A Twistor-Topological Approach to the Einstein Equations

In my articles in TN 31 (pp. 6–8) and TN 32 (1–5) (cf. also RP(1992)) the suggestion was made that the appropriate twistor concept for vacuum space-times is as a charge for helicity 3/2 fields. This idea has been further pursued by RP, TN 33 (1–6), RP(1994), LJM & RP, TN 37 (1–6), and JF, TN 37 (7–9). In particular, in RP(1992) and TN 32, I described how, in flat space-time $\mathbb{M}$, one may use a "chopping and pasting" method to work out the "twistor charge", where this charge appears as a "second kind" of gauge freedom for the potentials for the helicity 3/2 field, namely that freedom in the gauge quantities which does not affect the potentials. This is best exhibited in terms of an exact sequence

$$0 \to \underset{\text{charges}}{\begin{Bmatrix}\Omega\\ \Pi\end{Bmatrix}} \to \underset{\text{gauges}}{\begin{Bmatrix}\omega\\ \pi\end{Bmatrix}} \to \underset{\text{potentials}}{\begin{Bmatrix}\rho\\ \sigma\end{Bmatrix}} \to \underset{\text{fields}}{\begin{Bmatrix}0\\ \psi\end{Bmatrix}} \to 0 \qquad \text{(A)}$$

(these actually being local twistor sheaves: LJM & RP, TN 37). I am here using the "Dirac gauge" for the potentials $\rho_{A'BC}$, $\sigma_{A'B'C}$, which implies that they are symmetric, as is the field $\psi_{A'B'C'}$:

$$\rho_{A'BC} = \rho_{A'(BC)}, \quad \sigma_{A'B'C} = \sigma_{(A'B')C}, \quad \psi_{A'B'C'} = \psi_{(A'B'C')},$$

and we have the equations

$$\nabla_A^{A'}\rho_{A'BC} = 0, \qquad \text{(B)}$$

$$\nabla_{B'}^{B}\rho_{A'BC} = 2i\,\sigma_{A'B'C}, \quad \nabla_B^{B'}\sigma_{A'B'C} = 0, \qquad \text{(C)}$$

$$\nabla_{C'}^{C}\sigma_{A'B'C'} = \psi_{A'B'C'}, \quad \nabla_A^{A'}\psi_{A'B'C'} = 0. \qquad \text{(D)}$$

(The factor $2i$ is chosen for later convenience.) The gauge freedom is given by

$$\rho_{A'BC} \mapsto \rho_{A'BC} - i\varepsilon_{BC}\pi_{A'} + \nabla_{CA'}\omega^{B}, \quad \sigma_{A'B'C} \mapsto \sigma_{A'B'C} + \nabla_{CB'}\pi_{A'} \qquad \text{(E)}$$

where

$$\nabla_{A'}^{B}\omega_B = 2i\,\pi_{A'}, \qquad \nabla_B^{A'}\pi_{A'} = 0, \qquad \text{(F)}$$

so $\pi_{A'}$ satisfies the Dirac-Weyl neutrino equation for helicity ½, and ω^A is a "potential" for π (or, equivalently, a "neutrino" field of helicity $-\frac{1}{2}$ which has π as a source).

The "second kind" of gauge freedom, leaving ρ and σ alone, is given by

$$\omega_A \longrightarrow \omega_A + \Omega_A, \quad \pi_{A'} \longrightarrow \pi_{A'} + \Pi_{A'} \qquad \text{(G)}$$

where

$$\nabla_{AA'}\Omega_B = -i\varepsilon_{AB}\Pi_{A'}, \quad \nabla_{AA'}\Pi_{B'} = 0, \qquad \text{(H)}$$

so $(\Omega^A, \Pi_{A'})$ are the spinor parts of a twistor (i.e. a constant local twistor).

