right hand sides of the equations are C-trees in the variables so that when we encounter any derivative operator other than L acting on a function we can replace it with a C-tree.

As for condition (i) again the same argument as in the case of the vacuum Bianchi identities holds: the field equations impose conditions on all expressions of the form $s_nO\phi$, where s_n is any string of length n in the derivative operators, O is any of the operators M, M' N and ϕ stands for any of the unknowns. There are no restrictions on the expressions of the form $s_nL\phi$. Also, the commutator relations and relation (5) above only link expressions $s_n\phi$ for which the string does not consist entirely of L's. Therefore, in all the relations generated by the commutators, identity (5) and the field equations there can never appear a power and, hence, the powers are all independent. This proves exactness of the set of fields consisting of the four functions Ψ, Φ, λ and χ.

In this formal setting the characteristic initial value problem for the Bach equations is well posed. The initial data are all the powers $L^k\Psi$, $L^k\Phi$, $L^k\lambda$ and $L^k\chi$ corresponding to the totally symmetric spinor derivatives of the spinor fields at the vertex of the initial light cone.

Let us now assume that Λ and all its derivatives vanish. Additionally, we will assume that we have existence and uniqueness of solutions to equations which give rise to an exact set. This is certainly true in the formal sense. It is still not known whether the characteristic initial value problem is well posed for any reasonable function space. Suppose we give Ψ and all its symmetrized derivatives at the vertex of the initial light cone. Then evolution with the vacuum Bianchi identity $M'\Psi = 0$ (which is an exact set) produces a vacuum spacetime which necessarily provides a solution to the Bach equations. On the other hand, evolution of the same initial data together with $L^k\Phi = 0$, $L^k\chi = 0$ and $L^k\lambda = 0$ with the Bach equations gives a spacetime which is necessarily the same vacuum spacetime because of the uniqueness of the solution. This argument shows that one obtains solutions to the vacuum equations by appropriately restricting the initial data for the Bach equations.

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Hierarchy of Conservation Laws for Self-Dual Gravity

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Abstract

An infinite hierarchy of non-local conservation laws is constructed for the self-dual vacuum equations. Further, it is shown that the construction of such conserved currents has a natural description in terms of Penrose's non-linear graviton construction of such self-dual vacuum metrics.

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Twistor classification of type D vacuum space-times

Thomas von Schroeter

In the framework of the Yang-Mills twistor approach, stationary axisymmetric space-times can be characterized in terms of holomorphic rank 2 bundles over projective twistor space (Ward 1983). If the space-time extends analytically into a neighbourhood of an axis or horizon, then the patching data of the corresponding bundle consist of a single 2×2 matrix

$$P(z) = \frac{1}{f_0} \begin{pmatrix} 1 & -\psi_0 \\ -\psi_0 & f_0^2 + \psi_0^2 \end{pmatrix}$$

holomorphic in z which is entirely determined by the values of the associated *Ernst potential*

$$E(z,r) = f(z,r) + \mathrm{i}\psi(z,r)\,; \quad f_0(z) \equiv f(z,0)\,, \quad \psi_0(z) \equiv \psi(z,0)$$

on the axis/horizon $r = 0$ (Woodhouse and Mason 1988, Fletcher and Woodhouse 1990). The patching matrices for the ***Weyl solutions*** – i.e. those type D metrics for which the induced metric J on the space of Killing vectors can be diagonalized ($\omega = 0$ in (1) below) – were derived in an earlier note in TN 35. This note is concerned with the ***non-diagonal*** case, for which the metric takes the full stationary axisymmetric form

$$\mathrm{d}s^2 = f(z,r)\Big(\mathrm{d}t - \omega(z,r)\mathrm{d}\theta\Big)^2 - \frac{r^2}{f(z,r)}\mathrm{d}\theta^2 - \Omega^2(z,r)\left(\mathrm{d}z^2 + \mathrm{d}r^2\right)\,. \tag{1}$$

Here, f, ω, and Ω are real analytic functions on a Riemann surface Σ with complex coordinate $w = z + \mathrm{i}r$ which is the space of orbits of the Killing vectors, $\partial/\partial t$ (timelike) and $\partial/\partial\theta$ (spacelike). The imaginary part ψ of the Ernst potential is determined (up to an additive constant) by

$$\mathrm{d}\psi = \frac{f^2}{r} * \mathrm{d}\omega\,, \tag{2}$$

$$\mathrm{d}\omega \wedge \mathrm{d}\psi + \mathrm{d}\left(r * \frac{\mathrm{d}f}{f}\right) = 0\,, \tag{3}$$

and

$$\partial_w \log\left(f\Omega^2\right) = \frac{\mathrm{i}r}{f^2}(\partial_w E)(\partial_w \bar{E})\,, \tag{4}$$

where d is the exterior derivative and $*$ the star operator on Σ. The existence of such a function ψ is a consequence of the vacuum equations for the metric (1).

