# A higher spin generalization of the Dirac equation to arbitrary curved manifolds

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In this note I want to discuss a class of equations that can be considered as generalizations of the Dirac equation to higher spin fields. There have already been several proposals to this end (see [2] and references therein). The present approach grew out of joint work with George Sparling on higher spin massless spinor fields. In [1] we considered spinor fields of the type $\psi^{AB\ldots C}_{B'\ldots C'}$ with $p$ primed and $p+1$ unprimed indices subject to the equation

$$\partial_{A(A'}\psi^{AB\ldots C}_{B'\ldots C')} \doteq 0. \tag{1}$$

For $p = 0$ this is just the Weyl equation for a neutrino.

We were able to show that this equation has the following properties:

- There exists a variational principle which yields (1).
- The equation is conformally invariant.
- In flat space the solutions to (1) satisfy the equation $\Box^{p+1}\psi = 0$. This equation is the key to the theory of these massless fields and although it is not strictly hyperbolic (only if $p = 0$) one can still give existence and uniqueness proofs.
- The Cauchy problem for (1) is well posed: given initial data on a spacelike hypersurface $S$, then there exists a strip $N = S \times [-T, T]$ on which (1) has a unique solution with the specified initial values.
- There exists an equivalent exact set for equation (1) , i.e., the characteristic initial value problem is formally well posed.
- The system coupled to gravity via $G_{ab} = 8\pi T_{ab}$ forms an exact set as well.
- The general solution in flat space is characterized by a totally symmetric dual twistor $\phi_{\alpha_1\ldots\alpha_p}(k_A, k_{A'})$ defined on the future null cone of the origin of Minkowski space. There is some gauge freedom left: let $K_\alpha$ be the operator pair $(k_A, \partial/\partial_{k_{A'}})$, then $\phi_{\alpha_1\ldots\alpha_p} + K_{(\alpha_1}\gamma_{\ldots\alpha_p)}$ defines the same solution as $\phi_{\alpha_1\ldots\alpha_p}$.
- More in the spirit of the twistor programme we have the result that analytic solutions of (1) are described by the sheaf cohomology group $H^1(U, O(p,p))$ for suitable domains in twistor space. $O(p,p)$ is the sheaf of germs of rank $p$ totally symmetric covariant tensors on projective twistor space taking values in $O(p)$.

Using the same type of spinor fields one can write down a system of equations which extend the Dirac equation and which have several of the above properties:

$$\begin{aligned}\partial_{A(A'}\psi^{AB\ldots C}_{B'\ldots C')} &= -\frac{\mu}{p+1}\chi^{\;B\ldots C}_{A'B'\ldots C'}\\ \partial_{A'(A}\chi^{A'B'\ldots C'}_{B\ldots C)} &= -\frac{\mu}{p+1}\psi^{B'\ldots C'}_{AB\ldots C}\end{aligned} \tag{2}$$

The first thing to note is, of course, that for $p = 0$ this is just the Dirac equation in two-component spinor notation. As in that case, we have as many complex equations as we have unknown complex functions, namely $2(p+1)(p+2)$. Next, it is easy to see that there are no constraints to be satisfied on an initial spacelike surface. If we decompose the covariant derivative operator into a timelike part $D$ and a spacelike part $D_{AB} = D_{BA}$ with respect to a unit timelike covector field $t_{AA'}$, $\partial_{AA'} = t_{AA'}D + t^B_{A'}D_{AB}$, then none of the equations is purely spatial. I.e., all equations contain the timelike derivative $D$.

