Update on the massive propagator in twistor diagrams

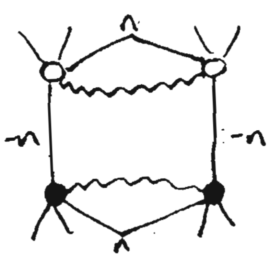
Here follows a sketch of recent developments regarding mass, following on from the ideas described in TN 32 for a possible solution to the problem of describing the massive Feynman propagator in terms of standard twistor diagram elements. The general idea there was to make finite sense of each term in the formal expression:

$$(\Box + m^2)^{-1} = \sum_0^\infty (-m^2)^n \, \Box^{-n-1}$$

by using the Barnes integral representation of the Feynman propagator function as evaluated on suitable test-fields, and then to find twistor integrals corresponding to each such term. The key new idea was to exploit the properties of the inhomogeneous *boundary* at infinity, i.e. a boundary on $\overline{XZ} = m$ or $\underline{WY} = m$, rather than use the poles $(\overline{XZ} - m)^{-1}$, $(\underline{WY} - m)^{-1}$ which had featured in my 1985 paper. This gets us from the "on-shell" amplitudes, i.e. Hankel functions satisfying the Klein-Gordon equation, to the more grown-up propagator "off-shell" functions. This is a particularly satisfactory idea because since 1985 it has become clear that boundaries of this kind are essential elements of the diagram formalism.

