Reduced Hypersurface Twistor Spaces — M. Dunn.

In Woodhouse & Mason (1988) twistor space **PT** was factored out by the Killing vector symmetries ∂_t and ∂_θ to obtain a reduced twistor space R with non-Hausdorff structure as shown in Figure 1.

In this article it will be shown that for a *curved* vacuum space-time M which is cylindrically symmetric (i.e. has spacelike Killing vectors ∂_z and ∂_θ), its hypersurface twistor space can be reduced by those symmetries to give a reduced hypersurface twistor space $\mathcal{R}$ whose structure is identical to that of R. It will also be shown how the initial data of the metric on the hypersurface can be encoded into this structure.

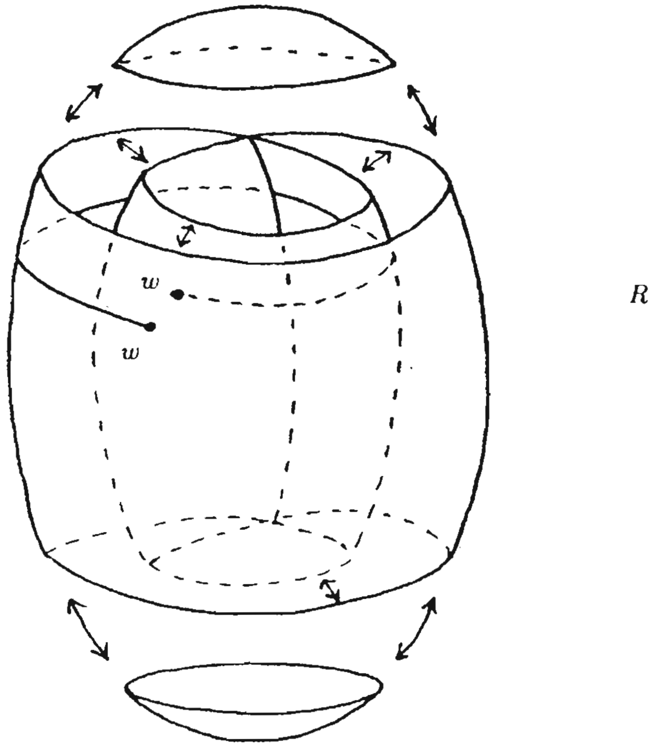

Figure 1

The metric of a cylindrically symmetric space-time can be written in *Weyl's canonical coordinates*:

$$ds^2 = \Omega^2(dt^2 - dr^2) - f(dz + \omega d\theta)^2 - \frac{r^2}{f}d\theta^2, \tag{1}$$

where Ω, f and ω are functions of t, r only (see Kramer *et al.* (1980)). The vacuum field equations $R_{ab} = 0$ imply that there exists a potential ψ such that

$$\psi_t = \frac{f^2}{r}\omega_r, \quad \psi_r = \frac{f^2}{r}\omega_t. \tag{2}$$

Defining the *Ernst potential* $\mathcal{E} = f + i\psi$, the field equations take the form

$$\mathcal{E}_{rr} + \tfrac{1}{r}\mathcal{E}_r - \mathcal{E}_{tt} = \tfrac{2}{\mathcal{E}+\overline{\mathcal{E}}}(\mathcal{E}_r^2 - \mathcal{E}_t^2); \tag{3a}$$

$$\left.\begin{aligned} k_r &= \tfrac{r}{(\mathcal{E}+\overline{\mathcal{E}})^2}(\mathcal{E}_r\overline{\mathcal{E}}_r + \mathcal{E}_t\overline{\mathcal{E}}_t), \\ k_t &= \tfrac{r}{(\mathcal{E}+\overline{\mathcal{E}})^2}(\mathcal{E}_r\overline{\mathcal{E}}_t + \overline{\mathcal{E}}_r\mathcal{E}_t). \end{aligned}\right\} \tag{3b}$$

The equation (3a), the *evolution equation*, is the integrability condition for equations (3b), the *constraint equations*, so all the information of the space-time is contained in $\mathcal{E}$. The values of $\mathcal{E}$ and $\mathcal{E}_t$ on a spacelike hypersurface $\mathcal{H}$ constitute the initial data for the space-time.

