A twistor construction of irreducible torsion-free G-structures

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0. Introduction. One of the most useful characteristics of an affine connection on a manifold M is its (restricted) holonomy group which is defined, up to a conjugation, as a subgroup of $GL(T_tM)$ consisting of all automorphisms of the tangent space T_tM at a point $t \in M$ induced by parallel translations along the t-based contractable loops in M. Which groups can occur as holonomies of affine connections? By Hano and Ozeki [H-O], any closed subgroup of a general linear group can be realized as a holonomy of some affine connection (which in general has non-vanishing torsion tensor). The same question, when restricted to the class of *torsion-free* affine connections only, is not yet answered. According to Berger [B], the list of all possible irreducibly acting holonomies of torsion-free affine connections is very restricted. How much is known about this list? In his seminal paper [B], Berger found a list of groups which embraces all possible holonomies of torsion-free *metric* connections, though his approach provides no method to distinguish which entries can indeed be realized as holonomies and which are superfluous. Later much work has been done to refine this list and to prove existence of Riemannian metrics with special holonomies [Al, Br1, Br2, S]. In the same paper Berger presented also a list of all but a finite number of possible candidates to irreducible holonomies of "non-metric" torsion-free affine connections. How many holonomies are missing from this second list is not known, but, as was recently shown by Bryant [Br3], the set of missing, or *exotic*, holonomies is non-empty. As usual in the representation theory, in order to get a deeper understanding of all irreducible real holonomies one should first try to address a complex version of the problem. The main result announced in this paper asserts that any torsion-free holomorphic affine connection with irreducibly acting holonomy group can be generated by twistor methods.

1. Complex contact structures. Let Y be a complex $(2n+1)$-dimensional manifold. A complex contact structure on Y is a rank $2n$ holomorphic subbundle $D \subset TY$ of the holomorphic tangent bundle to Y such that the Frobenius form

$$\begin{aligned} \Phi : D \times D &\longrightarrow TY/D \\ (v,w) &\longrightarrow [v,w] \bmod D \end{aligned}$$

is non-degenerate. A complex n-dimensional submanifold X of the complex contact manifold Y is called a *Legendre submanifold* if $TX \subset D$. The normal bundle of a Legendre submanifold $X \hookrightarrow Y$ is isomorphic to J^1L_X [L2], where $L_X = L|_X$ and L is the contact line bundle on Y defined by the exact sequence

$$0 \longrightarrow D \longrightarrow TY \longrightarrow L \longrightarrow 0.$$

Given a Legendre subnamifold $X \hookrightarrow Y$, there is a naturally associated "flat" model, $X \hookrightarrow J^1 L_X$, consisting of the total space of the vector bundle $J^1 L_X$ together with its canonical contact structure and the Legendre submanifold X realized as a zero section of $J^1 L_X \to X$. The Legendre submanifold $X \hookrightarrow Y$ is called *k-flat* if the kth-order Legendre jet [L2] of X in Y is isomorphic to the kth-order Legendre jet of X in $J^1 L_X$. Every complex Legendre submanifold is 1-flat, while the obstruction to be 2-flat is a cohomology class in $H^1(X, L_X \otimes S^2(J^1 L_X)^*)$.

2. Irreducible G-structures. Let M be an m-dimensional complex manifold and $\mathcal{L}^* M$ the holomorphic coframe bundle $\pi : \mathcal{L}^* M \to M$ whose fibers $\mathcal{L}^*_t M = \pi^{-1}(t)$ consist of all $\mathbb{C}$-linear isomorphisms $e : \mathbb{C}^m \to \Omega^1_t M$. The space $\mathcal{L}^* M$ is a principle right $GL(m, \mathbb{C})$-bundle with the right action given by $R_g(e) = e \circ g$. If G is a closed subgroup of $GL(m, \mathbb{C})$, then a (holomorphic) G-structure on M is a principle subbundle $\mathcal{G}$ of $\mathcal{L}^* M$ with the group G. It is clear that there is a one-to-one correspondence between the set of G-structures on M and holomorphic sections σ of the quotient bundle $\tilde{\pi} : \mathcal{L}^* M/G \to M$ whose typical fibre is isomorphic to $GL(m, \mathbb{C})/G$. A G-structure on M is called *locally flat* if $\mathcal{L}^* M/G$ can be trivialized over a sufficiently small neighbourhood, U, of each point $t \in M$ in such a way that the associated section σ of $\mathcal{L}^* M/G$ is represented over U by a constant $GL(m, \mathbb{C})/G$-valued function. A G-structure is called *1-flat* if, for each $t \in M$, the first jet of the associated section σ of $\mathcal{L}^* M/G$ at t is isomorphic to the first jet of some locally flat section of $\mathcal{L}^* M/G$. It is easy to show that a G-structure admits a torsion-free affine connection if and only if it is 1-flat (cf. [Br2]). A G-structure on M is called *irreducible* if the action of G on $\mathbb{C}^m$ leaves no non zero invariant subspaces.

3. Main theorem. Recall that a generalized flag variety X is a compact simply connected homogeneous Kähler manifold [B-E]). Any such a manifold is of the form $X = H/P$, where H is a complex semisimple Lie group and $P \subset H$ a fixed parabolic subgroup.

Theorem 1 *Let X be a generalised flag varity embedded as a Legendre submanifold into a complex contact manifold Y with contact line bundle L such that $h^0(X, L_X) = m > 0$. Then*

(i) There exists a complete family $\{X_t \hookrightarrow Y \mid t \in M\}$ of compact complex Legendre submanifolds obtained by holomorphic deformations of X inside Y. Each submanifold X_t is isomorphic to X. The moduli space M, called a Legendre moduli space, is an m-dimensional complex manifold.