In order to analyze the situation further I want to introduce the homogeneous (in $\pi^A$, $\pi^{A'}$) functions $\psi \equiv \psi_{AB\ldots CB'\ldots C'}\pi^A\pi^B\ldots\pi^C\pi^{B'}\ldots\pi^{C'}$ and $\chi$ defined in a similar way. Also we define the derivative operators $L \equiv \pi^A\pi^{A'}\partial_{AA'}$, $M \equiv \pi^A\partial^{A'}\partial_{AA'}$, $M' \equiv \partial^A\pi^{A'}\partial_{AA'}$ and $N \equiv \partial^A\partial^{A'}\partial_{AA'}$. Then it is easy to see that the system (2) can be written as

$$\begin{aligned} M'\psi &= -\mu\chi, \\ M\chi &= -\mu\psi. \end{aligned} \tag{3}$$

The derivative operators obey certain commutation relations involving the curvature of spacetime and the wave operator. In flat space these relations are trivial apart from $[L,N]\phi = -\frac{1}{2}(p+p'+2)\Box\phi$, $[M,M']\phi = -\frac{1}{2}(p-p')\Box\phi$ and the relation $LN\phi - MM'\phi = \frac{1}{2}p(p'+1)\Box\phi$ for a function $\phi$ with homogeneities $(p,p')$.

Using this formalism it is quite easy to derive the following curious result for the flat case, which is the analogue to the "key equation" in the massless case mentioned above.

**Theorem:** If $(\psi,\chi)$ is a $p$-solution, i.e., functions with resp. homogeneities $(p+1,p)$ and $(p,p+1)$ satisfying the system (3) then they also satisfy the equation ($m^2 = 2\mu^2$):

$$\left(\Box + m^2\right)\left(\Box + \frac{m^2}{4}\right)\ldots\left(\Box + \frac{m^2}{(p+1)^2}\right)\phi = 0. \tag{4}$$

*Proof:* The proof is by induction on $p$. For $p = 0$ we have the Dirac system and so we obtain $(\Box + m^2)\psi = (\Box + m^2)\chi = 0$. Suppose the claim is true for $p$-solutions and let $(\psi,\chi)$ be a $(p+1)$-solution. Then $(N\psi, N\chi)$ is a $p$-solution because $N$ commutes with $M$ and $M'$. Therefore, defining $P \equiv \prod_{j=0}^{p}(\Box + m^2/(j+1)^2)$, we have $P(N\psi) = 0$. Since $L$ commutes with $\Box$ and using the relation above we also have $0 = P(LN\psi) = P(MM' + \frac{1}{2}(p+2)^2\Box)\psi$. Using (3) we finally obtain $\prod_{j=0}^{p+1}(\Box + m^2/(j+1)^2)\,\psi = 0$. The same argument applies to $\chi$ and so the proof is complete.

So the fields $(\psi,\chi)$ represent some kind of "mass multiplet". In the curved case the right hand side is no longer zero but contains (derivatives of) the curvature and lower order derivatives of the fields. Just like in the massless case, the differential operator on the left hand side is not strictly hyperbolic. It is, however, the product of strictly hyperbolic operators. This property enables us to use the existence theory of Leray and Ohya [3] to prove local existence and uniqueness of solutions to the Cauchy problem for this system (the details will be given elsewhere).

In the same way as in [1] we can also prove that there exists an equivalent exact set for the system (2) and thus we arrive at the statement that the characteristic initial value problem is formally well posed. The null data to be prescribed consist of the following fields

$$\left\{M^{l-j}N^j\psi, M'^{l-j}N^j\chi : 0 \le l \le p, 0 \le j \le l\right\}$$

which makes $(p+1)(p+2)$ freely specifyable functions. This is in agreement with the observation that the number of characteristic data is half the number of Cauchy data. In order to make the same statement for the system coupled to gravity, we need to define an energy momentum tensor for (2). But this is easily done, once we have established a variational principle. Consider then the four-form

$$\mathcal{L} = \mathrm{Im}\left\{\bar\psi^{A'B'\ldots C'}_{B\ldots C}\partial_{A'A}\psi^{AB\ldots C}_{B'\ldots C'} - \bar\chi^{A'B'\ldots C'}_{B\ldots C}\partial_{AA'}\chi^{AB\ldots C}_{B'\ldots C'} + \left(\frac{\mu}{p+1}\right)\bar\psi^{A'B'\ldots C'}_{B\ldots C}\chi^{AB\ldots C}_{B'\ldots C'}\right\}\Sigma,$$

