Another view at the spin (3/2) equation

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In this note I want to draw attention to another point of view for the spin (3/2) equations which have been discussed in various places because they might provide a way of defining twistors for curved but Ricci flat spacetimes (see [4], [3], [2]). The two main properties are that the consistency condition for the existence of solutions are the vacuum equations $G_{ab} = 0$ and that in flat space twistors emerge as the charges of the fields.

Here I want to focus on the equation

$$\partial_{A(A'}\gamma^{AB}_{B')} = 0 \tag{1}$$

and show how it relates to the point of view in [3]. In particular, I will show how to obtain "charges" in a consistent way also in curved spacetimes. Recent work with George Sparling [1] shows that equation (1) has the following properties (among others): it is consistent in arbitrary curved spacetimes, i.e., the Cauchy problem is well posed; it is conformally invariant and can be obtained from the action

$$\mathcal{A} = \operatorname{Im}\left\{\int \bar{\gamma}^{A'B'}_{B}\partial_{A'A}\gamma^{AB}_{B'}\right\}\Sigma.$$

As it stands (1) is insensitive towards the vacuum equations because it has solutions on arbitrary manifolds. Let me denote the solution space of (1) by [3/2] and the solution space of the Weyl neutrino equation by [1/2]. Note, that the Weyl equation also has a well posed Cauchy problem on arbitrary manifolds. In flat space we have the following structure relating these solution spaces. Given any solution of (1) we obtain a solution of the neutrino equation by taking its divergence: $\partial^{B'}_{B}\gamma^{AB}_{B'} \in [1/2]$ for all $\gamma \in [3/2]$. Call this map $N : [3/2] \to [1/2]$. On the other hand, given $\nu^A \in [1/2]$ we can obtain a solution of (1) by taking a symmetrized derivative, $\partial^{(A}_{A'}\nu^{B)} \in [3/2]$ for all $\nu \in [1/2]$. Call that map $L : [1/2] \to [3/2]$. It is easy to see that $\operatorname{im} L \subset \ker N$. So we get a sequence of maps

$$[1/2] \xrightarrow{L} [3/2] \xrightarrow{N} [1/2] \tag{2}$$

which is not exact. In fact, from a crude argument, counting free functions for the Cauchy problem, it can be seen that [3/2] is characterized by six free functions of three variables whereas [1/2] amounts to two free functions. This shows that $\ker N$ amounts to four functions so that there are two free functions that do not correspond to the image of L. This can be made more precise using the Fourier representation of solutions of (1) in flat space.

If we now ask how much of this sequence can be carried over to curved manifolds we find that the L-part can only be defined if the tracefree part of the Ricci tensor vanishes, i.e., only on spacetimes with $\Phi_{ab} = 0$ will the derivative of a neutrino field be a solution of equation (1). Similarly, the divergence of a solution of equation (1) will be a solution of the Weyl equation only if $\Phi_{ab} = 0$. If, in addition, we insist on the property that $\operatorname{im} L \subset \ker N$ then also the scalar curvature has to vanish. In summary then, we find that we have the same sequence (2) iff $G_{ab} = 0$.

At this stage, the natural question to be asked is, of course: why do the Einstein equations favour this structure? Surprisingly, it is exactly this structure that is necessary to define "charges" as surface integrals in the curved case. Recall, that equation (1) arises from a variational principle. This means that its solution space comes equipped with a natural symplectic structure. The symplectic form on [3/2] is

$$\omega(\gamma_1, \gamma_2) = i \int_{\mathcal{H}} \left(\bar{\gamma}_{1B}^{A'B'} \gamma_{2B'}^{AB} - \bar{\gamma}_{2B}^{A'B'} \gamma_{1B'}^{AB} \right) \Sigma_{AA'}.$$

The integral is taken over a hypersurface $\mathcal{H}$. It is hypersurface independent because the integrand is closed iff the γ's satisfy equation (1).

