Deformed Twistor spaces and the KP equation

While twistor theory continues to come to terms with the geometry of the KP equation, one closely related $(2+1)$-dimensional integrable system that does possess such a description is the dispersionless KP (or dKP) equation:

$$(u_{2,t} - u_2 u_{2,x})_x = u_{2,yy}\,. \tag{1}$$

This arises from the following equations for the 2-form $\omega(\lambda)$:

$$\begin{aligned} \omega(\lambda) \wedge \omega(\lambda) &= 0\,, \\ d\omega(\lambda) &= 0\,, \end{aligned} \tag{2}$$

where $\omega = d\mathcal{B}_1 \wedge dx + d\mathcal{B}_2 \wedge dy + d\mathcal{B}_3 \wedge dt$, and

$$\begin{aligned} \mathcal{B}_1 &= \lambda\,, \\ \mathcal{B}_2 &= \frac{\lambda^2}{2} + u_2(x,y,t)\,, \\ \mathcal{B}_3 &= \frac{\lambda^3}{3} + \lambda u_2(x,y,t) + u_3(x,y,t)\,. \end{aligned}$$

Equation (2) is equivalent to the zero-curvature equation

$$\frac{\partial \mathcal{B}_2}{\partial t} - \frac{\partial \mathcal{B}_3}{\partial y} + \{\mathcal{B}_2, \mathcal{B}_3\} = 0 \tag{3}$$

with $\{\ ,\ \}$ being the Poisson bracket

$$\{f,g\} = \frac{\partial f}{\partial \lambda}\frac{\partial g}{\partial x} - \frac{\partial g}{\partial \lambda}\frac{\partial f}{\partial x}\,.$$

Equating powers of λ in (3) gives, on eliminating $u_3(x,y,t)$ the dKP equation (1). Equation (2) imply (locally) the existence of functions $\mathcal{P}(\lambda)$ and $\mathcal{Q}(\lambda)$ such that $\omega = d\mathcal{P} \wedge d\mathcal{Q}$, and any two such set of coordinates are connected by a Riemann-Hilbert problem.

Conventional approaches to the KP equation use the algebra of pseudo-differential operators. However, an alternative approach which is closer to the above derivation of the dKP equation may be obtained by replacing the Poisson bracket in (3) with the Moyal bracket [1]:

$$\{f,g\}_\kappa = \sum_{s=0}^{\infty} \frac{(-1)^s \kappa^{2s}}{(2s+1)!} \sum_{j=0}^{2s+1} (-1)^j \binom{2s+1}{j} (\partial_x^j \partial_\lambda^{2s+1-j} f)(\partial_x^{2s+1-j} \partial_\lambda^j g)\,. \tag{4}$$

With the same $\mathcal{B}_2$ and $\mathcal{B}_3$ as before equation (3) gives (on replacing the Poisson bracket with the Moyal bracket and on eliminating u_3) the KP equation

$$\left(\tfrac{1}{3}\kappa^2 u_{2,xxx} + u_{2,t} - u_2 u_{2,x} \right)_x = u_{2,yy}\,. \tag{5}$$

In the $\kappa \to 0$ limit the Moyal bracket collapses to the Poisson bracket and one recovers the dKP equation. Thus we have a description of the KP equation which avoids the use of pseudo-differential operators. A similar Moyal-algebraic deformation of the self-dual vacuum equation was introduced in [2] and shown by Takasaki [3] to be integrable via a Riemann-Hilbert problem in the corresponding Moyal loop group.

At first sight the definition (4) looks somewhat unwieldy, but it is, in many ways, very natural. If one wants to deform the Poisson bracket by introducing higher-order derivative terms, the Jacobi identity turns out to be highly restrictive, and one is automatically lead to the Moyal bracket. Moreover, the bracket may be written in terms of an associative $\star$-product

$$\{f,g\}_{\kappa} = f \star g - g \star f \,.$$

Such $\star$-products have a long history, having been studied by both Moyal [1] and Weyl [4].

Thus, in conclusion, one possible approach to the understanding of the geometry of the KP equation might be to try to formulate a version of twistor theory which makes use of this deformed Poisson bracket.

References

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