1

Spin 3/2 fields and local twistors

L.J. Mason & R.Penrose

1 Introduction

In TN's 31 and 33 one of us (RP) proposed that helicity 3/2 fields might provide a suitable vehicle for the definition of a twistor in vacuum space-times. On the one hand, in flat space-time, twistors emerge as the charges of helicity 3/2-fields, and on the other, the vacuum equations are the consistency condition for the existence of such fields (in potential form, e.g. that given by Rarita-Schwinger) so that they can exist only when the field equations are satisfied.

There are still fundamental obstacles. The difficulties of defining charges in curved space were discussed in Penrose (1992). However, there are intriguing issues that are already present in a flat background. We have to deal with the description of the field as a potential modulo gauge rather than as field. (The helicity 3/2 'field' associated with a Rarita-Schwinger potential is not gauge invariant in curved space.) Thus, a deeper examination of the R-S potential, its gauge freedom, and its relation to twistor theory should prove fruitful.

A neat form of the R-S equations can be given in terms of spinor-indexed forms. Let $\sigma_{A'}$ be a spinor-indexed 1-form and d be the exterior derivative extended to act on spinor-indexed quantities. Then the gauge freedom is $\sigma_{A'} \to \sigma_{A'} + d\nu_{A'}$ where $\nu_{A'}$ is a spinor-indexed function, and the R-S equations are

$$dx^{AA'} \wedge d\sigma_{A'} = 0. \tag{1.1}$$

One can see directly that the gauge transformations will be consistent with the field equations iff

$$dx^{AA'} \wedge d^2\nu_{A'} = dx^{AA'} \wedge R_{A'}^{B'}\nu_{A'} = 0$$

for all $\nu_{A'}$. This follows iff the Ricci tensor vanishes (this is effectively the same calculation as that required to show that the Sparling 3-form is closed when the vacuum equations are satisfied—see Penrose & Rindler 1986, Vol 2 chapter 6). There are 8 equations for 8 unknowns, and two free functions worth of gauge freedom, so we require two relations between the field equations for the system to be consistent (this can be seen by removing two of the unknowns using the gauge freedom to set, for example, $V\rfloor\sigma_{A'} = 0$ for some vector field V so that there are 2 more equations than unknowns). The relations follow by taking the exterior derivative of (1.1). This vanishes identically iff $dx^{AA'} \wedge R_{A'}^{B'} = 0$ so the equations are consistent in vacuum.

In flat space, the field is given by

$$d\sigma_{A'} = \psi_{A'B'C'}dx^{BB'} \wedge dx_B^{C'}$$

and is gauge invariant, but in curved space a gauge transformation will add $\Psi_{A'B'C'D'}\nu^{D'}$ to $\psi_{A'B'C'}$, making it non-invariant.

Since the helicity 3/2 field is only directly descibed in potential form in curved space, in Penrose (1992), a description of charges was given that only involves the potential. We shall consider the general case with a 1-form potential γ for the dual field ${}^*F$ possibly taking values in some flat vector bundle. Introduce a covering of the relevant region of space-time M by open sets $\{U_i\}$. The (dual) field on this region is equivalent to a collection of potentials γ_i on each U_i such that $d\gamma_i = {}^*F$ so that on $U_i \cap U_j$, γ_i and γ_j differ by a gauge transformation (the collection $\{\gamma_i\}$ are taken to be defined modulo global gauge transformations). The charge associated with $\{\gamma_i\}$ takes values in the space of *gauge freedom of the second kind.* For electromagnetism, this formulation follows from the de-Rham sequence for potentials of the dual field

$$0 \to \mathbb{C} \to \mathcal{E}(M) \to \Omega^1(M) \to \Omega^2(M) \to \ldots$$

where $\mathcal{E}(M)$ is the space of smooth functions on M and the second map is the injection of constant functions and the third and subsequent maps are the standard exterior derivative. The charge q of an electromagnetic field is obtained by integrating the (closed) 2-form ${}^*F$ over some given 2-surface, S. The closed 2-form ${}^*F$ is equivalent to a collection $\{\gamma_i\}$ with $\gamma_i \in \Omega^1(U_i)$ and $d(\gamma_i - \gamma_j) = 0$ on $U_i \cap U_j$ defined up to $\gamma_i \to \gamma_i + df_i$ for some $f_i \in C^\infty(M)$. By diagram chasing (or direct integration), one can see that q lies in the $\mathbb{C}$ of the above sequence playing the role of a *gauge transformation of the second kind*, which is to say a gauge transformation that leaves the *potential* invariant. This follows by defining f_{ij} by $\gamma_i - \gamma_j = df_{ij}$. It then follows, with $q_{ijk} = f_{ij} + f_{jk} - f_{ik}$, that $dq_{ijk} = 0$ so q_{ijk} is a Čech cocycle with constant coefficients and one can see that its evaluation on S is the charge (up to the usual 4π etc.).

