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On the X-ray, Radon and Penrose transforms

It has long been known that there is a close link between the Penrose transform and the X-ray transform and, furthermore, that the twistor correspondence for Minkowski space $\mathbb{M}$ of signature $(2,2)$ is the background geometry associated to the Radon transform in 3-dimensions. In the last issue of $\mathbb{TN}$, I showed how the Radon transform could be understood as the standard Penrose transform globalized so that the (2,2) signature Minkowski space is double covered and twistor space is the non-Hausdorff space obtained by gluing together two copies of $\mathbb{CP}^3$ together over some small thickeing of $\mathbb{RP}^3$. In this note I show how to write the various Radon transform formulae (and their inverses) in an invariant way that motivates some generalizations and brings out the projective invariances of the correspondence.

The Radon transform is a transform from functions f on $\mathbb{RP}^n$ to functions g on $\mathbb{RP}^{n*}$, the space of hyperplanes in $\mathbb{RP}^n$. Let Z^α be homogeneous coordinates on $\mathbb{RP}^n$ and W_α homogeneous coordinates on the dual space $\mathbb{RP}^{n*}$. The function g at a point $[W] \in \mathbb{RP}^{n*}$ is obtained by integrating f over the corresponding plane $W \cdot Z = 0$ in $\mathbb{RP}^n$. More generally one can transform from functions f on $\mathbb{RP}^n$ to functions g on $Gr(k,n)$ the Grassmanian of projective k-planes in $\mathbb{RP}^n$ by integration of f over each projective k-plane. This can be reduced to the former type of Radon transform by restricting the correspondence to some $\mathbb{RP}^{k+1}$ in $\mathbb{RP}^n$ and its dual $\mathbb{RP}^{k+1*}$ in $Gr(k,n)$. Ordinarily this correpondence is treated affine linearly (i.e. the points at infinity are thrown away) and the ordinary Euclidean measure on $\mathbb{R}^n$ is used. However, it is clear that the transform is projectively invariant.

In order to be able to integrate these functions invariantly we must introduce some line bundles: $\mathbb{RP}^n$ has two families of line bundles on it, $\mathcal{O}(p)$ and $\tilde{\mathcal{O}}(p)$. Let Z^α be homogeneous coordinates on $\mathbb{RP}^n$, then sections f of $\mathcal{O}(p)$ and $\tilde{\mathcal{O}}(p)$ are homogeneous functions of weight p, $f(aZ) = a^p f(Z)$ for $a \in \mathbb{R}^+$, but sections of $\mathcal{O}(p)$ satisfy $f(-Z) = (-1)^p f(Z)$ whereas sections of $\tilde{\mathcal{O}}(p)$ satisfy $f(-Z) = (-1)^{p-1} f(Z)$. Clearly, on restriction to a projective linear subspace, these line bundles become the corresponding bundles for that subspace.

The bundle of densities (i.e. quantities that can be integrated) on $\mathbb{RP}^n$ is $\mathcal{O}(-n-1)$ in odd dimensions, and $\tilde{\mathcal{O}}(-n-1)$ in even dimensions. One can think of integration in the non-orientable even-dimensional case as being performed by taking the section $\tilde{f}$ of $\tilde{\mathcal{O}}(-n-1)$, multiplying it by $\varepsilon_{\alpha\beta\cdots\delta} Z^\alpha dZ^\beta{}_\wedge \ldots{}_\wedge\, dZ^\delta$, (where $\varepsilon_{\alpha_0\cdots\alpha_n} = \varepsilon_{[\alpha_0\cdots\alpha_n]}$ is the volume element on $\mathbb{R}^{n+1}$) and then pulling back to the sphere $S^n \longrightarrow \mathbb{RP}^n$ and integrating on the sphere and then finally dividing the result by 2 to take account of the double cover. If we had used instead a section of $\mathcal{O}(-n-1)$ we would have automatically obtained zero.

