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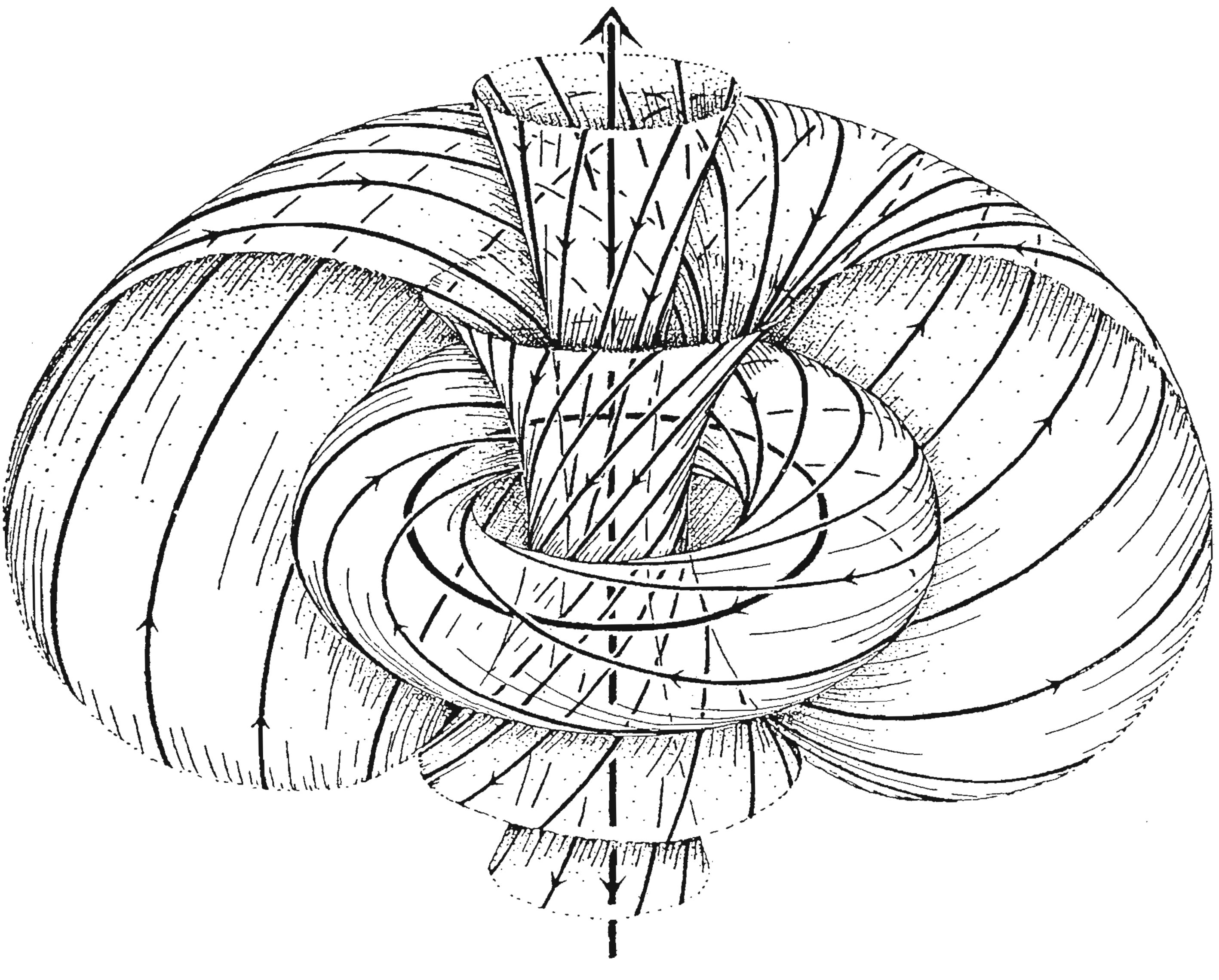

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Spin 3/2 fields and local twistors

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1 Introduction

In TN's 31 and 33 one of us (RP) proposed that helicity 3/2 fields might provide a suitable vehicle for the definition of a twistor in vacuum space-times. On the one hand, in flat space-time, twistors emerge as the charges of helicity 3/2-fields, and on the other, the vacuum equations are the consistency condition for the existence of such fields (in potential form, e.g. that given by Rarita-Schwinger) so that they can exist only when the field equations are satisfied.

There are still fundamental obstacles. The difficulties of defining charges in curved space were discussed in Penrose (1992). However, there are intriguing issues that are already present in a flat background. We have to deal with the description of the field as a potential modulo gauge rather than as field. (The helicity 3/2 'field' associated with a Rarita-Schwinger potential is not gauge invariant in curved space.) Thus, a deeper examination of the R-S potential, its gauge freedom, and its relation to twistor theory should prove fruitful.

A neat form of the R-S equations can be given in terms of spinor-indexed forms. Let $\sigma_{A'}$ be a spinor-indexed 1-form and d be the exterior derivative extended to act on spinor-indexed quantities. Then the gauge freedom is $\sigma_{A'} \to \sigma_{A'} + d\nu_{A'}$ where $\nu_{A'}$ is a spinor-indexed function, and the R-S equations are

$$dx^{AA'} \wedge d\sigma_{A'} = 0. \tag{1.1}$$

One can see directly that the gauge transformations will be consistent with the field equations iff

$$dx^{AA'} \wedge d^2\nu_{A'} = dx^{AA'} \wedge R^{B'}_{A'}\nu_{A'} = 0$$

for all $\nu_{A'}$. This follows iff the Ricci tensor vanishes (this is effectively the same calculation as that required to show that the Sparling 3-form is closed when the vacuum equations are satisfied—see Penrose & Rindler 1986, Vol 2 chapter 6). There are 8 equations for 8 unknowns, and two free functions worth of gauge freedom, so we require two relations between the field equations for the system to be consistent (this can be seen by removing two of the unknowns using the gauge freedom to set, for example, $V \rfloor \sigma_{A'} = 0$ for some vector field V so that there are 2 more equations than unknowns). The relations follow by taking the exterior derivative of (1.1). This vanishes identically iff $dx^{AA'} \wedge R^{B'}_{A'} = 0$ so the equations are consistent in vacuum.

In flat space, the field is given by

$$d\sigma_{A'} = \psi_{A'B'C'} dx^{BB'} \wedge dx_B^{C'}$$

and is gauge invariant, but in curved space a gauge transformation will add $\Psi_{A'B'C'D'}\nu^{D'}$ to $\psi_{A'B'C'}$, making it non-invariant.

Since the helicity 3/2 field is only directly descibed in potential form in curved space, in Penrose (1992), a description of charges was given that only involves the potential. We shall consider the general case with a 1-form potential γ for the dual field ${}^*F$ possibly taking values in some flat vector bundle. Introduce a covering of the relevant region of space-time M by open sets $\{U_i\}$. The (dual) field on this region is equivalent to a collection of potentials γ_i on each U_i such that $d\gamma_i = {}^*F$ so that on $U_i \cap U_j$, γ_i and γ_j differ by a gauge transformation (the collection $\{\gamma_i\}$ are taken to be defined modulo global gauge transformations). The charge associated with $\{\gamma_i\}$ takes values in the space of *gauge freedom of the second kind.* For electromagnetism, this formulation follows from the de-Rham sequence for potentials of the dual field

$$0 \to \mathbb{C} \to \mathcal{E}(M) \to \Omega^1(M) \to \Omega^2(M) \to \ldots$$

where $\mathcal{E}(M)$ is the space of smooth functions on M and the second map is the injection of constant functions and the third and subsequent maps are the standard exterior derivative. The charge q of an electromagnetic field is obtained by integrating the (closed) 2-form ${}^*F$ over some given 2-surface, S. The closed 2-form ${}^*F$ is equivalent to a collection $\{\gamma_i\}$ with $\gamma_i \in \Omega^1(U_i)$ and $d(\gamma_i - \gamma_j) = 0$ on $U_i \cap U_j$ defined up to $\gamma_i \to \gamma_i + df_i$ for some $f_i \in C^\infty(M)$. By diagram chasing (or direct integration), one can see that q lies in the $\mathbb{C}$ of the above sequence playing the role of a *gauge transformation of the second kind*, which is to say a gauge transformation that leaves the *potential* invariant. This follows by defining f_{ij} by $\gamma_i - \gamma_j = df_{ij}$. It then follows, with $q_{ijk} = f_{ij} + f_{jk} - f_{ik}$, that $dq_{ijk} = 0$ so q_{ijk} is a Čech cocycle with constant coefficients and one can see that its evaluation on S is the charge (up to the usual 4π etc.).