This point of view for helicity 3/2 fields led to a new difficulty even in flat space: if one only has the Rarita-Schwinger field, then the analogue of the above sequence is now

$$0 \to \mathbb{S}_{A'} \to \mathcal{E}_{A'} \to \Omega^1_{A'} \to \Omega^2_{A'} \to \ldots$$

where the third and fourth maps are the covariant exterior derivative. The problem is that the gauge transformations of the second kind are just constant spinors $\mathbb{S}_{A'}$ whereas we were hoping to obtain a whole twistor. This spinor is the secondary part of the charge of the corresponding helicity 3/2 field. In Penrose (1992,1994) it was noted that if one wishes to encode the primary part of the charge, one requires the next potential down the potential chain. A proposal for an exact sequence involving this second potential in the R-S case was also given in Penrose (1994).

The purpose of this note is to give an improved version of this sequence, with unrestricted gauge freedom, and relate it to local twistors.

2 Local twistors and the helicity 3/2 equations

In Woodhouse (1985) and Mason (1990) it was noted that the potential chains for zero rest mass fields have a natural formulation in terms of local twistors. Perhaps the simplest formulation of the potential chain for the helicity 3/2 equation is as follows

$$\psi_{A'B'C'} = \nabla^{A}_{A'}\sigma_{B'C'A}, \qquad \sigma_{AA'B'} = \nabla^{B}_{A'}\rho_{B'AB}, \qquad \rho_{ABA'} = \nabla^{C}_{A'}\phi_{ABC}$$

where each potential is a symmetric spinor, and is subject to field equations and gauge freedom (cf. Penrose and Rindler vol. 2, section 6.4).

A slightly different formulation of the potential chain, without its first and last parts, can be encoded in a local-twistor-valued 1-form with $n-2$ symmetric indices for helicity n. This is easily understood from the positive homogeneity twistor-function description: for helicity 3/2 one takes a dual twistor function R of homogeneity one and then evaluates $R^\alpha = \partial R/\partial W_\alpha$ as a self-dual Maxwell Field with a local twistor index. Denote this field by $\mathcal{R}^\alpha$. (Since, at this stage, we are working in conformally flat space-time, the bundle of local twistors is flat and so we can just evaluate such an object in a covariantly constant frame and then transform to the standard one.) Then $\mathcal{R}^\alpha$ is defined modulo gauge transformations, $\mathcal{R}^\alpha \to \mathcal{R}^\alpha + DZ^\alpha$ where Z^α is an arbitrary function with values in local twistors and D denotes the exterior derivative extended to act on quantities with a local twistor index using the local twistor connection. We also have the self-duality equation $(D\mathcal{R}^\alpha)^- = 0$ where the superscript '$-$' on a 2-form denotes its ASD part. As a consequence of the fact that R^α is of the special form $\partial R/\partial W_\alpha$, $D\mathcal{R}^\alpha$ also has vanishing primary part. This follows from the fact that the 'field' integral formula for the primary part is the expression

$$\oint \frac{\partial^3 R}{\partial\tilde\pi_A \partial\tilde\omega^{A'}\partial\tilde\omega^{B'}}\tilde\pi^C d\tilde\pi_C .$$

However, this is of the the form of the integral of $\partial f_{-1}/\partial\tilde\pi_A\tilde\pi^B d\tilde\pi_B$ where f_{-1} has homogeneity -1 (and has indices) but this is an exact form being proportional to $d(\tilde\pi^A f_{-1})$ so that its integral vanishes.

This last condition can be written $X_{\alpha\beta}D\mathcal{R}^\beta = 0$ where

$$X_{\alpha\beta} = \begin{pmatrix} \varepsilon_{AB} & 0 \\ 0 & 0 \end{pmatrix} \tag{2.2}$$

is the 'position twistor' i.e. the canonical section (up to scale) of $\mathbb{T}_{[\alpha\beta]}$ over Minkowski space given by expressing points of Minkowski space as elements of the projectivisation of $\mathbb{T}_{[\alpha\beta]}$ via the Klein correspondence.