For the "chopping and pasting" procedure, it is helpful to make a comparison with the electromagnetic case. Here, I refer to the magnetic charge that arises from Dirac's procedure which makes use of the non-global nature of the usual electromagnetic potential for a magnetic monopole. The relevant exact sequence is now (part of) the deRham sequence

$$0 \to \mathbb{C} \xrightarrow{i} \Omega^0 \xrightarrow{d} \Omega^1 \xrightarrow{d} \Omega^2 \qquad \text{(I)}$$

(Ω^r being the sheaf of **holomorphic** r-forms) and the procedure amounts to evaluating a closed 2-form (the Maxwell field) on a 2-cycle which lies in an open region **R** (of homotopy type S^2) surrounding a world-tube of magnetic charge. Breaking the sequence up as

$$0 \to d\Omega^0 \to \Omega^1 \to d\Omega^1 \to 0$$
$$0 \to \mathbb{C} \to \Omega^0 \to d\Omega^0 \to 0$$

(the relevant region being **R**, in each case), we get

$$H^0(d\Omega^1) \to H^1(d\Omega^0) \to H^1(\Omega^1)$$

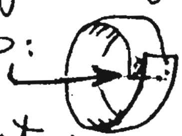
and

$$H^1(\Omega^0) \longrightarrow H^1(d\Omega^0) \to H^2(\mathbb{C}) \to H^2(\Omega^0).$$

Since $H^1(\Omega^1) = H^1(\Omega^0) = H^2(\Omega^0) = 0$, and $H^2(\mathbb{C}) = \mathbb{C}$, we have a map from $H^0(d\Omega^1)$ onto $\mathbb{C}$, this being the evaluation of the total magnetic charge of the world-tube. Although this could have been obtained as a Gauss integral of the field over S^2, the above procedure is seen more explicitly in terms of the following ("chopping and pasting") operation. The Maxwell field F is a closed (whence locally exact) 2-form, and is therefore in $H^0(d\Omega^1)$. Think of **R** being chopped into two (slightly overlapping) hemispherical, topologically trivial, regions $\mathbf{R}_1$, $\mathbf{R}_2$. In each region $\mathbf{R}_i$, we have $F = d\overset{i}{A}$, whence

$$d(\overset{2}{A}-\overset{1}{A}) = d\overset{2}{A} - d\overset{1}{A} = F - F = 0$$

on the overlap (homotopy type S^1).

We can then chop this overlap into two — or just cut it once, with an overlap:  — and try to write $\overset{2}{A}-\overset{1}{A}$ as $d\varphi$. We find that the scalar field φ jumps as we go around, the difference between the two φ-values $\overset{\mathrm{I}}{\varphi}$, $\overset{\mathrm{II}}{\varphi}$ at the overlap being a constant (since $d(\overset{\mathrm{II}}{\varphi}-\overset{\mathrm{I}}{\varphi}) = d\overset{\mathrm{II}}{\varphi} - d\overset{\mathrm{I}}{\varphi} = (\overset{2}{A}-\overset{1}{A})-(\overset{2}{A}-\overset{1}{A}) = 0$), this constant being the required magnetic charge μ. In each case, we simply follow back along the de Rham sequence — which we can express graphically as the (Zeeman) diagram:

$$0 \to \{\mu\} \xrightarrow{i} \{\varphi\} \xrightarrow{d} \{A\} \xrightarrow{d} \{F\} \to 0 \qquad \text{(J)}$$

We wish to apply this to the helicity $3/2$ sequence (A)

$$0 \to \begin{Bmatrix}\Omega\\ \Pi\end{Bmatrix} \to \begin{Bmatrix}\omega\\ \pi\end{Bmatrix} \to \begin{Bmatrix}\rho\\ \sigma\end{Bmatrix} \to \begin{Bmatrix}0\\ \psi\end{Bmatrix} \to 0$$

with a view to using it in curved Ricci-flat space-time $\mathcal{M}$. In $\mathcal{M}$, however, the space $\begin{Bmatrix}\Omega\\ \Pi\end{Bmatrix}$ does not exist (or becomes trivial) since the equations (H) become inconsistent. Moreover, the equations (B) and (D) also become inconsistent, and must be abandoned. We can retain (C), (E), and (F), but in dropping (B), we can obtain an additional "field" quantity which lies in the extra gauge freedom

$$\rho_{A'BC} \mapsto \rho_{A'BC} + \overset{\bullet}{\rho}_{A'BC}$$

where

$$\nabla^{B}_{B'}\overset{\bullet}{\rho}_{A'BC} = 0.$$

This "field" is a helicity $-3/2$ quantity $\tilde{\psi}_{ABC}\,(=\tilde{\psi}_{(ABC)})$, in flat space-time, defined by