This point of view for helicity 3/2 fields led to a new difficulty even in flat space: if one only has the Rarita-Schwinger field, then the analogue of the above sequence is now

$$0 \to \mathbb{S}_{A'} \to \mathcal{E}_{A'} \to \Omega^1_{A'} \to \Omega^2_{A'} \to \ldots$$

where the third and fourth maps are the covariant exterior derivative. The problem is that the gauge transformations of the second kind are just constant spinors $\mathbb{S}_{A'}$ whereas we were hoping to obtain a whole twistor. This spinor is the secondary part of the charge of the corresponding helicity 3/2 field. In Penrose (1992,1994) it was noted that if one wishes to encode the primary part of the charge, one requires the next potential down the potential chain. A proposal for an exact sequence involving this second potential in the R-S case was also given in Penrose (1994).

The purpose of this note is to give an improved version of this sequence, with unrestricted gauge freedom, and relate it to local twistors.

2 Local twistors and the helicity 3/2 equations

In Woodhouse (1985) and Mason (1990) it was noted that the potential chains for zero rest mass fields have a natural formulation in terms of local twistors. Perhaps the simplest formulation of the potential chain for the helicity 3/2 equation is as follows

$$\psi_{A'B'C'} = \nabla^A_{A'}\sigma_{B'C'A}, \qquad \sigma_{AA'B'} = \nabla^B_{A'}\rho_{B'AB}, \qquad \rho_{ABA'} = \nabla^C_{A'}\phi_{ABC}$$

where each potential is a symmetric spinor, and is subject to field equations and gauge freedom (cf. Penrose and Rindler vol. 2, section 6.4).

A slightly different formulation of the potential chain, without its first and last parts, can be encoded in a local-twistor-valued 1-form with $n-2$ symmetric indices for helicity n. This is easily understood from the positive homogeneity twistor-function description: for helicity 3/2 one takes a dual twistor function R of homogeneity one and then evaluates $R^\alpha = \partial R/\partial W_\alpha$ as a self-dual Maxwell Field with a local twistor index. Denote this field by $\mathcal{R}^\alpha$. (Since, at this stage, we are working in conformally flat space-time, the bundle of local twistors is flat and so we can just evaluate such an object in a covariantly constant frame and then transform to the standard one.) Then $\mathcal{R}^\alpha$ is defined modulo gauge transformations, $\mathcal{R}^\alpha \to \mathcal{R}^\alpha + DZ^\alpha$ where Z^α is an arbitrary function with values in local twistors and D denotes the exterior derivative extended to act on quantities with a local twistor index using the local twistor connection. We also have the self-duality equation $(D\mathcal{R}^\alpha)^- = 0$ where the superscript '$-$' on a 2-form denotes its ASD part. As a consequence of the fact that R^α is of the special form $\partial R/\partial W_\alpha$, $D\mathcal{R}^\alpha$ also has vanishing primary part. This follows from the fact that the 'field' integral formula for the primary part is the expression

$$\oint \frac{\partial^3 R}{\partial\tilde{\pi}_A\partial\tilde{\omega}^{A'}\partial\tilde{\omega}^{B'}}\tilde{\pi}^C d\tilde{\pi}_C.$$

However, this is of the the form of the integral of $\partial f_{-1}/\partial\tilde{\pi}_A\tilde{\pi}^B d\tilde{\pi}_B$ where f_{-1} has homogeneity -1 (and has indices) but this is an exact form being proportional to $d(\tilde{\pi}^A f_{-1})$ so that its integral vanishes.

This last condition can be written $X_{\alpha\beta}D\mathcal{R}^\beta = 0$ where

$$X_{\alpha\beta} = \begin{pmatrix} \varepsilon_{AB} & 0 \\ 0 & 0 \end{pmatrix} \tag{2.2}$$

is the 'position twistor' i.e. the canonical section (up to scale) of $\mathbb{T}_{[\alpha\beta]}$ over Minkowski space given by expressing points of Minkowski space as elements of the projectivisation of $\mathbb{T}_{[\alpha\beta]}$ via the Klein correspondence.

We have that the primary and secondary parts are the potentials for the helicity 3/2 field as follows

$$\mathcal{R}^\alpha := \begin{pmatrix} \rho^A \\ \sigma_{A'} \end{pmatrix} := \begin{pmatrix} \rho^A_b dx^b \\ \sigma_{A'b}dx^b \end{pmatrix}.$$

The equations $(D\mathcal{R}^\alpha)^- = 0$ and $X_{\alpha\beta}D\mathcal{R}^\beta = 0$ become

$$d\rho^A + idx^{AA'}\wedge\sigma_{A'} = 0 \text{ and } (d\sigma_{A'})^- = 0.$$

The first equation can be seen to imply the second by first taking the covariant exterior derivative of the first (in flat space still) to obtain

$$dx^{AA'} \wedge d\sigma_{A'} = 0,$$

the form of the R-S equation given in the introduction, and then writing out this system in full. The helicity 3/2 field is then the secondary part of $D\mathcal{R}^\alpha$.

Thus, locally at least, $\mathcal{R}^\alpha$ modulo DZ^α satisfying $X_{\alpha\beta}D\mathcal{R}^\alpha = 0$ is equivalent to a helicity 3/2 field and is the $\mathcal{P}$-transform of $\partial R/\partial W_\alpha$.

This formulation of the potential for a helicity 3/2 field is sufficient to encode the full charge of the field in the context of gauge freedom of the second kind since we now have the exact sequence

$$0 \to \mathbb{T}^\alpha \to \mathcal{E}^\alpha \to \Omega^{1\alpha} \to \Omega^{2\alpha} \to \ldots$$

where the second map is the injection of the covariantly constant local twistors into general local twistor fields, the third and fourth maps are the exterior derivative extended to act on local twistors using the local twistor connection. Thus the gauge freedom of the second kind is given precisely by constant twistors. (A sequence of this nature was given in Penrose 1994, but with a restricted gauge freedom.)

3 Generalization to curved space

The above equations do not work as they stand in curved space as, when $D^2 \neq 0$, the equation $X_{\alpha\beta}D\mathcal{R}^\beta = 0$ is no longer compatible with the gauge freedom. However, a weaker equation is compatible with the gauge freedom in vacuum, namely

$$DX_{\alpha\beta} \wedge D\mathcal{R}^\beta = 0. \tag{3.3}$$

There is a subtlety here as $X_{\alpha\beta}$ is only defined up to scale by the conformal metric. The scale is fixed by choosing a particular conformal factor and representing $X_{\alpha\beta}$ by equation (2.2). Equivalently, a choice of conformal scale leads to an infinity twistor

$$I_{\alpha\beta} = \begin{pmatrix} 0 & 0 \\ 0 & \varepsilon^{A'B'} \end{pmatrix}$$

which can be used to normalize the position twistor by means of the relation

$$X_{\alpha\beta}I^{\alpha\beta} = 2.$$

Equation (3.3) is only conformally invariant when $X_{\alpha\beta}D\mathcal{R}^\alpha$ vanishes also. This additional equation can be consistent only in flat space, however.

The compatibility of the gauge freedom with equation (3.3) follows iff we have the relation $DX_{\alpha\beta}\wedge\mathcal{K}^\beta_\gamma = 0$ where $\mathcal{K}^\alpha_\beta$ is the curvature of the local twistor connection and $X_{\alpha\beta}$ is normalized as above by a choice of conformal factor. This relation holds iff $\nabla^A_{A'}\Psi_{ABCD} = 0$ and thus requires the Einstein equations. As before, there are as many equations as unknowns (in this case 16 rather than 8) but the gauge freedom can be used to set some (in this case 4) of the unknowns to zero leading to an overdetermined system. However, if the relation $DX_{\alpha\beta} \wedge \mathcal{K}^{\beta}_{\gamma} = 0$ holds, the covariant exterior derivative of equation (3.3) vanishes identically leading to four relations between the 16 equations so that they are not overdetermined in vacuum (or more generally when the trace free part of the Ricci tensor vanishes). (Conversely, if $\nabla^{A}_{A'}\Psi_{ABCD} \neq 0$ the equations are inconsistent.)