Recall that hypersurface twistors (strictly speaking, *dual* hypersurface twistors, which I am using for convenience of notation, avoiding a lot of primes!) correspond to curves in the complex "thickening" of $\mathcal{H}$, $\mathbf{C}\mathcal{H}$, called β-curves. These are the analogue of β-planes in flat space, which do not in general exist in curved space-times, since the curvature causes integrability problems. However, the restriction of the β-plane distribution (the vector fields spanned by $\{\eta^A\nabla_{AA'}\}$) to $\mathbf{C}\mathcal{H}$ is just a single vector field

$$\mathbf{V}_{\mathbf{C}\mathcal{H}} = \eta^A N^{BA'}\eta_B\nabla_{AA'}, \tag{4}$$

where N^a is a normal vector field to $\mathcal{H}$. The integral curves of $\mathbf{V}_{\mathbf{C}\mathcal{H}}$ are the β-curves.

The spinor η^A is parallelly propagated along the β-curve, which tells us that the lift of $\mathbf{V}_{\mathbf{C}\mathcal{H}}$ to the spin bundle S^A is

$$\mathbf{V}_{S^A} = \eta^{\mathbf{A}} N^{\mathbf{DA'}}\eta_{\mathbf{D}}\left(\nabla_{\mathbf{AA'}} - \gamma_{\mathbf{AA'B}}{}^{\mathbf{C}}\eta^{\mathbf{B}}\frac{\partial}{\partial\eta^{\mathbf{C}}}\right), \tag{5}$$

where $\gamma_{\mathbf{AA'B}}{}^{\mathbf{C}}$ are the *spin-coefficients* (see Penrose & Rindler (1984)).

We next factor out by the Euler vector field $\Upsilon = \eta^{\mathbf{A}}\partial/\partial\eta^{\mathbf{A}}$ to obtain a vector field $\mathbf{V}_{\mathbf{P}S^A}$ on the projective spin bundle.

Hypersurface twistor space $\mathbf{P}\mathcal{T}^*$ is the space of β-curves, i.e. the quotient of $\mathbf{P}S^A$ by $\mathbf{V}_{\mathbf{P}S^A}$. We factor out this space by the symmetries induced by ∂_z and ∂_θ to obtain $\mathcal{R}$.

Alternatively, we could first reduce $\mathbf{P}S^A$ by the symmetries to obtain a space $\mathcal{F}$ (and an associated vector field $\mathbf{V}_{\mathcal{F}}$, tangent to the β-curves on $\mathcal{F}$), and then take the quotient by $\mathbf{V}_{\mathcal{F}}$ to get $\mathcal{R}$. This route will give us the structure of $\mathcal{R}$.

The natural null tetrad to use for a cylindrically symmetric space-time is:

$$\begin{aligned} D &= \tfrac{1}{\sqrt{2\Omega}}(\partial_t + \partial_r), & D' &= \tfrac{1}{\sqrt{2\Omega}}(\partial_t - \partial_r), \\ \delta &= \lambda\left(\left(\tfrac{r}{f} + i\omega\right)\partial_z - i\partial_\theta\right), & \delta' &= \lambda\left(\left(\tfrac{r}{f} - i\omega\right)\partial_z + i\partial_\theta\right), \end{aligned} \tag{6}$$

where $\lambda = r^{-1}\sqrt{f/2}$. We take our hypersurface to be one of constant time, so that the normal vector field to $\mathcal{H}$ is $N^a = o^A o^{A'} + \iota^A\iota^{A'}$. The tangent vector field to the β-curves on $\mathbf{C}\mathcal{H}$ is

$$\mathbf{V}_{\mathbf{C}\mathcal{H}} = -uvD - v^2\delta' + u^2\delta + uvD', \tag{7}$$