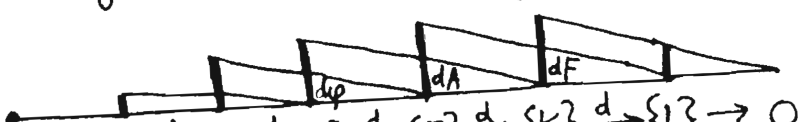
$$\nabla_A^{A'}\dot{\rho}_{A'BC} = \tilde{\psi}_{ABC}$$

(see LJM&RP, TN 37), the gauge freedom in $\dot{\rho}$ being part of that (the "π-part") which is already a freedom in ρ. In $\mathbb{M}$, we still have an exact sequence

$$0 \to \left\{\begin{matrix}\Omega\\ \Pi\end{matrix}\right\} \to \left\{\begin{matrix}\omega\\ \pi\end{matrix}\right\} \to \left\{\begin{matrix}\rho\\ \sigma\end{matrix}\right\} \to \left\{\begin{matrix}\tilde{\psi}\\ \psi\end{matrix}\right\} \to 0, \qquad (K)$$

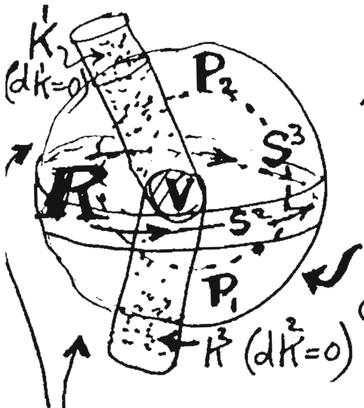
but now the twistor (Ω, Π) is not merely the charge Z^α for ψ, but contains a contribution from the projection part (i.e. secondary part) of the charge (dual twistor) W_α for $\tilde{\psi}$. Thus, (Ω, Π) has the form "$Z^\alpha + I^{\alpha\beta}W_\beta$".

Of course, we require some sort of non-linear twistor space for $\mathcal{M}$, so it is just as well that a (Ω, Π) defined simply by (H) will not do. For various suggestive (but, as yet, inadequate) reasons, it appears that it would be greatly helpful if the flat-space sequence (K) could be extended to a longer one. (Roughly: each "twistor (Ω,Π)" exists, in some sense, "somewhere" — say at $\mathbb{C}\mathscr{I}^+$ — but different twistors exist at different places, so they cannot be added. The vacuum equations allow this twistor concept to be "spread" about the space-time. However, (K) really "looks at" regions of $\mathcal{M}$ that are essentially no more than two-dimensional, and we need at least three dimensions to "feel out" the space-time.) The idea is that we try to mirror what goes on with the de Rham sequence (I) extended further to the right, which I write as



$$0 \to \{\mu\} \xrightarrow{i} \{\varphi\} \xrightarrow{d} \{A\} \xrightarrow{d} \{F\} \xrightarrow{d} \{K\} \xrightarrow{d} \{L\} \to 0$$

Here, first, we allow the "Maxwell field" F to have a magnetic charge density K (a 3-form $K = dF$), so the standard Maxwell equations are violated. At this stage, this density is still conserved, however ($dK = 0$, i.e. $\nabla_a {}^*K^a = 0$), owing to the existence of F. But we can also allow some region **V** of violation of magnetic charge conservation, this charge violation being expressed by the 4-form $L = dK$, where K is not now of the form dF. Suppose that **V** is a small, topologically trivial, region of space-time,



where we can imagine world tubes of conserved magnetic charge density entering and leaving **V**. Now, surround **V** by an (open) region **P** of space-time (homotopy type S^3). We can imagine **P** to be built up from two topologically trivial "hemispherical" regions $\mathbf{P}_1$ ("past hemisphere") and $\mathbf{P}_2$ ("future hemisphere") which overlap in a region **R** (homotopy type S^2) containing an "equatorial" S^2. In $\mathbf{P}_i$, we can find $\overset{i}{F}$ such that $d\overset{i}{F} = \overset{i}{K}$, for each i, and in **R** we have $F = \overset{2}{F} - \overset{1}{F}$, with $dF = 0$.