If we write out this system in terms of the primary and secondary parts of $\mathcal{R}^\alpha$, we obtain the equations

$$dx^{B'}_{A} \wedge d\rho^A + i dx^{B'}_{A} \wedge dx^{AA'} \wedge \sigma_{A'} = 0 \text{ and } dx^{AA'} \wedge \sigma_{A'} = 0.$$

In flat space we see that we acquire an extra helicity-3/2 field, since

$$D\mathcal{R}^\alpha = \begin{pmatrix} \tilde{\psi}^{A}_{BC} dx^{BB'} \wedge dx^{C}_{B'} \\ \psi_{A'B'C'} dx^{BB'} \wedge dx^{C'}_{B} \end{pmatrix}$$

with the ψ's totally symmetric and $\nabla^{A}_{A'}\tilde{\psi}_{ABC} = 0 = \nabla^{A'}_{A}\psi_{A'B'C'}$. Conversely given such ψ's, we can write down the twistor indexed 2-form as above which is closed and so can be written as $D\mathcal{R}^\alpha$ for some $\mathcal{R}^\alpha$ and automatically satisfies $DX_{\alpha\beta} \wedge D\mathcal{R}^\alpha = 0$.

However, the gauge freedom of the second kind is still just the constant local twistors, since the fields sit in the same exact sequence as the single helicity 3/2 fields did as before.

4 Where do we go from here?

Connections with other aspects of twistor theory

There are some intriguing connections with other aspects of twistor theory. Firstly, the connections between twistor theory and integrable systems highlight the role played by linear systems in twistor correspondences and so one might hope that the above linear system might lead to some kind of twistor construction. See Mason (1994) for an exploration of this line of reasoning. Secondly, the identities that guarantee the consistency of the R-S equations are precisely those that lead to the characterization of the vacuum equations by means of the Sparling 3-form. This suggests that a (local) twistorial analogue of the Sparling 3-form should be taken to be $DZ^\alpha \wedge DZ^\beta \wedge DX_{\alpha\beta}$. It is certainly closed iff the vacuum Bianchi identities are satisfied. This expression (with some modification to eliminate the terms that contain the Ricci tensor explicitly) also has an interpretation as a Hamiltonian that generates a translation along the vector field $\pi^{A'}\omega^A$ together with a spin-frame rotation generated by $\pi_{A'}\pi^{B'}$. These ideas should extend to a local twistor generalization of the connections between the Sparling 3-form, the canonical formalism and quasi-local mass described in Mason & Frauendiener (1990).

Towards a definition of twistors in vacuum space-times

The aforementioned considerations serve to clarify the role of twistors and their relationship to the R-S equations in flat space, by exhibiting them as gauge quantities of the second kind.

However, the full generalization of these ideas to Ricci-flat curved space remains elusive. So far, these considerations suffer from being 'too linear', since a twistor space without a vector space structure ought eventually to arise. One promising route to achieving this would be to examine a role for twistors as providing a 'charge' in the *active* sense rather than passive sense, similarly to the way in which the electric charge features in the electromagnetic connection $\nabla_a - ieA_a$. Combining the active with the passive roles for a twistor might provide a route to the required non-linearity. Work is in progress.

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Another view at the spin (3/2) equation

Jörg Frauendiener

In this note I want to draw attention to another point of view for the spin (3/2) equations which have been discussed in various places because they might provide a way of defining twistors for curved but Ricci flat spacetimes (see [4], [3], [2]). The two main properties are that the consistency condition for the existence of solutions are the vacuum equations $G_{ab} = 0$ and that in flat space twistors emerge as the charges of the fields.

Here I want to focus on the equation

$$\partial_{A(A'}\gamma^{AB}_{B')} = 0 \tag{1}$$

and show how it relates to the point of view in [3]. In particular, I will show how to obtain "charges" in a consistent way also in curved spacetimes. Recent work with George Sparling [1] shows that equation (1) has the following properties (among others): it is consistent in arbitrary curved spacetimes, i.e., the Cauchy problem is well posed; it is conformally invariant and can be obtained from the action

$$\mathcal{A} = \operatorname{Im}\left\{ \int \bar{\gamma}^{A'B'}_{B} \partial_{A'A} \gamma^{AB}_{B'} \right\} \Sigma.$$

As it stands (1) is insensitive towards the vacuum equations because it has solutions on arbitrary manifolds. Let me denote the solution space of (1) by [3/2] and the solution space of the Weyl neutrino equation by [1/2]. Note, that the Weyl equation also has a well posed Cauchy problem on arbitrary manifolds. In flat space we have the following structure relating these solution spaces. Given any solution of (1) we obtain a solution of the neutrino equation by taking its divergence: $\partial^{B'}_{B}\gamma^{AB}_{B'} \in [1/2]$ for all $\gamma \in [3/2]$. Call this map $N : [3/2] \to [1/2]$. On the other hand, given $\nu^A \in [1/2]$ we can obtain a solution of (1) by taking a symmetrized derivative, $\partial^{(A}_{A'}\nu^{B)} \in [3/2]$ for all $\nu \in [1/2]$. Call that map $L : [1/2] \to [3/2]$. It is easy to see that $\operatorname{im} L \subset \ker N$. So we get a sequence of maps

$$[1/2] \xrightarrow{L} [3/2] \xrightarrow{N} [1/2] \tag{2}$$

which is not exact. In fact, from a crude argument, counting free functions for the Cauchy problem, it can be seen that [3/2] is characterized by six free functions of three variables whereas [1/2] amounts to two free functions. This shows that $\ker N$ amounts to four functions so that there are two free functions that do not correspond to the image of L. This can be made more precise using the Fourier representation of solutions of (1) in flat space.

If we now ask how much of this sequence can be carried over to curved manifolds we find that the L-part can only be defined if the tracefree part of the Ricci tensor vanishes, i.e., only on spacetimes with $\Phi_{ab} = 0$ will the derivative of a neutrino field be a solution of equation (1). Similarly, the divergence of a solution of equation (1) will be a solution of the Weyl equation only if $\Phi_{ab} = 0$. If, in addition, we insist on the property that $\operatorname{im} L \subset \ker N$ then also the scalar curvature has to vanish. In summary then, we find that we have the same sequence (2) iff $G_{ab} = 0$.

At this stage, the natural question to be asked is, of course: why do the Einstein equations favour this structure? Surprisingly, it is exactly this structure that is necessary to define "charges" as surface integrals in the curved case. Recall, that equation (1) arises from a variational principle. This means that its solution space comes equipped with a natural symplectic structure. The symplectic form on [3/2] is

$$\omega(\gamma_1, \gamma_2) = i \int_{\mathcal{H}} \left(\bar{\gamma}_{1B}^{A'B'} \gamma_{2B'}^{AB} - \bar{\gamma}_{2B}^{A'B'} \gamma_{1B'}^{AB} \right) \Sigma_{AA'}.$$

The integral is taken over a hypersurface $\mathcal{H}$. It is hypersurface independent because the integrand is closed iff the γ's satisfy equation (1).

If we think of [1/2] as inducing transformations on [3/2] via $\gamma \mapsto \gamma + L\nu$, then we may ask for the Hamiltonians that generate these transformations. This means that we have to solve the equation

$$\omega(L\nu, \gamma) = -\delta H_\nu(\gamma) \qquad \text{for all } \gamma \in [3/2] \tag{3}$$

for H_ν, given $\nu \in [1/2]$. Now the left hand side is

$$\operatorname{Im} \int \partial_{B'}^{(A} \nu^{B)} \bar{\gamma}_B^{A'B'} \Sigma_{AA'}$$

which is equal to (up to factors)

$$\operatorname{Im} \int D\nu^B \wedge \bar{\gamma}_B^{A'B'} \Sigma_{A'B'},$$

$\Sigma_{A'B'}$ being the selfdual two-forms on spacetime and D being the covariant exterior derivative. Integrating by parts we find that the left hand side of (3) is

$$\omega(L\nu, \gamma) = \operatorname{Im} \int_{\partial\mathcal{H}} \nu^B \bar{\gamma}_B^{A'B'} \Sigma_{A'B'} - \operatorname{Im} \int_{\mathcal{H}} \nu^B D\bar{\gamma}_B^{A'B'} \wedge \Sigma_{A'B'},$$

consisting of a hypersurface integral and a two dimensional boundary integral. Examination of the hypersurface integrand shows that it is equal to $\left(\nu^A \partial_{B'}^B \bar{\gamma}_B^{A'B'} \Sigma_{AA'}\right)$. This shows that this integral will vanish iff we restrict ourselves to the subspace of [3/2] consisting of divergence free solutions of (1). The Einstein equations ensure that [1/2] acts on $\ker N$. The upshot of all this is that provided the Einstein equations hold, we have an action of [1/2] on the symplectic submanifold $\ker N$ which can be considered as gauge transformations. The corresponding Hamiltonians or "charges" are given by the surface integral

$$H_\nu(\gamma) = \operatorname{Im} \int_{\partial\mathcal{H}} \nu^B \bar{\gamma}_B^{A'B'} \Sigma_{A'B'},$$

where now γ is restricted to be a divergence free solution of (1), i.e., a solution of the equation $\partial_{AA'}\gamma^{AB}_{B'} = 0$.