The scheme I indicated in TN32 was that $\Box^{-n-1}$

should correspond to the twistor integral



[$W \overset{n}{\frown} Y$ means: $(\underline{WY})^{-n-1}$

~~~ means: boundary on $\underline{WY} = m$]

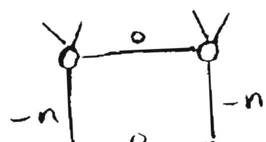
and I checked that the terms for n=0, n=1, were correct. One needs to choose k = exp (- γ), where γ is Euler's constant, if the mass parameter entering into the boundary specification is to be the same as that occurring in the formal power series (though this is not essential.) It seems something of a miracle that the n=1 term can be made to agree by a suitable contour choice: this is not something that follows just from its satisfaction of a differential relation.

Recently Stephen Spence has checked my calculations, and gone on to consider the case of n=2. He finds in that case the correct terms plus an unwanted extra term which is a multiple of k^2/m^2

The same feature occurred in my earlier (1985) work. That is, I worked out

[W —— Y

means $\dfrac{1}{(\underline{WY} - m)}$]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and found terms of this kind arising which prevented the correct Hankel function being formed. Indeed by considering the operation $\frac{d}{dm}$ which turns the boundary into a pole, one can see that these unwanted terms *must* arise in the new context.

In the 1985 work I found a way of getting rid of these terms, but at the cost of changing the diagram formalism. In those days it was not so clear how the inhomogeneous elements ought to be defined, and it seemed all right to abandon the natural "boundary" definition for a logarithmic expression. What I found can be expressed like this: we could get the right Hankel function if the (−n)-line were to be re-defined as

$$\frac{(w.z-\epsilon)^{n-1}}{(n-1)!}\left\{\log\left(\frac{w.z-\epsilon}{k}\right)-1-\frac{1}{2}-\frac{1}{3}\cdots-\frac{1}{n-1}\right\}$$

Now this expression is crying out for the limit $\epsilon \to 0$ to be taken, and indeed it's only in the $\epsilon = 0$ limit that we get exactly the right answer. Unfortunately the contours disappear at that limit!

It is fairly clear that the same thing could be applied in the new context. That is, if we were to use these logarithmic expressions, now attached to the inhomogeneous boundary, we should lose the unwanted terms and regain just the Feynman propagator function.

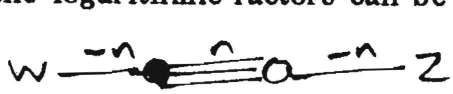
But what are we to make of these logarithmic factors, and of the contour that disappears in the limit? This has required a sequence of new ideas.

1. A contour re-emerges if we add another boundary line, i.e take the combination

$$\oint_{w.z=k}\frac{(w.z)^{n-1}}{(n-1)!}\left\{\log\left(\frac{w.z}{k}\right)-1-\frac{1}{2}-\frac{1}{3}\cdots-\frac{1}{n-1}\right\}$$

2. Observe that  $= \log\left(\frac{w.z}{k}\right)$

This suggests that the logarithmic factors can be supplied as required by taking

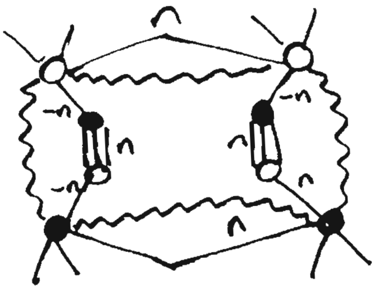


This is not quite the case, as in fact this is

$$\frac{(w.z)^{n-1}}{(n-1)!}\left\{\log\left(\frac{w.z}{k}\right)-1-\frac{1}{2}\cdots-\frac{1}{n-1}\right\}-\sum_{r=0}^{n-2}\frac{k^{n-1-r}(-1)^r(w.z)^r}{r!(n-1-r)!(n-1-r)}$$

However, these extra terms don't affect the subsequent integration and so it seems that

$$\sum_{n=0}^{\infty} m^{2n}$$



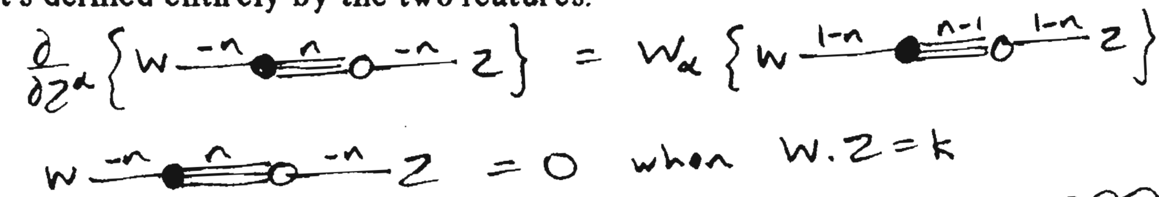
(✱)

gives the Feynman propagator function. The expression for 

looks appalling but it can be rewritten as $\displaystyle\int_k^{W.Z} dx_{n-1}\int_k^{x_{n-1}} dx_{n-2}\cdots\int_k^{x_3} dx_2\int_k^{x_2}\frac{dx_1}{x_1}$

$$= \int_k^{W.Z} \frac{(W.Z-x)^{n-1}}{(n-1)!}\,\frac{dx}{x}$$

and it's defined entirely by the two features:

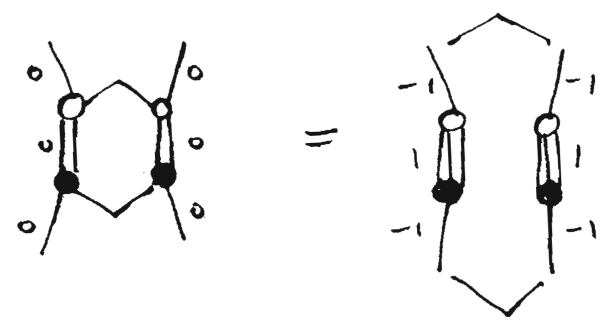


These two properties ensure that 

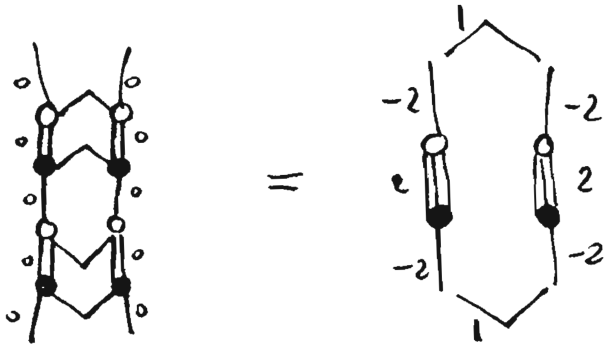
aand it follows from this that (✱) solves the inhomogeneous Klein-Gordon equation.

3. These observations succeed in giving us a scheme in which only standard diagram elements play a role. However a further development gets us much closer to a more fundamental picture in which the nth twistor diagram corresponds to the picture of n successive interactions with the constant Higgs field.

To do this note that

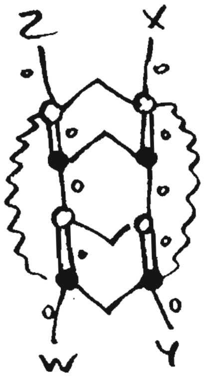


as one can see by an integration-by-parts argument. This might suggest that



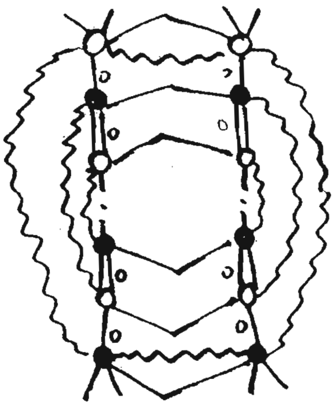
etc.

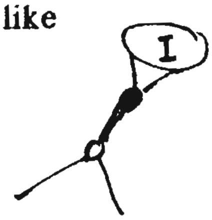
but in fact the requisite contours do not seem to exist for this. Instead, one must again add in further boundary lines. I have got as far as showing that



$$= \frac{1}{(WY\,XZ)^2}\left\{W.Z\left(\log\left(-\frac{W.Z}{k}\right)-1\right)\right\}\left\{Y.X\left(\log\left(-\frac{Y.X}{k}\right)-1\right)\right\}$$

by a method that should extend to the nth case. The upshot of this idea would be that each term in the series is represented by a diagram like



[each  is now  — an evaluation of the Higgs field]

which could then be modified to take correct account of the spin-projection which has so far been neglected, and to adapt to the case of spins other than zero.

It is encouraging that these contour-integral techniques are similar to those used by me and Lewis O'Donald in earlier work on ultra-violet divergences (his article in TN32, and his thesis). This suggests to me that the problem of representing general Feynman diagrams will be solved by exploiting these same ideas.

Refs:

Proc. Roy. Soc. A397, 375-396 (1985)

S.T. Spence: Qualifying dissertation (1994) (to whom many thanks!)

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