where $u = \eta^0, v = \eta^1$. The only non-zero spin-coefficients are

$$\begin{aligned}
\varepsilon = -\gamma' &= \tfrac{1}{2\Omega} D\Omega + \tfrac{if}{4r} D\omega, \\
\varepsilon' = -\gamma &= \tfrac{1}{2\Omega} D'\Omega - \tfrac{if}{4r} D'\omega, \\
\rho &= -\tfrac{1}{2r} Dr, \\
\rho' &= -\tfrac{1}{2r} D'r, \\
\sigma &= \tfrac{f}{2r} D\left(\tfrac{r}{f} + i\omega\right), \\
\sigma' &= \tfrac{f}{2r} D'\left(\tfrac{r}{f} - i\omega\right),
\end{aligned} \tag{8}$$

where $\varepsilon = \gamma_{00'0}{}^{0}$, etc., as in Penrose & Rindler (1984).

This gives us the tangent vector field to the β-curves on $\mathcal{F}$ as:

$$\begin{aligned}
\mathbf{V}_{\mathcal{F}} &= -2\partial_r + \sqrt{2}\Omega\left(\sigma'\zeta^3 - 2(\varepsilon+\varepsilon')\zeta + \sigma\zeta^{-1}\right)\partial_\zeta \\
&= \tfrac{\mathrm{d}r}{\mathrm{d}s}\partial_r + \tfrac{\mathrm{d}\zeta}{\mathrm{d}s}\partial_\zeta,
\end{aligned} \tag{9}$$

where s is a parameter along the β-curves and $\zeta = v/u$. The equation of the β-curves on $\mathcal{F}$ is therefore

$$\frac{\mathrm{d}}{\mathrm{d}r}(\zeta^2) = f_2\zeta^4 + f_1\zeta^2 + f_0, \tag{10}$$

where

$$\begin{aligned}
f_0 &= \tfrac{1}{\mathcal{E}+\overline{\mathcal{E}}}\mathcal{E}_t + \left(\tfrac{1}{\mathcal{E}+\overline{\mathcal{E}}}\mathcal{E}_r - \tfrac{1}{2r}\right), \\
f_1 &= \tfrac{2r}{(\mathcal{E}+\overline{\mathcal{E}})^2}\left(\mathcal{E}_r\overline{\mathcal{E}}_t + \overline{\mathcal{E}}_r\mathcal{E}_t\right) - \tfrac{2}{\mathcal{E}+\overline{\mathcal{E}}}\overline{\mathcal{E}}_t, \\
f_2 &= \tfrac{1}{\mathcal{E}+\overline{\mathcal{E}}}\overline{\mathcal{E}}_t - \left(\tfrac{1}{\mathcal{E}+\overline{\mathcal{E}}}\overline{\mathcal{E}}_r - \tfrac{1}{2r}\right),
\end{aligned} \tag{11}$$

and where the constraint equations (3b) were used to obtain f_1 in terms of the initial data.

Equation (10) is a Riccati equation, with solutions of the form

$$\zeta^2 = -\frac{1}{f_2}\left(\frac{g_1' + wg_2'}{g_1 + wg_2}\right), \tag{12}$$

where w is a constant and g_1, g_2 are linearly independent solutions of

$$g'' - \left(\frac{f_2'}{f_2} + f_1\right)g' + f_0 f_2 g = 0, \tag{13}$$

the prime denoting differentiation with respect to r (see Hille (1969)). We can use w as a coordinate on $\mathcal{R}$.

This give us a picture of the β-curves on $\mathcal{F}$ as in Figure 2. We are interested in solving the field equations on one connected coordinate patch (in general not intersecting the $r = 0$ axis), corresponding to some interval V on the r-axis in $\mathcal{F}$. A value of w corresponds to one point of $\mathcal{R}$ if, for any $r \in V$, one can move continuously along the β-curve Q_w, staying within V, from one root of (12) to the other. For example, in Figure 2, w_1 corresponds to one point of $\mathcal{R}$, whereas w_2 corresponds to two, since Q_{w_2} has two disconnected leaves in V. The curve Q_∞ is degenerate and always corresponds to two points.