Let us now see how this fits in with the flat space expression for the charges in [3]:

$$q = \int_{\partial\mathcal{H}} \mu^{C'} \psi_{A'B'C'} \Sigma^{A'B'}$$

where $\psi_{A'B'C'} = \partial_{AA'}\bar{\gamma}^A_{B'C'}$ is the helicity (3/2)-field defined by $\bar{\gamma}$ and $\mu^{A'}$ is the primary part of a dual twistor $(\mu^{A'}, \lambda_A)$, so $\partial_{AA'}\mu^{B'} = i\epsilon_{A'}{}^{B'}\lambda_A$ and $\partial_{AA'}\lambda_B = 0$. Consider now the complex one-form $\alpha = \bar{\gamma}_{A'B'A}\mu^{B'}\theta^{AA'}$. Then

$$\begin{aligned} d\alpha &= \partial_B^{A'}\left(\bar{\gamma}_{A'C'A}\mu^{C'}\right)\Sigma^{AB} + \partial^A_{B'}\left(\bar{\gamma}_{A'C'A}\mu^{C'}\right)\Sigma^{A'B'} \\ &= \left(\partial_B^{A'}\bar{\gamma}_{A'C'A}\right)\mu^{C'}\Sigma^{AB} + \bar{\gamma}_{A'C'A}\left(\partial_B^{A'}\mu^{C'}\right)\Sigma^{AB} \\ &\quad + \left(\partial^A_{B'}\bar{\gamma}_{A'C'A}\right)\mu^{C'}\Sigma^{A'B'} + \bar{\gamma}_{A'C'A}\left(\partial^A_{B'}\mu^{C'}\right)\Sigma^{A'B'}. \end{aligned}$$

The first two terms vanish because of (1) and the twistor equation and so

$$d\alpha = \psi_{A'B'C'}\mu^{C'}\Sigma^{A'B'} + i\bar{\gamma}_{A'B'A}\lambda^A\Sigma^{A'B'}.$$

Integration over $\partial\mathcal{H}$ shows that the flat space expression and the symplectic expression for the charges agree up to signs and taking the imaginary part if we identify ν^A and λ^A.

All this looks encouraging but there are still many open questions. Why do we get only an imaginary part in the symplectic charge integral? How do twistors appear in the curved space formula? What is the role of the solution ν^A of the Weyl equation in relation to the projection part of a dual twistor? Maybe some of these questions can be answered once the role of the second potential has been clarified.

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A higher spin generalization of the Dirac equation to arbitrary curved manifolds

Jörg Frauendiener

In this note I want to discuss a class of equations that can be considered as generalizations of the Dirac equation to higher spin fields. There have already been several proposals to this end (see [2] and references therein). The present approach grew out of joint work with George Sparling on higher spin massless spinor fields. In [1] we considered spinor fields of the type $\psi^{AB\ldots C}_{B'\ldots C'}$ with p primed and $p+1$ unprimed indices subject to the equation

$$\partial_{A(A'}\psi^{AB\ldots C}_{B'\ldots C')} \doteq 0. \tag{1}$$

For $p=0$ this is just the Weyl equation for a neutrino.

We were able to show that this equation has the following properties:
- There exists a variational principle which yields (1).
- The equation is conformally invariant.
- In flat space the solutions to (1) satisfy the equation $\Box^{p+1}\psi = 0$. This equation is the key to the theory of these massless fields and although it is not strictly hyperbolic (only if $p=0$) one can still give existence and uniqueness proofs.
- The Cauchy problem for (1) is well posed: given initial data on a spacelike hypersurface S, then there exists a strip $N = S\times[-T,T]$ on which (1) has a unique solution with the specified initial values.
- There exists an equivalent exact set for equation (1) , i.e., the characteristic initial value problem is formally well posed.
- The system coupled to gravity via $G_{ab} = 8\pi T_{ab}$ forms an exact set as well.
- The general solution in flat space is characterized by a totally symmetric dual twistor $\phi_{\alpha_1\ldots\alpha_p}(k_A, k_{A'})$ defined on the future null cone of the origin of Minkowski space. There is some gauge freedom left: let K_α be the operator pair $(k_A, \partial/\partial_{k_{A'}})$, then $\phi_{\alpha_1\ldots\alpha_p} + K_{(\alpha_1}\gamma_{\ldots\alpha_p)}$ defines the same solution as $\phi_{\alpha_1\ldots\alpha_p}$.
- More in the spirit of the twistor programme we have the result that analytic solutions of (1) are described by the sheaf cohomology group $H^1(U, O(p,p))$ for suitable domains in twistor space. $O(p,p)$ is the sheaf of germs of rank p totally symmetric covariant tensors on projective twistor space taking values in $O(p)$.

Using the same type of spinor fields one can write down a system of equations which extend the Dirac equation and which have several of the above properties:

$$\begin{aligned}\partial_{A(A'}\psi^{AB\ldots C}_{B'\ldots C')} &= -\frac{\mu}{p+1}\chi^{B\ldots C}_{A'B'\ldots C'}\\ \partial_{A'(A}\chi^{A'B'\ldots C'}_{B\ldots C)} &= -\frac{\mu}{p+1}\psi^{B'\ldots C'}_{AB\ldots C}\end{aligned} \tag{2}$$

The first thing to note is, of course, that for $p = 0$ this is just the Dirac equation in two-component spinor notation. As in that case, we have as many complex equations as we have unknown complex functions, namely $2(p+1)(p+2)$. Next, it is easy to see that there are no constraints to be satisfied on an initial spacelike surface. If we decompose the covariant derivative operator into a timelike part D and a spacelike part $D_{AB} = D_{BA}$ with respect to a unit timelike covector field $t_{AA'}$, $\partial_{AA'} = t_{AA'}D + t^B_{A'}D_{AB}$, then none of the equations is purely spatial. I.e., all equations contain the timelike derivative D.

In order to analyze the situation further I want to introduce the homogeneous (in π^A, $\pi^{A'}$) functions $\psi \equiv \psi_{AB\ldots CB'\ldots C'}\pi^A\pi^B\ldots\pi^C\pi^{B'}\ldots\pi^{C'}$ and χ defined in a similar way. Also we define the derivative operators $L \equiv \pi^A\pi^{A'}\partial_{AA'}$, $M \equiv \pi^A\partial^{A'}\partial_{AA'}$, $M' \equiv \partial^A\pi^{A'}\partial_{AA'}$ and $N \equiv \partial^A\partial^{A'}\partial_{AA'}$. Then it is easy to see that the system (2) can be written as

$$\begin{aligned} M'\psi &= -\mu\chi, \\ M\chi &= -\mu\psi. \end{aligned} \tag{3}$$

The derivative operators obey certain commutation relations involving the curvature of spacetime and the wave operator. In flat space these relations are trivial apart from $[L,N]\,\phi = -\frac{1}{2}(p+p'+2)\Box\phi$, $[M,M']\,\phi = -\frac{1}{2}(p-p')\Box\phi$ and the relation $LN\phi - MM'\phi = \frac{1}{2}p(p'+1)\Box\phi$ for a function ϕ with homogeneities (p,p').

Using this formalism it is quite easy to derive the following curious result for the flat case, which is the analogue to the "key equation" in the massless case mentioned above.

Theorem: If (ψ,χ) is a p-solution, i.e., functions with resp. homogeneities $(p+1,p)$ and $(p,p+1)$ satisfying the system (3) then they also satisfy the equation ($m^2 = 2\mu^2$):

$$\left(\Box + m^2\right)\left(\Box + \frac{m^2}{4}\right)\ldots\left(\Box + \frac{m^2}{(p+1)^2}\right)\phi = 0. \tag{4}$$

Proof: The proof is by induction on p. For $p = 0$ we have the Dirac system and so we obtain $(\Box + m^2)\psi = (\Box + m^2)\chi = 0$. Suppose the claim is true for p-solutions and let (ψ,χ) be a $(p+1)$-solution. Then $(N\psi, N\chi)$ is a p-solution because N commutes with M and M'. Therefore, defining $P \equiv \prod_{j=0}^{p}(\Box + m^2/(j+1)^2)$, we have $P(N\psi) = 0$. Since L commutes with $\Box$ and using the relation above we also have $0 = P(LN\psi) = P(MM' + \frac{1}{2}(p+2)^2\Box)\psi$. Using (3) we finally obtain $\prod_{j=0}^{p+1}(\Box + m^2/(j+1)^2)\,\psi = 0$. The same argument applies to χ and so the proof is complete.