Since one is integrating over codimension one projective hyperplanes, the integrand has to be a density for the hyperplanes. The standard Radon transform in even dimensions is therefore a map

$$\mathcal{R} : \Gamma(\mathbb{RP}^n, \mathcal{O}(-n)) \to \Gamma(\mathbb{RP}^{n*}, \tilde{\mathcal{O}}(-1))$$

and in odd dimensions

$$\mathcal{R} : \Gamma(\mathbb{RP}^n, \tilde{\mathcal{O}}(-n)) \to \Gamma(\mathbb{RP}^n, \tilde{\mathcal{O}}(-1)).$$

To see this invariantly, let W_α be homogeneous coordinates on $\mathbb{RP}^{n*}$, then on restriction to $Z \cdot W = 0$, we have:

$$\varepsilon_{\alpha\beta\gamma\cdots\delta} Z^\beta dZ^\gamma{}_\wedge \cdots{}_\wedge dZ^\delta|_{Z\cdot W=0} = W_\alpha \nu$$

for some $(n-1)$–form ν of weight n in Z and -1 in W. This follows from ε identities and $W_\beta Z^{[\beta} dZ^\gamma{}_\wedge \cdots{}_\wedge dZ^{\delta]}|_{Z\cdot W=0} = 0$ which in turn follows from $W \cdot Z = 0$ and $W \cdot dZ = 0$ on $W \cdot Z = 0$.

Let $f \in \Gamma(\mathbb{RP}^n, \mathcal{O}(-n))$ for n even or $f \in \Gamma(\mathbb{RP}^n, \tilde{\mathcal{O}}(-n))$ for n odd and then define $\mathcal{R}f(W)$ by:

$$W_\alpha \mathcal{R}f(W) = \int_{Z\cdot W=0} f \varepsilon_{\alpha\beta\gamma\cdots\delta} Z^\beta dZ^\gamma{}_\wedge \cdots{}_\wedge dZ^\delta.$$

It is clear that $\mathcal{R}f$ must have homogeneity -1 since the left hand side of the equation must have overall homogeneity zero. That it is a section of $\tilde{\mathcal{O}}(-1)$ rather than $\mathcal{O}(-1)$ follows from the fact that a choice of orientation for the plane $Z \cdot W = 0$ needs to be made in order to integrate. Such a choice can be made by a choice of $W_\alpha \in S^n$ covering $[W]$ in $\mathbb{RP}^n$ since that provides a normal direction to the plane $W \cdot Z = 0$ in $\mathbb{R}^{n+1}$ and hence to its intersection with S^n on which we can perform the integration (as mentioned above). Clearly the sign of the integral reverses if the sign of W and hence the orientation is reversed. This means that $\mathcal{R}f$ does not reverse its sign under $W \to -W$ as required for a section of $\tilde{\mathcal{O}}(-1)$.

Generalizations

One can write down generalizations in a similar spirit to the formulae for the Penrose transform for different helicities. For $f \in \Gamma(\mathbb{RP}^n, \mathcal{O}(-s-n))$, n even, or $f \in \Gamma(\mathbb{RP}^n, \tilde{\mathcal{O}}(-s-n))$, n odd, and $s > 0$, we have that there exists g such that:

$$W_{\beta_0} \partial^{\alpha_1} \cdots \partial^{\alpha_s} g(W) = \int_{Z\cdot W=0} Z^{\alpha_1} Z^{\alpha_2} \cdots Z^{\alpha_s} f(Z) \varepsilon_{\beta_0\beta_1\cdots\beta_n} Z^{\beta_1} dZ^{\beta_2}{}_\wedge \cdots{}_\wedge dZ^{\beta_n}$$

where $g \in \Gamma(\mathbb{RP}^{n*}, \tilde{\mathcal{O}}(s-1))$. For $s < 0$ we have:

$$W_{\beta_0} W_{\alpha_1} \cdots W_{\alpha_{|s|}} g(W) = \int_{Z\cdot W=0} \partial_{\alpha_1} \partial_{\alpha_2} \cdots \partial_{\alpha_{|s|}} f(Z) \varepsilon_{\beta_0\beta_1\cdots\beta_n} Z^{\beta_1} dZ^{\beta_2}{}_\wedge \cdots{}_\wedge dZ^{\beta_n}$$

where again $g \in \Gamma(\mathbb{RP}^{n*}, \tilde{\mathcal{O}}(s-1))$. To see that such a g exists, in both formulae it is sufficient to prove it in the $s = 1$ case since we can reduce the general case to this using the symmetry of the integrand over the $\alpha_1 \cdots \alpha_s$ indices and by contracting the $s-1$ other free indices off with constants A_{α_2} etc.. In the second formula it suffices to show that the integral vanishes when the β_0 and α_1 indices are skewed since the right hand side is manifestly proportional to W_{β_0}. This follows by expressing the integrand skewed over β_0 and α_1 as an exact form $d(f \varepsilon_{\alpha_1\beta_0\beta_1\cdots\beta_{n-1}} Z^{\beta_1} dZ^{\beta_2}{}_\wedge \cdots{}_\wedge dZ^{\beta_{n-1}})$ so that the integral vanishes. For the first formula we must show that $\partial^{[\alpha} g^{\beta]} = 0$ where g^α is defined by:

$$W_\alpha g^{\alpha_1}(W) = \int_{Z\cdot W=0} Z^{\alpha_1} f(Z) \varepsilon_{\alpha\beta\gamma\cdots\delta} Z^\beta dZ^\gamma{}_\wedge \cdots{}_\wedge dZ^\delta.$$