From there on, we proceed as before, to obtain the magnetic charge μ — which now measures the total charge creation in the region **V**. More abstractly, we extend the original evaluation procedure to obtain a map from $H^0(\mathbf{P}, d\Omega^2)$ onto $H^3(\mathbf{P}, \mathbb{C}) \cong \mathbb{C}$. The field K, being locally of the form dF, is an element of $H^0(\mathbf{P}, d\Omega^2)$.

To extend the sequence (K), we make use of various ideas, and, most specifically, one due to Jörge Frauendiner (JF in TN 37). It will simplify matters if we first consider merely the projection part of (K) suitably extended. In JF's scheme, instead of using a $\sigma_{A'B'C}$ subject to (C), we consider the weaker (conformally invariant) equation

$$\nabla^{B'}_{(B}\sigma_{C)A'B'} = 0 \qquad \text{(L)}$$

(with the symmetry $\sigma_{A'B'C} = \sigma_{(A'B')C}$ still holding), and then we define

$$\beta_{A'} = \nabla^{BB'}\sigma_{A'B'B} \qquad \text{(M)}$$

which, by virtue of the vacuum equations satisfies

$$\nabla^{AA'}\beta_{A'} = 0 \qquad \text{(N)}$$

with $\beta_{A'}$ invariant under the same gauge freedom as before:

$$\sigma_{A'B'C} \mapsto \sigma_{A'B'C} + \nabla_{B'C}\pi_{A'}\,, \quad \nabla^{AA'}\pi_{A'} = 0. \qquad \text{(O)}$$

However, the sequence is now not exact, and we have a Zeeman diagram looking like

$$0 \to \{\Pi\} \xrightarrow{i} \{\pi\} \xrightarrow{\nabla} \{\sigma\} \xrightarrow{\nabla\cdot} \{\beta\} \to 0 \,.$$

Exactness can be restored by introducing the somewhat odd-looking sheaf of "σ moddedout by π" to describe the "sourced spin 3/2 field":

$$0 \to \{\Pi\} \to \{\pi\} \to \{\sigma\} \to \{\sigma\}/\{\pi\} \to \{\beta\} \to 0 \qquad \text{(P)}$$

and now β describes a kind of charge density for this field.

We can try to understand this by resorting to the abstract procedures adopted previously, where all homology groups refer to the region **P** described earlier, and we imagine some sort of "twistor creation" taking place in the region **V** that **P** surrounds. We have (in an obvious notation)

$$H^0(\sigma/\pi) \to H^0(\beta) \to H^1(\nabla\sigma) \to H^1(\sigma/\pi) \to H^1(\beta) \qquad \text{(Q)}$$

from $\quad 0 \to \{\nabla\sigma\} \to \{\sigma\}/\{\pi\} \to \{\beta\} \to 0\,.$

In $\mathbb{M}$, the full "twistor charge" for helicity 3/2 should be found in $H^1(\nabla\sigma)$, since this contains the helicity raised "Gauss integral" for the field $\psi = \nabla\sigma$ defined by the potential σ according to (D). It can be seen that $H^1(\beta) = 0$, but we must examine $H^1(\sigma/\pi)$. Using

$$0 \to \{\pi\}/\{\Pi\} \to \{\sigma\} \to \{\sigma\}/\{\pi\} \to 0$$

we obtain

$$H^1(\sigma) \to H^1(\sigma/\pi) \to H^2(\pi/\Pi) \to H^2(\sigma)\,.$$

By results due to JF and GAJS, it follows that $H^1(\sigma) = H^2(\sigma) = 0$, whence $H^1(\sigma/\pi) \cong H^2(\pi/\Pi)$. Using

$$0 \to \{\Pi\} \to \{\pi\} \to \{\pi\}/\{\Pi\} \to 0$$

we obtain
$$H^2(\pi) \to H^2(\pi/\Pi) \to H^3(\Pi) \to H^3(\pi).$$
We have $H^2(\pi) = H^3(\pi) = 0$, but $H^3(\Pi)$ is just the primed spin space $\mathbb{S}_{A'}$ $(\cong \mathbb{C}^2)$. This gives us back the "charge integral" that we would have obtained by directly applying a "chop and paste" procedure to σ. It delivers (in the absence of the second potential ρ) merely the projection part $\Pi_{A'}$ of the twistor $(\Omega^A, \Pi_{A'})$. However, the full twistor must be in $H^1(\nabla\sigma)$, whence it follows that the primary part lies in $H^0(\beta)$ (when $\Pi_{A'} = 0$). Somehow, in the sequence (P), the twistor is "split" between the two ends of the sequence, Ω^A being found in H^0 of the right-hand end and $\Pi_{A'}$ in H^3 of the left-hand end.