So the fields (ψ,χ) represent some kind of "mass multiplet". In the curved case the right hand side is no longer zero but contains (derivatives of) the curvature and lower order derivatives of the fields. Just like in the massless case, the differential operator on the left hand side is not strictly hyperbolic. It is, however, the product of strictly hyperbolic operators. This property enables us to use the existence theory of Leray and Ohya [3] to prove local existence and uniqueness of solutions to the Cauchy problem for this system (the details will be given elsewhere).

In the same way as in [1] we can also prove that there exists an equivalent exact set for the system (2) and thus we arrive at the statement that the characteristic initial value problem is formally well posed. The null data to be prescribed consist of the following fields

$$\left\{M^{l-j}N^j\psi, M'^{l-j}N^j\chi : 0 \le l \le p, 0 \le j \le l\right\}$$

which makes $(p+1)(p+2)$ freely specifyable functions. This is in agreement with the observation that the number of characteristic data is half the number of Cauchy data. In order to make the same statement for the system coupled to gravity, we need to define an energy momentum tensor for (2). But this is easily done, once we have established a variational principle. Consider then the four-form

$$\mathcal{L} = \operatorname{Im}\left\{\bar{\psi}^{A'B'\ldots C'}_{B\ldots C}\,\partial_{A'A}\psi^{AB\ldots C}_{B'\ldots C'} - \bar{\chi}^{A'B'\ldots C'}_{B\ldots C}\,\partial_{AA'}\chi^{AB\ldots C}_{B'\ldots C'} + \left(\frac{\mu}{p+1}\right)\bar{\psi}^{A'B'\ldots C'}_{B\ldots C}\,\chi^{AB\ldots C}_{B'\ldots C'}\right\}\Sigma,$$

where Σ is the volume form of spacetime. Then we define the action $\mathcal{A}[\psi,\bar{\psi},\chi,\bar{\chi},\theta^a] \equiv \int_M \mathcal{L}$. The dependence on a tetrad (or more appropriately the canonical one-form of the bundle of orthonormal frames) θ^a is implicit in Σ and via the torsion free condition also in the connection. Varying $\mathcal{A}$ with respect to $\bar{\psi}$ and $\bar{\chi}$ yields the system (2), variation with respect to ψ and χ gives the complex conjugate system and the variation with respect to θ^a contains the energy momentum tensor. This is explained in more detail in [1].

Using the Einstein equation $G_{ab} = 8\pi T_{ab}$ we can consider the tracefree part of the Ricci tensor Φ_{ab} and the scalar curvature Λ as expressed in terms of the fields ψ, χ and their first derivatives. Then adding the Bianchi identity

$$\partial^A_{A'}\Psi_{ABCD} = \partial^{B'}_{(B}\Phi_{CD)A'B'}$$

to the system, we can prove the existence of an equivalent exact set for the coupled system using the same type of recursive argument as in [1]. This will also be given in detail elsewhere.

The system (2) is another example of a system that is not symmetric hyperbolic but still gives rise to an exact set.

Finally, I want to briefly discuss the structure of the solution space of the system (3) in the flat case. This, however, should be considered only as a presentation of preliminary results. Let me denote by $[p]$ the space of p-solutions of (3). Given a p-solution we can construct a $(p+1)$-solution by taking a symmetrized derivative, i.e., by applying the operator L. Thus $L[p-1] \subset [p]$. On the other hand, taking the divergence of a p-solution by applying N results in a $(p-1)$-solution and so

$N[p] \subset [p-1]$. Let (ψ, χ) be a p-solution and consider the identity $LN\psi - MM'\psi = \frac{1}{2}(p+1)^2\Box\psi$. Using (3) we get $LN\psi = \frac{1}{2}(p+1)^2\left(\Box + m^2/(p+1)^2\right)\psi$ and so we find that the operator $(\Box + m^2/(p+1)^2)$ maps a p-solution into the image of L.

The following picture therefore emerges: in $[p]$ there exists a distinguished class of solutions, the image of $[p-1]$ under the map L. These are considered as "unimportant" or gauge and we consider each p-solution as being defined only up to an element of $L[p-1]$. This leads to the factor space $F_p \equiv [p]\Big|_{L[p-1]}$. Since $\Box$ commutes with L it maps $L[p-1]$ into itself and is therefore a well defined operator on F_p and we can define the Klein-Gordon operator $\Box + \frac{m^2}{(p+1)^2}$ on F_p. Due to the calculation above this operator vanishes identically on F_p. So we are led to consider the elements of the factor space F_p as the higher spin analogues of the Dirac wave functions. They do fit into this picture because they are in the trivial factor space $[0]\Big|_{\{0\}}$.

Although there exist solutions to the system (2) on arbitrary curved manifolds, it remains to be seen how much of this factor space structure can be carried over to curved manifolds. Work on this is still in progress.

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J. Frauendiener

Twistors and the Einstein Equations

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Abstract It has been proposed that the appropriate global definition of a twistor, applicable to general curved vacuum space-times, would be as a charge for a massless field of helicity $^3/_2$. In flat space-time, using the Dirac form of these potentials, these twistor charges arise as the "gauge freedom of the second kind" in a long exact sequence involving the first and second potentials for the field.

A construction due to Ward is recalled, in which potentials for massless fields can act as partial connections on non-linear bundles, integrable on β-planes. This is generalized, in the case of helicity $^3/_2$, to provide a full connection on a vector bundle of rank 3, leading to an expression whereby the usual Rarita-Schwinger potential is supplemented by a second potential.

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NEW MASSLESS FREE FIELDS IN OLD SPACETIMES

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Abstract

A kind of massless free field, a "symmetric recurrent" spinor field is defined. The principal spinors of such a field define shear-free ray congruences. A vacuum solution of Einstein's equations is type {2,2},{4} or conformally flat iff its Weyl spinor is symmetric recurrent. The massless free fields of the Robinson-Sommers theorem are symmetric recurrent. A spacetime with a certain kind of symmetric recurrent spinor admits a Killing spinor. A massless free field associated with a Killing spinor is symmetric recurrent. Symmetric recurrent fields are constructed for spacetimes with certain types of Killing spinor, essentially one per Killing spinor.

to appear in 'Classical and Quantum Gravity'.

On the X-ray, Radon and Penrose transforms

It has long been known that there is a close link between the Penrose transform and the X-ray transform and, furthermore, that the twistor correspondence for Minkowski space $\mathbb{M}$ of signature $(2,2)$ is the background geometry associated to the Radon transform in 3-dimensions. In the last issue of $\mathbb{TN}$, I showed how the Radon transform could be understood as the standard Penrose transform globalized so that the (2,2) signature Minkowski space is double covered and twistor space is the non-Hausdorff space obtained by gluing together two copies of $\mathbb{CP}^3$ together over some small thickeing of $\mathbb{RP}^3$. In this note I show how to write the various Radon transform formulae (and their inverses) in an invariant way that motivates some generalizations and brings out the projective invariances of the correspondence.

The Radon transform is a transform from functions f on $\mathbb{RP}^n$ to functions g on $\mathbb{RP}^{n*}$, the space of hyperplanes in $\mathbb{RP}^n$. Let Z^α be homogeneous coordinates on $\mathbb{RP}^n$ and W_α homogeneous coordinates on the dual space $\mathbb{RP}^{n*}$. The function g at a point $[W] \in \mathbb{RP}^{n*}$ is obtained by integrating f over the corresponding plane $W \cdot Z = 0$ in $\mathbb{RP}^n$. More generally one can transform from functions f on $\mathbb{RP}^n$ to functions g on $Gr(k,n)$ the Grassmanian of projective k-planes in $\mathbb{RP}^n$ by integration of f over each projective k-plane. This can be reduced to the former type of Radon transform by restricting the correspondence to some $\mathbb{RP}^{k+1}$ in $\mathbb{RP}^n$ and its dual $\mathbb{RP}^{k+1*}$ in $Gr(k,n)$. Ordinarily this correpondence is treated affine linearly (i.e. the points at infinity are thrown away) and the ordinary Euclidean measure on $\mathbb{R}^n$ is used. However, it is clear that the transform is projectively invariant.