If we think of [1/2] as inducing transformations on [3/2] via $\gamma \mapsto \gamma + L\nu$, then we may ask for the Hamiltonians that generate these transformations. This means that we have to solve the equation

$$\omega(L\nu, \gamma) = -\delta H_\nu(\gamma) \qquad \text{for all } \gamma \in [3/2] \tag{3}$$

for H_ν, given $\nu \in [1/2]$. Now the left hand side is

$$\operatorname{Im} \int \partial_{B'}^{(A} \nu^{B)} \bar{\gamma}_B^{A'B'} \Sigma_{AA'}$$

which is equal to (up to factors)

$$\operatorname{Im} \int D\nu^B \wedge \bar{\gamma}_B^{A'B'} \Sigma_{A'B'},$$

$\Sigma_{A'B'}$ being the selfdual two-forms on spacetime and D being the covariant exterior derivative. Integrating by parts we find that the left hand side of (3) is

$$\omega(L\nu, \gamma) = \operatorname{Im} \int_{\partial \mathcal{H}} \nu^B \bar{\gamma}_B^{A'B'} \Sigma_{A'B'} - \operatorname{Im} \int_{\mathcal{H}} \nu^B D\bar{\gamma}_B^{A'B'} \wedge \Sigma_{A'B'},$$

consisting of a hypersurface integral and a two dimensional boundary integral. Examination of the hypersurface integrand shows that it is equal to $\left(\nu^A \partial_{B'}^B \bar{\gamma}_B^{A'B'} \Sigma_{AA'}\right)$. This shows that this integral will vanish iff we restrict ourselves to the subspace of [3/2] consisting of divergence free solutions of (1). The Einstein equations ensure that [1/2] acts on $\ker N$. The upshot of all this is that provided the Einstein equations hold, we have an action of [1/2] on the symplectic submanifold $\ker N$ which can be considered as gauge transformations. The corresponding Hamiltonians or "charges" are given by the surface integral

$$H_\nu(\gamma) = \mathrm{Im} \int_{\partial\mathcal{H}} \nu^B \bar{\gamma}_B^{A'B'} \Sigma_{A'B'},$$

where now γ is restricted to be a divergence free solution of (1), i.e., a solution of the equation $\partial_{AA'}\gamma^{AB}_{B'} = 0$.

Let us now see how this fits in with the flat space expression for the charges in [3]:

$$q = \int_{\partial\mathcal{H}} \mu^{C'} \psi_{A'B'C'} \Sigma^{A'B'}$$

where $\psi_{A'B'C'} = \partial_{AA'}\bar{\gamma}^A_{B'C'}$ is the helicity (3/2)-field defined by $\bar{\gamma}$ and $\mu^{A'}$ is the primary part of a dual twistor $(\mu^{A'}, \lambda_A)$, so $\partial_{AA'}\mu^{B'} = i\epsilon_{A'}{}^{B'}\lambda_A$ and $\partial_{AA'}\lambda_B = 0$. Consider now the complex one-form $\alpha = \bar{\gamma}_{A'B'A}\mu^{B'}\theta^{AA'}$. Then

$$\begin{aligned} d\alpha &= \partial_B^{A'}\left(\bar{\gamma}_{A'C'A}\mu^{C'}\right)\Sigma^{AB} + \partial^A_{B'}\left(\bar{\gamma}_{A'C'A}\mu^{C'}\right)\Sigma^{A'B'} \\ &= \left(\partial^{A'}_B\bar{\gamma}_{A'C'A}\right)\mu^{C'}\Sigma^{AB} + \bar{\gamma}_{A'C'A}\left(\partial^{A'}_B\mu^{C'}\right)\Sigma^{AB} \\ &\quad + \left(\partial^A_{B'}\bar{\gamma}_{A'C'A}\right)\mu^{C'}\Sigma^{A'B'} + \bar{\gamma}_{A'C'A}\left(\partial^A_{B'}\mu^{C'}\right)\Sigma^{A'B'}. \end{aligned}$$

The first two terms vanish because of (1) and the twistor equation and so

$$d\alpha = \psi_{A'B'C'}\mu^{C'}\Sigma^{A'B'} + i\bar{\gamma}_{A'B'A}\lambda^A\Sigma^{A'B'}.$$

Integration over $\partial\mathcal{H}$ shows that the flat space expression and the symplectic expression for the charges agree up to signs and taking the imaginary part if we identify ν^A and λ^A.

All this looks encouraging but there are still many open questions. Why do we get only an imaginary part in the symplectic charge integral? How do twistors appear in the curved space formula? What is the role of the solution ν^A of the Weyl equation in relation to the projection part of a dual twistor? Maybe some of these questions can be answered once the role of the second potential has been clarified.

References

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