To proceed further, we must re-introduce the second potential ρ. This gives the whole scheme another (short) exact sequence structure, involving

$$0 \to \{\mathring{\dot\rho}\} \to \{\dot\rho\} \to \{\dot\sigma\} \to 0,$$

where (L) must be filled out to

$$\nabla^{B'}_{(B}\dot\sigma_{C)A'B'} = 0, \quad \nabla^{B}_{(B'}\dot\rho_{A')BC} = 2i\dot\sigma_{A'B'C} \qquad \text{(R)}$$

$\mathring{\dot\rho}_{A'BC}$ subject to $\nabla^{B}_{(B'}\mathring{\dot\rho}_{A')BC} = 0$ acting as freedom in $\dot\rho$ (each spinor symmetric in the relevant index pair). All this works in Ricci-flat $\mathcal{M}$, with gauge freedom for $\dot\rho$, $\dot\sigma$ the same as with ρ, σ, as before, in (E), (F). In $\mathbb{M}$, we get

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \{\mathring{\Omega}\} \arrow[r] \arrow[d] & \{\mathring{\omega}\} \arrow[r] \arrow[d] & \{\mathring{\rho}\} \arrow[r] \arrow[d] & \{\mathring{\dot\rho}\}/\{\mathring{\omega}\} \arrow[r] \arrow[d] & \{\mathring{\alpha}\} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \left\{\begin{matrix}\Omega\\ \Pi\end{matrix}\right\} \arrow[r] \arrow[d] & \left\{\begin{matrix}\omega\\ \pi\end{matrix}\right\} \arrow[r] \arrow[d] & \left\{\begin{matrix}\rho\\ \sigma\end{matrix}\right\} \arrow[r] \arrow[d] & \left\{\begin{matrix}\dot\rho\\ \dot\sigma\end{matrix}\right\}/\left\{\begin{matrix}\omega\\ \pi\end{matrix}\right\} \arrow[r] \arrow[d] & \left\{\begin{matrix}\alpha\\ \beta\end{matrix}\right\} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \{\Pi\} \arrow[r] \arrow[d] & \{\pi\} \arrow[r] \arrow[d] & \{\sigma\} \arrow[r] \arrow[d] & \{\dot\sigma\}/\{\pi\} \arrow[r] \arrow[d] & \{\beta\} \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 & 0 & 0 &
\end{tikzcd} \qquad \text{(S)}$$

as a commutative exact diagram of sheaves. Here, we have

$$\alpha_C = \tfrac{1}{2}\nabla^{BB'}\rho_{B'BC}\,,\quad \beta_{A'} = \nabla^{BB'}\sigma_{A'B'B} \qquad \text{(T)}$$

(the "freedom" in α and ρ being denoted $\mathring{\alpha}$, $\mathring{\rho}$, etc.) and the equations

$$\nabla_{A'}^{A}\alpha_A = 2i\,\beta_{A'}\,,\quad \nabla_{A}^{A'}\beta_{A'} = 0$$

(compare (F)). As sheaf sequences, everything seems to work consistently from the third ($\{\omega\}$, $\{{\omega \atop \pi}\}$, $\{\pi\}$) column out to the right, provided that $\mathcal{M}$ is Ricci-flat. In $\mathcal{M}$, we have, for the second column, the standard Poincaré invariant twistor short exact sequence

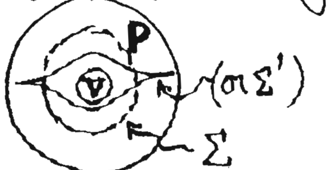
$$0 \to \mathbb{S}^A \to \mathbb{T}^\alpha \to \mathbb{S}_{A'} \to 0.$$

In fact, we find four versions of this sequence (two in dual form) running round the outside of the square