In order to be able to integrate these functions invariantly we must introduce some line bundles: $\mathbb{RP}^n$ has two families of line bundles on it, $\mathcal{O}(p)$ and $\tilde{\mathcal{O}}(p)$. Let Z^α be homogeneous coordinates on $\mathbb{RP}^n$, then sections f of $\mathcal{O}(p)$ and $\tilde{\mathcal{O}}(p)$ are homogeneous functions of weight p, $f(aZ) = a^p f(Z)$ for $a \in \mathbb{R}^+$, but sections of $\mathcal{O}(p)$ satisfy $f(-Z) = (-1)^p f(Z)$ whereas sections of $\tilde{\mathcal{O}}(p)$ satisfy $f(-Z) = (-1)^{p-1} f(Z)$. Clearly, on restriction to a projective linear subspace, these line bundles become the corresponding bundles for that subspace.

The bundle of densities (i.e. quantities that can be integrated) on $\mathbb{RP}^n$ is $\mathcal{O}(-n-1)$ in odd dimensions, and $\tilde{\mathcal{O}}(-n-1)$ in even dimensions. One can think of integration in the non-orientable even-dimensional case as being performed by taking the section $\tilde{f}$ of $\tilde{\mathcal{O}}(-n-1)$, multiplying it by $\varepsilon_{\alpha\beta\cdots\delta} Z^\alpha dZ^\beta{}_\wedge \ldots{}_\wedge dZ^\delta$, (where $\varepsilon_{\alpha_0\cdots\alpha_n} = \varepsilon_{[\alpha_0\cdots\alpha_n]}$ is the volume element on $\mathbb{R}^{n+1}$) and then pulling back to the sphere $S^n \to \mathbb{RP}^n$ and integrating on the sphere and then finally dividing the result by 2 to take account of the double cover. If we had used instead a section of $\mathcal{O}(-n-1)$ we would have automatically obtained zero.

Since one is integrating over codimension one projective hyperplanes, the integrand has to be a density for the hyperplanes. The standard Radon transform in even dimensions is therefore a map

$$\mathcal{R} : \Gamma(\mathbb{RP}^n, \mathcal{O}(-n)) \to \Gamma(\mathbb{RP}^{n*}, \tilde{\mathcal{O}}(-1))$$

and in odd dimensions

$$\mathcal{R} : \Gamma(\mathbb{RP}^n, \tilde{\mathcal{O}}(-n)) \to \Gamma(\mathbb{RP}^n, \tilde{\mathcal{O}}(-1)).$$

To see this invariantly, let W_α be homogeneous coordinates on $\mathbb{RP}^{n*}$, then on restriction to $Z \cdot W = 0$, we have:

$$\varepsilon_{\alpha\beta\gamma\cdots\delta} Z^\beta dZ^\gamma{}_\wedge \cdots{}_\wedge dZ^\delta|_{Z\cdot W=0} = W_\alpha \nu$$

for some $(n-1)$–form ν of weight n in Z and -1 in W. This follows from ε identities and $W_\beta Z^{[\beta} dZ^\gamma{}_\wedge \cdots{}_\wedge dZ^{\delta]}|_{Z\cdot W=0} = 0$ which in turn follows from $W \cdot Z = 0$ and $W \cdot dZ = 0$ on $W \cdot Z = 0$.

Let $f \in \Gamma(\mathbb{RP}^n, \mathcal{O}(-n))$ for n even or $f \in \Gamma(\mathbb{RP}^n, \tilde{\mathcal{O}}(-n))$ for n odd and then define $\mathcal{R}f(W)$ by:

$$W_\alpha \mathcal{R}f(W) = \int_{Z\cdot W=0} f \varepsilon_{\alpha\beta\gamma\cdots\delta} Z^\beta dZ^\gamma{}_\wedge \cdots{}_\wedge dZ^\delta.$$

It is clear that $\mathcal{R}f$ must have homogeneity -1 since the left hand side of the equation must have overall homogeneity zero. That it is a section of $\tilde{\mathcal{O}}(-1)$ rather than $\mathcal{O}(-1)$ follows from the fact that a choice of orientation for the plane $Z \cdot W = 0$ needs to be made in order to integrate. Such a choice can be made by a choice of $W_\alpha \in S^n$ covering $[W]$ in $\mathbb{RP}^n$ since that provides a normal direction to the plane $W \cdot Z = 0$ in $\mathbb{R}^{n+1}$ and hence to its intersection with S^n on which we can perform the integration (as mentioned above). Clearly the sign of the integral reverses if the sign of W and hence the orientation is reversed. This means that $\mathcal{R}f$ does not reverse its sign under $W \to -W$ as required for a section of $\tilde{\mathcal{O}}(-1)$.

Generalizations

One can write down generalizations in a similar spirit to the formulae for the Penrose transform for different helicities. For $f \in \Gamma(\mathbb{RP}^n, \mathcal{O}(-s-n))$, n even, or $f \in \Gamma(\mathbb{RP}^n, \tilde{\mathcal{O}}(-s-n))$, n odd, and $s > 0$, we have that there exists g such that:

$$W_{\beta_0} \partial^{\alpha_1} \cdots \partial^{\alpha_s} g(W) = \int_{Z\cdot W=0} Z^{\alpha_1} Z^{\alpha_2} \cdots Z^{\alpha_s} f(Z) \varepsilon_{\beta_0\beta_1\cdots\beta_n} Z^{\beta_1} dZ^{\beta_2}{}_\wedge \cdots{}_\wedge dZ^{\beta_n}$$

where $g \in \Gamma(\mathbb{RP}^{n*}, \tilde{\mathcal{O}}(s-1))$. For $s < 0$ we have:

$$W_{\beta_0} W_{\alpha_1} \cdots W_{\alpha_{|s|}} g(W) = \int_{Z\cdot W=0} \partial_{\alpha_1} \partial_{\alpha_2} \cdots \partial_{\alpha_{|s|}} f(Z) \varepsilon_{\beta_0\beta_1\cdots\beta_n} Z^{\beta_1} dZ^{\beta_2}{}_\wedge \cdots{}_\wedge dZ^{\beta_n}$$

where again $g \in \Gamma(\mathbb{RP}^{n*}, \tilde{\mathcal{O}}(s-1))$. To see that such a g exists, in both formulae it is sufficient to prove it in the $s = 1$ case since we can reduce the general case to this using the symmetry of the integrand over the $\alpha_1 \cdots \alpha_s$ indices and by contracting the $s-1$ other free indices off with constants A_{α_2} etc.. In the second formula it suffices to show that the integral vanishes when the β_0 and α_1 indices are skewed since the right hand side is manifestly proportional to W_{β_0}. This follows by expressing the integrand skewed over β_0 and α_1 as an exact form $d(f\varepsilon_{\alpha_1\beta_0\beta_1\cdots\beta_{n-1}} Z^{\beta_1} dZ^{\beta_2}{}_\wedge \cdots{}_\wedge dZ^{\beta_{n-1}})$ so that the integral vanishes. For the first formula we must show that $\partial^{[\alpha} g^{\beta]} = 0$ where g^α is defined by:

$$W_\alpha g^{\alpha_1}(W) = \int_{Z\cdot W=0} Z^{\alpha_1} f(Z) \varepsilon_{\alpha\beta\gamma\cdots\delta} Z^\beta dZ^\gamma{}_\wedge \cdots{}_\wedge dZ^\delta.$$

We use the fact that the integral of the Lie derivative of the integrand along $Z^\beta\partial_\gamma$ is $W_\gamma\partial^\beta$ acting on the integral (this follows from $SL(n,\mathbb{R})$ equivariance of the integral). Again one can show, with some manipulation, that $W_\alpha W_\beta \partial^{[\gamma}g^{\delta]}$ is given by the integral of an exact (indexed) form.

It is perhaps worth noting that the g in the first formula is unique as the kernel of the operator $\partial^\alpha \cdots \partial^\gamma$ consists of polynomials, but one cannot construct sections of $\tilde{\mathcal{O}}(r)$ out of polynomials as their sign does not change appropriately under inversion. The second formula, however, will have kernel consisting of homogeneous polynomials of degree $-s-n$.