We use the fact that the integral of the Lie derivative of the integrand along $Z^\beta\partial_\gamma$ is $W_\gamma\partial^\beta$ acting on the integral (this follows from $SL(n,\mathbb{R})$ equivariance of the integral). Again one can show, with some manipulation, that $W_\alpha W_\beta \partial^{[\gamma} g^{\delta]}$ is given by the integral of an exact (indexed) form.

It is perhaps worth noting that the g in the first formula is unique as the kernel of the operator $\partial^\alpha\cdots\partial^\gamma$ consists of polynomials, but one cannot construct sections of $\tilde{\mathcal{O}}(r)$ out of polynomials as their sign does not change appropriately under inversion. The second formula, however, will have kernel consisting of homogeneous polynomials of degree $-s-n$.

The inversion formulae and further generalizations

The difference between the Radon transform in odd and even dimensions is perhaps most manifest in the inversion formulae. In odd dimensions, the Radon transform and the above generalizations takes one from $\Gamma(\tilde{\mathcal{O}}(-n-s))$ to $\Gamma(\tilde{\mathcal{O}}(s-1))$. This is inverted up to a constant overall factor by the above generalized transform with s replaced by $1-n-s$. This clearly cannot work in even dimensions as the image of the Radon transform are sections of $\tilde{\mathcal{O}}(s-1)$ which cannot be integrated over hyperplanes even after the homogeneity has been raised or lowered by differentiation etc..

In even dimensions the inversion formula is, in formal terms,

$$f(Z) = \int \frac{g(W)}{(Z\cdot W)^{s+n}} \varepsilon^{\beta_0\beta_1\cdots\beta_n} W_{\beta_0} dW_{\beta_1} \wedge \cdots \wedge dW_{\beta_n}$$

where $g \in \Gamma(\mathbb{RP}^{n*}, \tilde{\mathcal{O}}(s-1))$, $f \in \Gamma(\mathbb{RP}^n, \mathcal{O}(-s-n))$. This is a singular integral and needs to be regularized[1]. The fact that it is possible to regularize it invariantly follows from the existence of the forward transform. When $s \le -n$ the formula must be understood by its $(1-s-n)$'th derivative with respect to Z^α or alternatively with an additional factor of $\log(Z\cdot W)$ in the integrand and and a polynomial ambiguity in the result for f.

This formula completes the story for scalar weighted functions in even dimensions. However, in odd dimensions it leads to a new transform

$$\widetilde{\mathcal{R}} : \Gamma(\mathbb{RP}^{n*}, \mathcal{O}(s-1)) \to \Gamma(\mathbb{RP}^n, \mathcal{O}(-s-n))$$

given by the above formula except with $g \in \Gamma(\mathbb{RP}^{n*}, \mathcal{O}(s-1))$ as is required for integration in odd dimensions and $f \in \Gamma(\mathbb{RP}^n, \mathcal{O}(-s-n))$. One expects that it is nontrivial and is inverted by the same formula with s replaced by $1-s-n$. The same comments as above concerning ambiguities will apply for $s \le -n$.

[1] In order to have a rigourously invariant formulation one must demonstrate that this regularization can be performed invariantly. In recent work with T.N.Bailey it has been possible to prove this directly although an invariant proof of the inversion property is still lacking

Connections with twistor theory

In my article in the last TN, I showed how the X-ray transform could be understood as a globalization of the standard Penrose transform. The novelty of the above in this context is that it shows that one can obtain an inverse Penrose transform by choosing a plane in twistor space and using the above inversion formula on the corresponding β-plane in space-time. Fritz-John avoids the awkwardness of choosing a β-plane by integrating over all possible choices.

The second example is the twistor transform for Minkowski space of signature $(2,2)$ from cohomology classes on twistor space to dual twistor space. The cohomology has to be that of the sheaf $\mathcal{O}(n)$ and hence the corresponding functions on $\mathbb{RP}^3$ are sections of $\mathcal{O}(n)$ rather than $\tilde{\mathcal{O}}(n)$ thus the version of the transform thats relevant. Thus the twistor transform in this case will be implemented by the non-local formula of the last section.

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