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 & \mathbb{S}_A \arrow[l] \arrow[d] & \mathbb{T}_\alpha \arrow[l] \arrow[d] & \mathbb{S}^A \arrow[l] \arrow[d] & 0 \arrow[l] \\
0 & \mathbb{T}^\alpha \arrow[l] \arrow[d] & \boxed{\text{charges for } \psi \text{ and } \tilde{\varphi}} \arrow[l] \arrow[d] & \mathbb{T}_\alpha \arrow[l] \arrow[d] & 0 \arrow[l] \\
0 & \mathbb{S}_{A'} \arrow[l] \arrow[d] & \mathbb{T}^\alpha \arrow[l] \arrow[d] & \mathbb{S}^A \arrow[l] \arrow[d] & 0 \arrow[l] \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

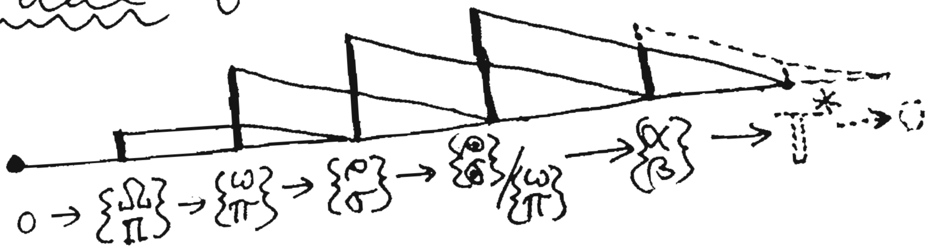
There is a certain "twist" in this diagram owing to the "$Z^\alpha + I^{\alpha\beta}W_\beta$" nature of (Ω, Π), as mentioned earlier (after (K)). The second column of this sequence comes from H^3 of the second column of (S), and the fourth column comes from H^0 of the sixth column of (S).

We can also introduce duals for the spaces of these various types of "field" in the region $\mathbf{P}$, constructing a divergence-free vector J^a, to be integrated over some topologically non-trivial compact 3-surface Σ ($\simeq S^3$) within $\mathbf{P}$. Remarkably, we find that the relevant dual sheaf sequence is simply the same sequence as (P) all

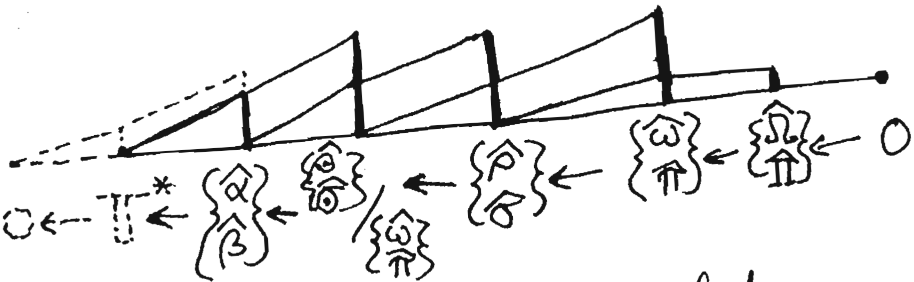


over again, but running in the opposite direction:

Dual of



is



where to express this duality we use

$$J^{AA'} = \omega^{A}\hat{\beta}^{A'} + \pi^{A'}\hat{\alpha}^{A} \quad \text{or} \quad \rho^{AB}_{B'}\hat{\odot\sigma}^{A'B'}_{B} + \sigma^{A'B'}_{B}\hat{\odot\rho}^{AB}_{B'}$$

$$\text{or} \quad \odot\rho^{AB}_{B'}\hat{\sigma}^{A'B'}_{B} + \odot\sigma^{A'B'}_{B}\hat{\rho}^{AB}_{B'} \quad \text{or} \quad \alpha^{A}\hat{\pi}^{A'} + \beta^{A'}\hat{\omega}^{A},$$

as the case may be (possibly modulo factors). The dotted part at the end of each sequence is there if we consider fields throughout **P** (i.e. $H^0(\mathbf{P}, \ )$), whereas the dotted part is absent if we merely consider sheaves. The existence of this duality serves to confirm some of the statements concerning $H^0(\mathbf{R}, \beta)$, etc., made earlier.

There is still a good deal that remains mysterious about what happens in a general Ricci-flat M. Work in progress.

Thanks to LJM and JF (and also GAJS) especially

Roger P.

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