The inversion formulae and further generalizations

The difference between the Radon transform in odd and even dimensions is perhaps most manifest in the inversion formulae. In odd dimensions, the Radon transform and the above generalizations takes one from $\Gamma(\tilde{\mathcal{O}}(-n-s))$ to $\Gamma(\tilde{\mathcal{O}}(s-1))$. This is inverted up to a constant overall factor by the above generalized transform with s replaced by $1-n-s$. This clearly cannot work in even dimensions as the image of the Radon transform are sections of $\tilde{\mathcal{O}}(s-1)$ which cannot be integrated over hyperplanes even after the homogeneity has been raised or lowered by differentiation etc..

In even dimensions the inversion formula is, in formal terms,

$$f(Z) = \int \frac{g(W)}{(Z\cdot W)^{s+n}} \varepsilon^{\beta_0\beta_1\cdots\beta_n} W_{\beta_0} dW_{\beta_1} \wedge \cdots \wedge dW_{\beta_n}$$

where $g \in \Gamma(\mathbb{RP}^{n*}, \tilde{\mathcal{O}}(s-1))$, $f \in \Gamma(\mathbb{RP}^n, \mathcal{O}(-s-n))$. This is a singular integral and needs to be regularized[1]. The fact that it is possible to regularize it invariantly follows from the existence of the forward transform. When $s \le -n$ the formula must be understood by its $(1-s-n)$'th derivative with respect to Z^α or alternatively with an additional factor of $\log(Z\cdot W)$ in the integrand and and a polynomial ambiguity in the result for f.

This formula completes the story for scalar weighted functions in even dimensions. However, in odd dimensions it leads to a new transform

$$\widetilde{\mathcal{R}} : \Gamma(\mathbb{RP}^{n*}, \mathcal{O}(s-1)) \to \Gamma(\mathbb{RP}^n, \mathcal{O}(-s-n))$$

given by the above formula except with $g \in \Gamma(\mathbb{RP}^{n*}, \mathcal{O}(s-1))$ as is required for integration in odd dimensions and $f \in \Gamma(\mathbb{RP}^n, \mathcal{O}(-s-n))$. One expects that it is nontrivial and is inverted by the same formula with s replaced by $1-s-n$. The same comments as above concerning ambiguities will apply for $s \le -n$.

[1] In order to have a rigourously invariant formulation one must demonstrate that this regularization can be performed invariantly. In recent work with T.N.Bailey it has been possible to prove this directly although an invariant proof of the inversion property is still lacking

Connections with twistor theory

In my article in the last TN, I showed how the X-ray transform could be understood as a globalization of the standard Penrose transform. The novelty of the above in this context is that it shows that one can obtain an inverse Penrose transform by choosing a plane in twistor space and using the above inversion formula on the corresponding β-plane in space-time. Fritz-John avoids the awkwardness of choosing a β-plane by integrating over all possible choices.

The second example is the twistor transform for Minkowski space of signature $(2,2)$ from cohomology classes on twistor space to dual twistor space. The cohomology has to be that of the sheaf $\mathcal{O}(n)$ and hence the corresponding functions on $\mathbb{RP}^3$ are sections of $\mathcal{O}(n)$ rather than $\tilde{\mathcal{O}}(n)$ thus the version of the transform thats relevant. Thus the twistor transform in this case will be implemented by the non-local formula of the last section.

Thanks to T.N.Bailey and G.A.J.Sparling for discussions.

L J Mason

ABSTRACT OF D.Phil THESIS by F.Müller

The ladder representations $\mathcal{H}_k$, $\overline{\mathcal{H}}_k$ ($k \in \mathbb{Z}$) of $SU(2,2)$ which via the Penrose transform correspond to free massless fields of helicity $\mp k\hbar/2$ have analogues for all $SU(p,q)$. We extend the formalism of twistor diagrams accordingly and construct projection operators from tensor products of such representations into irreducible subspaces by means of these generalised twistor diagrams. We first consider the case $p = q = 1$ where $\mathcal{H}_k$ ($\overline{\mathcal{H}}_k$) exhaust all discrete series representations with lowest (highest) weight. We formulate projection operators on $\mathcal{H}_k \otimes \mathcal{H}_l$ in algebraic, diagrammatic and analytic ways and obtain formulae to translate between these various guises. For realisations of $\mathcal{H}_k$ on spaces of sections we then establish an equivalence of diagram composition with the algebraic composition of operators which is expected to carry over to realisations on higher cohomology groups. We achieve this by explicit construction of a contour of integration on which a power series expansion of the linking segments is possible. We use this equivalence to prove that, as a representation of $SU(1,1)$, any finite tensor product $\otimes_{i=0}^{n} \mathcal{H}_{k_i}^{i}$ can be decomposed by compositions of 'box diagrams'. These constructions are shown to carry over to $SU(p,q)$ and it is indicated, by means of examples, how they might be complemented to form complete sets of projections in the general case. This requires some explicit decomposition formulae which we give in a restricted case for $SU(2,1)$. For the decomposition of $\mathcal{H}_k \otimes \mathcal{H}_l$ the $SU(1,1)$ projections are almost sufficient and we extend them to a complete set for $SU(p,q)$. The translation of algebraic expressions for projections back into diagrams is in general found to be a simplification.

We then use our techniques to formulate a number of conformally invariant first order scattering amplitudes in terms of projections on tensor products of ladder representations and make a few remarks on extensions to conformally non-invariant cases. Apart from giving invariant descriptions of such amplitudes the use of orthogonal states and diagrams in higher dimensions also simplifies the calculational aspects of diagram integration.

Franz Müller

submitted in January 1994.

Update on the massive propagator in twistor diagrams

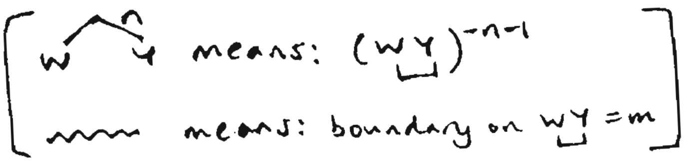
Here follows a sketch of recent developments regarding mass, following on from the ideas described in TN 32 for a possible solution to the problem of describing the massive Feynman propagator in terms of standard twistor diagram elements. The general idea there was to make finite sense of each term in the formal expression:

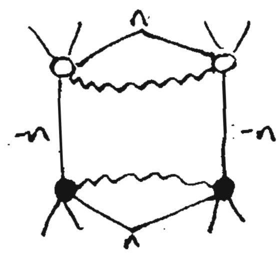
$$(\Box + m^2)^{-1} = \sum_{0}^{\infty} (-m^2)^n \, \Box^{-n-1}$$

by using the Barnes integral representation of the Feynman propagator function as evaluated on suitable test-fields, and then to find twistor integrals corresponding to each such term. The key new idea was to exploit the properties of the inhomogeneous *boundary* at infinity, i.e. a boundary on $XZ = m$ or $WY = m$, rather than use the poles $(XZ - m)^{-1}$, $(WY - m)^{-1}$ which had featured in my 1985 paper. This gets us from the "on-shell" amplitudes, i.e. Hankel functions satisfying the Klein-Gordon equation, to the more grown-up propagator "off-shell" functions. This is a particularly satisfactory idea because since 1985 it has become clear that boundaries of this kind are essential elements of the diagram formalism.

The scheme I indicated in TN32 was that $\Box^{-n-1}$

should correspond to the twistor integral

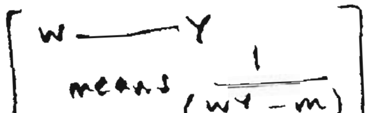


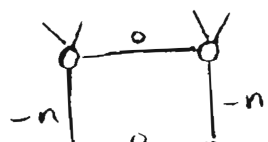


and I checked that the terms for n=0, n=1, were correct. One needs to choose k = exp (- γ), where γ is Euler's constant, if the mass parameter entering into the boundary specification is to be the same as that occurring in the formal power series (though this is not essential.) It seems something of a miracle that the n=1 term can be made to agree by a suitable contour choice: this is not something that follows just from its satisfaction of a differential relation.

Recently Stephen Spence has checked my calculations, and gone on to consider the case of n=2. He finds in that case the correct terms plus an unwanted extra term which is a multiple of k^2/m^2

The same feature occurred in my earlier (1985) work. That is, I worked out





and found terms of this kind arising which prevented the correct Hankel function being formed. Indeed by considering the operation $\frac{d}{dm}$ which turns the boundary into a pole, one can see that these unwanted terms *must* arise in the new context.