We have that the primary and secondary parts are the potentials for the helicity 3/2 field as follows

$$\mathcal{R}^\alpha := \begin{pmatrix} \rho^A \\ \sigma_{A'} \end{pmatrix} := \begin{pmatrix} \rho^A_b dx^b \\ \sigma_{A'b}dx^b \end{pmatrix}.$$

The equations $(D\mathcal{R}^\alpha)^- = 0$ and $X_{\alpha\beta}D\mathcal{R}^\beta = 0$ become

$$d\rho^A + i dx^{AA'} \wedge \sigma_{A'} = 0 \text{ and } (d\sigma_{A'})^- = 0.$$

The first equation can be seen to imply the second by first taking the covariant exterior derivative of the first (in flat space still) to obtain

$$dx^{AA'} \wedge d\sigma_{A'} = 0,$$

the form of the R-S equation given in the introduction, and then writing out this system in full. The helicity 3/2 field is then the secondary part of $D\mathcal{R}^\alpha$.

Thus, locally at least, $\mathcal{R}^\alpha$ modulo DZ^α satisfying $X_{\alpha\beta}D\mathcal{R}^\alpha = 0$ is equivalent to a helicity 3/2 field and is the $\mathcal{P}$-transform of $\partial R/\partial W_\alpha$.

This formulation of the potential for a helicity 3/2 field is sufficient to encode the full charge of the field in the context of gauge freedom of the second kind since we now have the exact sequence

$$0 \to \mathbb{T}^\alpha \to \mathcal{E}^\alpha \to \Omega^{1\alpha} \to \Omega^{2\alpha} \to \dots$$

where the second map is the injection of the covariantly constant local twistors into general local twistor fields, the third and fourth maps are the exterior derivative extended to act on local twistors using the local twistor connection. Thus the gauge freedom of the second kind is given precisely by constant twistors. (A sequence of this nature was given in Penrose 1994, but with a restricted gauge freedom.)

3 Generalization to curved space

The above equations do not work as they stand in curved space as, when $D^2 \neq 0$, the equation $X_{\alpha\beta}D\mathcal{R}^\beta = 0$ is no longer compatible with the gauge freedom. However, a weaker equation is compatible with the gauge freedom in vacuum, namely

$$DX_{\alpha\beta} \wedge D\mathcal{R}^\beta = 0. \tag{3.3}$$

There is a subtlety here as $X_{\alpha\beta}$ is only defined up to scale by the conformal metric. The scale is fixed by choosing a particular conformal factor and representing $X_{\alpha\beta}$ by equation (2.2). Equivalently, a choice of conformal scale leads to an infinity twistor

$$I_{\alpha\beta} = \begin{pmatrix} 0 & 0 \\ 0 & \varepsilon^{A'B'} \end{pmatrix}$$

which can be used to normalize the position twistor by means of the relation

$$X_{\alpha\beta}I^{\alpha\beta} = 2.$$

Equation (3.3) is only conformally invariant when $X_{\alpha\beta}D\mathcal{R}^\alpha$ vanishes also. This additional equation can be consistent only in flat space, however.

The compatibility of the gauge freedom with equation (3.3) follows iff we have the relation $DX_{\alpha\beta}\wedge\mathcal{K}^\beta_\gamma = 0$ where $\mathcal{K}^\alpha_\beta$ is the curvature of the local twistor connection and $X_{\alpha\beta}$ is normalized as above by a choice of conformal factor. This relation holds iff $\nabla^A_{A'}\Psi_{ABCD} = 0$ and thus requires the Einstein equations. As before, there are as many equations as unknowns (in this case 16 rather than 8) but the gauge freedom can be used to set some (in this case 4) of the unknowns to zero leading to an overdetermined system. However, if the relation $DX_{\alpha\beta} \wedge \mathcal{K}^{\beta}_{\gamma} = 0$ holds, the covariant exterior derivative of equation (3.3) vanishes identically leading to four relations between the 16 equations so that they are not overdetermined in vacuum (or more generally when the trace free part of the Ricci tensor vanishes). (Conversely, if $\nabla^{A}_{A'}\Psi_{ABCD} \neq 0$ the equations are inconsistent.)