This gives a structure for $\mathcal{R}$ identical to that of the reduced twistor space in Woodhouse & Mason (1988).

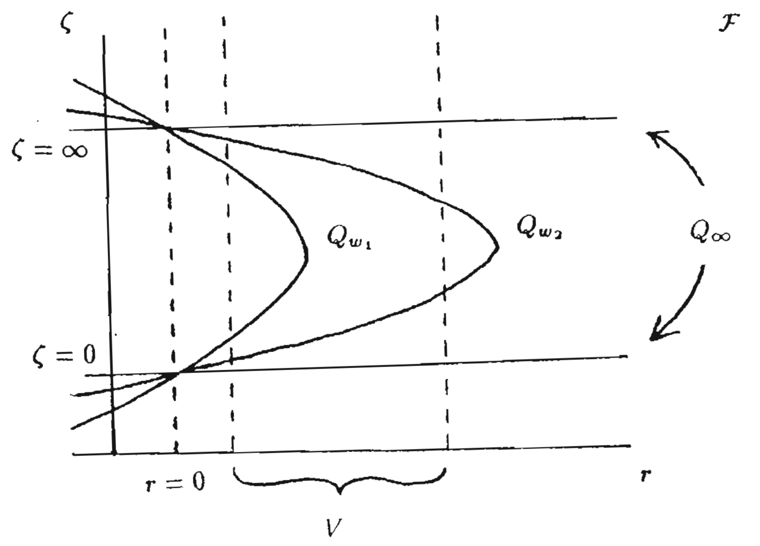

Figure 2

Now, the initial data correspond to a pair of curves on $\mathcal{R}$. Since, for a particular w, ζ has a double root when $w = -g_1'(r)/g_2'(r)$ ($\zeta = 0$) and when $w = -g_1(r)/g_2(r)$ ($\zeta = \infty$), we define the curves, parametrised by real values of r, on $\mathcal{R}$, to be

$$\begin{aligned} w &= -g_1'/g_2', \\ w &= -g_1/g_2. \end{aligned} \tag{14}$$

For each (real) value r_0 of r, the points of the curves on $\mathcal{R}$ are those corresponding to the two β-curves which *touch* the line $r = r_0$ in $\mathcal{F}$. There is one curve in each of the "glued down" caps in $\mathcal{R}$ (see Figure 1). In the axis-regular case, where the point $r = 0$ lies in V, the space $\mathcal{R}$ consists of two Riemann spheres glued down over one connected region. In this case, both curves lie in this region and intersect at $w = 0$, which corresponds to the line $r = 0$ in $\mathcal{F}$.

Indeed, any two functions which define such curves on $\mathcal{R}$ (in either case) contain the information of an initial data set for a cylindrically symmetric space-time M. For, given two arbitrary independent functions g_1, g_2 of r, we can obtain a second order differential equation

$$g'' + ag' + bg = 0 \tag{15}$$

by solving the simultaneous equations

$$\begin{aligned} g_1'' + ag_1' + bg_1 &= 0 \\ g_2'' + ag_2' + bg_2 &= 0 \end{aligned} \tag{16}$$

for a and b. Putting

$$-\left(\frac{f_2'}{f_2} + f_1\right) = a, \quad f_0 f_2 = b, \tag{17}$$

gives us two further simultaneous equations for $\mathcal{E}$ and $\mathcal{E}_t$.

These equations are, of course, rather hard to solve in practise!

It is hoped that the conformal scale of the space-time and the location of $\mathcal{H}$ within the space-time may be encoded as cohomology classes of twistor functions on $\mathcal{R}$, as these structures will generalise to a non-symmetric space-time. So far, the conformal scale has been encoded, but the "time function" has proved elusive.

References:

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