where $\Sigma$ is the volume form of spacetime. Then we define the action $\mathcal{A}[\psi,\bar\psi,\chi,\bar\chi,\theta^a] \equiv \int_M \mathcal{L}$. The dependence on a tetrad (or more appropriately the canonical one-form of the bundle of orthonormal frames) $\theta^a$ is implicit in $\Sigma$ and via the torsion free condition also in the connection. Varying $\mathcal{A}$ with respect to $\bar\psi$ and $\bar\chi$ yields the system (2), variation with respect to $\psi$ and $\chi$ gives the complex conjugate system and the variation with respect to $\theta^a$ contains the energy momentum tensor. This is explained in more detail in [1].

Using the Einstein equation $G_{ab} = 8\pi T_{ab}$ we can consider the tracefree part of the Ricci tensor $\Phi_{ab}$ and the scalar curvature $\Lambda$ as expressed in terms of the fields $\psi$, $\chi$ and their first derivatives. Then adding the Bianchi identity

$$\partial^A_{A'}\Psi_{ABCD} = \partial^{B'}_{(B}\Phi_{CD)A'B'}$$

to the system, we can prove the existence of an equivalent exact set for the coupled system using the same type of recursive argument as in [1]. This will also be given in detail elsewhere.

The system (2) is another example of a system that is not symmetric hyperbolic but still gives rise to an exact set.

Finally, I want to briefly discuss the structure of the solution space of the system (3) in the flat case. This, however, should be considered only as a presentation of preliminary results. Let me denote by $[p]$ the space of $p$-solutions of (3). Given a $p$-solution we can construct a $(p+1)$-solution by taking a symmetrized derivative, i.e., by applying the operator $L$. Thus $L[p-1] \subset [p]$. On the other hand, taking the divergence of a $p$-solution by applying $N$ results in a $(p-1)$-solution and so

$N[p] \subset [p-1]$. Let $(\psi, \chi)$ be a $p$-solution and consider the identity $LN\psi - MM'\psi = \frac{1}{2}(p+1)^2\Box\psi$. Using (3) we get $LN\psi = \frac{1}{2}(p+1)^2\,(\Box + m^2/(p+1)^2)\,\psi$ and so we find that the operator $(\Box + m^2/(p+1)^2)$ maps a $p$-solution into the image of $L$.

The following picture therefore emerges: in $[p]$ there exists a distinguished class of solutions, the image of $[p-1]$ under the map $L$. These are considered as "unimportant" or gauge and we consider each $p$-solution as being defined only up to an element of $L[p-1]$. This leads to the factor space $F_p \equiv [p]\Big|_{L[p-1]}$. Since $\Box$ commutes with $L$ it maps $L[p-1]$ into itself and is therefore a well defined operator on $F_p$ and we can define the Klein-Gordon operator $\Box + \frac{m^2}{(p+1)^2}$ on $F_p$. Due to the calculation above this operator vanishes identically on $F_p$. So we are led to consider the elements of the factor space $F_p$ as the higher spin analogues of the Dirac wave functions. They do fit into this picture because they are in the trivial factor space $[0]\Big|_{\{0\}}$.

Although there exist solutions to the system (2) on arbitrary curved manifolds, it remains to be seen how much of this factor space structure can be carried over to curved manifolds. Work on this is still in progress.

## References


[1] J. Frauendiener and G. A. J. Sparling. On a class of consistent linear higher spin equations on curved manifolds. *preprint*, 1994.

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[3] J. Leray and Y. Ohya. Systèmes linéaires, hyperboliques non-stricts. In M. Froissart, editor, *Hyperbolic equations and waves*, pages 283–322. Battelle Seattle 1968 Rencontres, Springer-Verlag Berlin, 1970.