If we write out this system in terms of the primary and secondary parts of $\mathcal{R}^{\alpha}$, we obtain the equations

$$dx^{B'}_{A} \wedge d\rho^{A} + i dx^{B'}_{A} \wedge dx^{AA'} \wedge \sigma_{A'} = 0 \text{ and } dx^{AA'} \wedge \sigma_{A'} = 0.$$

In flat space we see that we acquire an extra helicity-3/2 field, since

$$D\mathcal{R}^{\alpha} = \begin{pmatrix} \tilde{\psi}^{A}_{BC} dx^{BB'} \wedge dx^{C}_{B'} \\ \psi_{A'B'C'} dx^{BB'} \wedge dx^{C'}_{B} \end{pmatrix}$$

with the ψ's totally symmetric and $\nabla^{A}_{A'}\tilde{\psi}_{ABC} = 0 = \nabla^{A'}_{A}\psi_{A'B'C'}$. Conversely given such ψ's, we can write down the twistor indexed 2-form as above which is closed and so can be written as $D\mathcal{R}^{\alpha}$ for some $\mathcal{R}^{\alpha}$ and automatically satisfies $DX_{\alpha\beta} \wedge D\mathcal{R}^{\alpha} = 0$.

However, the gauge freedom of the second kind is still just the constant local twistors, since the fields sit in the same exact sequence as the single helicity 3/2 fields did as before.

4 Where do we go from here?

Connections with other aspects of twistor theory

There are some intriguing connections with other aspects of twistor theory. Firstly, the connections between twistor theory and integrable systems highlight the role played by linear systems in twistor correspondences and so one might hope that the above linear system might lead to some kind of twistor construction. See Mason (1994) for an exploration of this line of reasoning. Secondly, the identities that guarantee the consistency of the R-S equations are precisely those that lead to the characterization of the vacuum equations by means of the Sparling 3-form. This suggests that a (local) twistorial analogue of the Sparling 3-form should be taken to be $DZ^{\alpha} \wedge DZ^{\beta} \wedge DX_{\alpha\beta}$. It is certainly closed iff the vacuum Bianchi identities are satisfied. This expression (with some modification to eliminate the terms that contain the Ricci tensor explicitly) also has an interpretation as a Hamiltonian that generates a translation along the vector field $\pi^{A'}\omega^{A}$ together with a spin-frame rotation generated by $\pi_{A'}\pi^{B'}$. These ideas should extend to a local twistor generalization of the connections between the Sparling 3-form, the canonical formalism and quasi-local mass described in Mason & Frauendiener (1990).

Towards a definition of twistors in vacuum space-times

The aforementioned considerations serve to clarify the role of twistors and their relationship to the R-S equations in flat space, by exhibiting them as gauge quantities of the second kind.

However, the full generalization of these ideas to Ricci-flat curved space remains elusive. So far, these considerations suffer from being 'too linear', since a twistor space without a vector space structure ought eventually to arise. One promising route to achieving this would be to examine a role for twistors as providing a 'charge' in the *active* sense rather than passive sense, similarly to the way in which the electric charge features in the electromagnetic connection $\nabla_a - ieA_a$. Combining the active with the passive roles for a twistor might provide a route to the required non-linearity. Work is in progress.

Thanks to J. Frauendiener and G.A.J.Sparling for comments and discussion.

References

Mason, L.J. (1990) Local twistors and the Penrose transform for homogeneous bundles, in *Further adavnces in twistor theory*, §I.2.17 (see also §I.2.18), Vol. I, eds. L.J.Mason & L.P.Hughston, Pitman Research Notes in Mathematics No. 231, Longman.

Mason, L.J. & Frauendiener, J. (1990) The Sparling 3-form, Ashtekar variables and quasi-local mass, in *Twistors in mathematics and physics*, eds. T.N.Bailey and R.J.Baston, LMS lecture notes series 156, C.U.P..

Mason, L.J. (1994) to appear in proceedings of Devon conference in Seale Hayne, ed. S. Huggett.

Penrose, R. (1992) Twistors as spin 3/2 charges, in *Gravitation and Modern Cosmology* eds. A.Zichichi, N. de Sabbata, and N. Sánchez (Plenum Press, New York)

Penrose, R. (1994) to appear in proceedings of Devon conference in Seale Hayne, ed. S. Huggett

Woodhouse, N.M.J. (1985) Real methods in twistor theory, Class. & Quant. Grav. **2** 257-291