As before, our approach is based on the fact that all type D vacuum space-times have Killing spinors of valence 2 (Walker and Penrose 1970). Equations (3)–(5) of the earlier note remain valid, but now with

$$J = \begin{pmatrix} f\omega^2 - \frac{r^2}{f} & -f\omega \\ -f\omega & f \end{pmatrix}\,, \quad A = \begin{pmatrix} -\omega\beta - \frac{\mathrm{i}r}{f} * \beta \\ \beta \end{pmatrix}\,,$$

where $\beta = \beta_z \mathrm{d}z + \beta_r \mathrm{d}r$ is a complex[1] one-form on Σ, and translate into

$$\mathrm{d}\beta + \frac{3}{2}\beta \wedge \frac{\mathrm{d}E}{f} = 0 \tag{5}$$

[1] Unlike the Weyl case, the equations which β has to satisfy, i.e. (5)-(7), are not real and thus β_z and β_r cannot be taken to be real functions.

$$d * \beta - *\beta \wedge \left(\frac{dE}{2f} - 2\frac{dr}{r}\right) = 0 \tag{6}$$

$$\partial_w \log \frac{\beta_z - i\beta_r}{\Omega^2 f} = -\frac{i}{2}\frac{\partial_w \psi}{f} \tag{7a}$$

$$\partial_{\bar{w}} \log \frac{\beta_z + i\beta_r}{\Omega^2 f} = -\frac{i}{2}\frac{\partial_{\bar{w}} \psi}{f} . \tag{7b}$$

Expanding all quantities near the axis (or horizon), $r = 0$, and eliminating from (2)-(7) all functions but $E_0(z) = E(z, 0)$, one finds that E_0 satisfies the same simple ODE that $f_0(z)$ has to satisfy in the case of the Weyl solutions:

$$3E_0^{(4)}E_0'' - 4(E_0''')^2 = 0 . \tag{8}$$

Using the freedom $z \mapsto \pm z + \text{const.}$, $\psi \mapsto \psi + \text{const.}$, and $\psi \mapsto \psi + \text{const.}\ z$ (all constants real)[2], the general solution of (8) can be reduced to

$$E_0(z) = \begin{cases} \gamma z^2 + ez + g & \text{if } E_0''' = 0 \\ \alpha(z + ib)^{-1} + c + dz & \text{if } E_0''' \neq 0 \end{cases}$$

where $\alpha, \gamma \in \mathbf{C}$, $b, c, d, e, g \in \mathbf{R}$. As a (positive) real overall factor in E can be incorporated into the metric components by homothetic transformations in the space of Killing vectors, the two sets of parameters, $[\gamma, e, g]$ and $[\alpha, c, d]$, can be regarded as homogeneous coordinates of the solution spaces, and thus there are at most 4 real parameters. This is, of course, the expected number for the general type D vacuum (acceleration, rotation, mass, and NUT parameter).

One of the most prominent examples is *Kerr space-time.* Fletcher and Woodhouse (1990, eqn. (55)) find

$$f_0(z) = \frac{z^2 - m^2 + a^2}{(z+m)^2 + a^2} , \quad \psi_0(z) = -\frac{2am}{(z+m)^2 + a^2}$$

where m and a are, respectively, the mass and angular momentum parameter, and $a < m$. Replacing $z + m$ by z, one obtains $E_0(z) = 1 - 2m(z - ia)^{-1}$ and thus

$$\alpha = -2m , \quad b = -a , \quad c = 1 , \quad d = 0 .$$

Further examples are under investigation. One of the aims is, of course, to relate the parameters to physical properties of the space-times.

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[2]The freedom to make these changes arises, respectively, from the definitions of z and ψ in terms of their differentials (z is the "harmonic conjugate" of r on Σ) and from the freedom to make linear transformations in the space of Killing vectors.

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