In the 1985 work I found a way of getting rid of these terms, but at the cost of changing the diagram formalism. In those days it was not so clear how the inhomogeneous elements ought to be defined, and it seemed all right to abandon the natural "boundary" definition for a logarithmic expression. What I found can be expressed like this: we could get the right Hankel function if the $(-n)$-line were to be re-defined as

$$\frac{(w.z-\epsilon)^{n-1}}{(n-1)!}\left\{\log\left(\frac{w.z-\epsilon}{k}\right)-1-\frac{1}{2}-\frac{1}{3}\cdots-\frac{1}{n-1}\right\}$$

Now this expression is crying out for the limit $\epsilon \to 0$ to be taken, and indeed it's only in the $\epsilon = 0$ limit that we get exactly the right answer. Unfortunately the contours disappear at that limit!

It is fairly clear that the same thing could be applied in the new context. That is, if we were to use these logarithmic expressions, now attached to the inhomogeneous boundary, we should lose the unwanted terms and regain just the Feynman propagator function.

But what are we to make of these logarithmic factors, and of the contour that disappears in the limit? This has required a sequence of new ideas.

1. A contour re-emerges if we add another boundary line, i.e take the combination

$$\oint_{w.z=k}\frac{(w.z)^{n-1}}{(n-1)!}\left\{\log\left(\frac{w.z}{k}\right)-1-\frac{1}{2}-\frac{1}{3}\cdots-\frac{1}{n-1}\right\}$$

2. Observe that

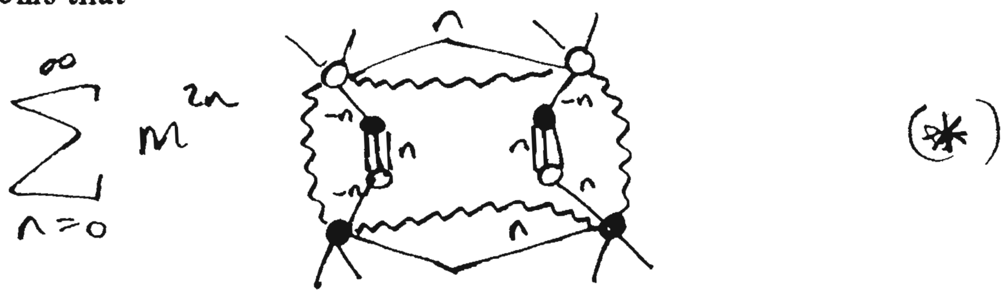
$= \log\left(\frac{w.z}{k}\right)$

This suggests that the logarithmic factors can be supplied as required by taking

This is not quite the case, as in fact this is

$$\frac{(w.z)^{n-1}}{(n-1)!}\left\{\log\left(\frac{w.z}{k}\right)-1-\frac{1}{2}\cdots-\frac{1}{n-1}\right\}-\sum_{r=0}^{n-2}\frac{k^{n-1-r}(-1)^r(w.z)^r}{r!(n-1-r)!(n-1-r)}$$

However, these extra terms don't affect the subsequent integration and so it seems that

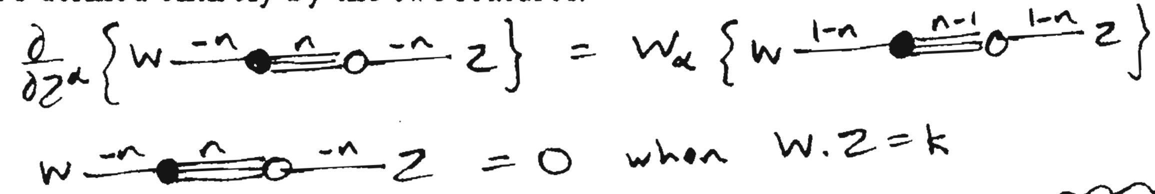


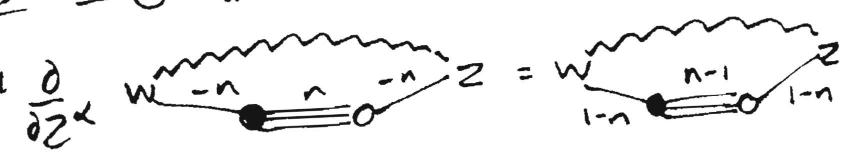
gives the Feynman propagator function. The expression for 

looks appalling but it can be rewritten as $\int_k^{W.Z} dx_{n-1} \int_k^{x_{n-1}} dx_{n-2} \cdots \int_k^{x_3} dx_2 \int_k^{x_2} \frac{dx_1}{x_1}$

$$= \int_k^{W.Z} \frac{(W.Z-x)^{n-1}}{(n-1)!} \frac{dx}{x}$$

and it's defined entirely by the two features:

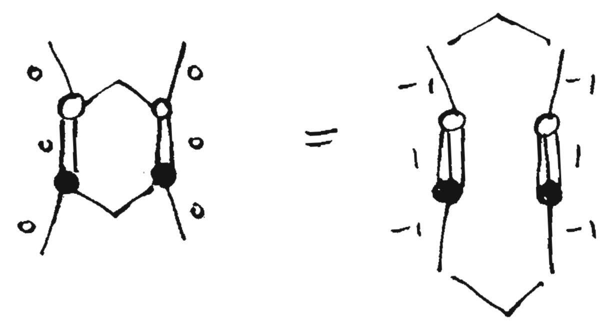


These two properties ensure that 

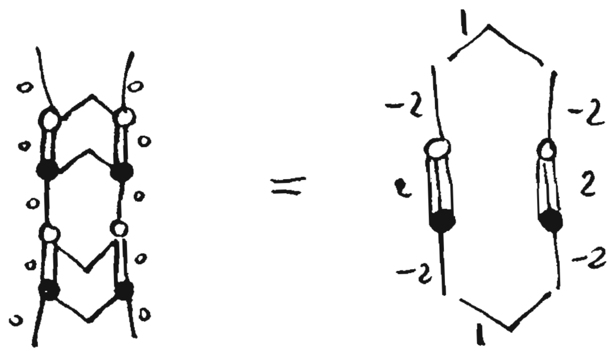
aand it follows from this that (*) solves the inhomogeneous Klein-Gordon equation.

3. These observations succeed in giving us a scheme in which only standard diagram elements play a role. However a further development gets us much closer to a more fundamental picture in which the nth twistor diagram corresponds to the picture of n successive interactions with the constant Higgs field.

To do this note that

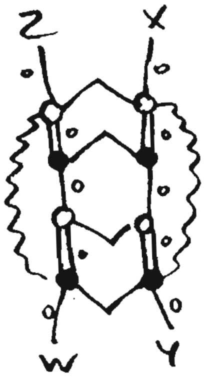


as one can see by an integration-by-parts argument. This might suggest that



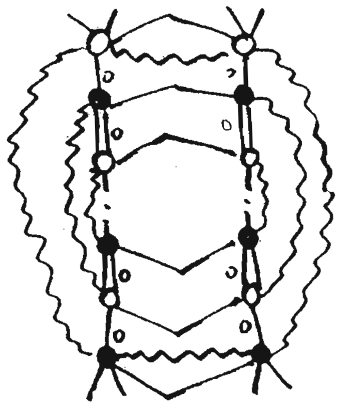
etc.

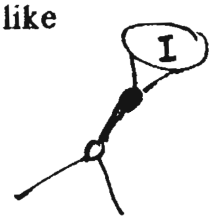
but in fact the requisite contours do not seem to exist for this. Instead, one must again add in further boundary lines. I have got as far as showing that



$$= \frac{1}{(\overline{WY}\ \overline{XZ})^2}\left\{W.Z\left(\log\left(-\frac{W.Z}{k}\right)-1\right)\right\}\left\{Y.X\left(\log\left(-\frac{Y.X}{k}\right)-1\right)\right\}$$

by a method that should extend to the nth case. The upshot of this idea would be that each term in the series is represented by a diagram like



[each  is now 

— an evaluation of the Higgs field]

which could then be modified to take correct account of the spin-projection which has so far been neglected, and to adapt to the case of spins other than zero.

It is encouraging that these contour-integral techniques are similar to those used by me and Lewis O'Donald in earlier work on ultra-violet divergences (his article in TN32, and his thesis). This suggests to me that the problem of representing general Feynman diagrams will be solved by exploiting these same ideas.

Refs:

Proc. Roy. Soc. A397, 375-396 (1985)

S.T. Spence: Qualifying dissertation (1994) (to whom many thanks!)

Andrew Hodges

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Contents

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