(ii) The Legendre submanifold $X \hookrightarrow Y$ is stable under holomorphic deformations of the contact structure on (the tubular neighbourhood of X in) Y.

(iii) For each $t \in M$, there is a canonical isomorphism $s : T_t M \to H^0(X_t, L_{X_t})$ representing a tangent vector at t as a global holomorphic section of the line bundle $L_{X_t} = L|_{X_t}$.

(iv) The Legendre moduli space M comes equipped with an induced irreducible G-structure, $\mathcal{G}_{ind} \to M$, with G isomorphic to the group of all global biholomorphisms $\phi : L_X \to L_X$ which commute with the projection $\pi : L_X \to X$.

(v) *The induced G-structure on M is 1-flat (i.e. torsion-free) if and only if the complete family* $\{X_t \hookrightarrow Y \mid t \in M\}$ *consists of 2-flat Legendre submanifolds. The obstruction for the induced torsion-free G-structure to be locally flat is given by a tensor field on M whose value at each point* $t \in M$ *is represented by a cohomology class* ρ_t *in* $H^1(X_t, L_{X_t} \otimes S^3(J^1 L_{X_t})^*)$.

(vi) *Let* $G \subset GL(m, \mathbb{C})$ *be one of the following groups: (a)* $SO(2n+1, \mathbb{C})$ *when* $m = 2n+2 \geq 8$*; (b)* $Sp(2n+2, \mathbb{C})$ *when* $m = 2n+2 \geq 4$*; (c)* G_2 *when* $m = 7$*. If* $\mathcal{G}$ *is any torsion-free* $G \times \mathbb{C}^*$*-structure on an m-dimensional manifold M, then there exists a complex contact manifold (Y, L) and a generalized flag variety X embedded into Y as a Legendre submanifold such that, at least locally, M is canonically isomorphic to the associated Legendre moduli space and* $\mathcal{G} \subset \mathcal{G}_{ind}$*. In the case (a)* $X = SO(2n+2, \mathbb{C})/U(n+1)$ *and* $\mathcal{G}_{ind}$ *is a* $CO(2n+2, \mathbb{C})$*-structure; in the case (b)* $X = \mathbb{CP}^{2n+1}$ *and* $\mathcal{G}_{ind}$ *is a* $GL(2n+2, \mathbb{C})$*-structure; and in the case (c)* $X = Q_5$ *and* $\mathcal{G}_{ind}$ *is a* $CO(7, \mathbb{C})$*-structure.*

(vii) *Let* $G \subset GL(m, \mathbb{C})$ *be an aribtrary semisimple Lie subgroup except the ones considered in (vi). If* $\mathcal{G}$ *is any torsion-free* $G \times \mathbb{C}^*$*-structure on an m-dimensional manifold M, then there exists a complex contact manifold (Y, L) and a Legendre submanifold* $X \hookrightarrow Y$ *with* $X = G/P$ *for some parabolic subgroup* $P \subset G$ *such that, at least locally, M is canonically isomorphic to the associated Legendre moduli space and* $\mathcal{G} = \mathcal{G}_{ind}$.

Remarks:

1. The Lie algebra of the group G of all global biholomorphisms $L_X \to L_X$ which commute with the projection $\pi : L_X \to X$ is exactly the vector space $H^0(X, L_X \otimes (J^1 L_X)^*)$ with its natural Lie algebra structure [Me1]. If $X = H/P$, then the induced G-structure on the associated Legendre moduli space is often isomorphic to $H \times \mathbb{C}^*$, but there are exceptions [A] which are considered in Theorem 1(vi). In these exceptional cases the originial G-structure may not be equal to the induced one, and one might try to identify some additional structures on the associated twistor spaces (Y, L) which ensure that $\mathcal{G}_{ind}$ admits a necessary reduction. However, in the context of problems discussed in the introduction there is no need in such a study, because these "exceptional" G-structures are fairly well understood by now [B, Br1, Br2, S]. If there is an exotic torsion-free G-structure other than Bryant's G_3 [Br3], it must be covered, up to a $\mathbb{C}^*$ action, by the "generic" clause (vii) in Theorem 1.

2. Two particular examples of this general construction have been considered earlier [L1, Br3]. The first example is a pair $X \hookrightarrow Y$ consisting of an n-quadric Q_n embedded into a $(2n+1)$-dimensional contact manifold (Y, L) with $L|_X \simeq i^* \mathcal{O}_{\mathbb{CP}^{n+1}}(1)$, $i : Q_n \hookrightarrow \mathbb{CP}^{n+1}$ being a standard projective realization of Q_n. It is easy to check that in this case $H^0(X, L_X \otimes (J^1 L_X)^*)$ is precisely the conformal algebra implying that the associated $(n+2)$-dimensional Legendre moduli space M comes equipped canonically with a conformal structure. This is in accord with LeBrun's paper [L1], where it has been shown how a conformal Weyl connection can be encoded into complex contact structure on the space of complex null geodesics. Since $H^1(X, L_X \otimes S^2(J^1 L_X)^*) = 0$, the induced conformal structure must be torsion-free in agreement with the classical result of differential geometry. Easy calculations show that the vector space

$H^1(X, L_X \otimes S^3(J^1L_X)^*)$ is exactly the subspace of $TM \otimes \Omega^1 M \otimes \Omega^2 M$ consisting of tensors with Weyl curvature symmetries. Thus Theorem 1(v) implies the well-known Schouten conformal flatness criterion.

The second example, which also was among motivations behind the present work, is Bryant's [Br3] relative deformation problem $X \hookrightarrow Y$ with X being a rational Legendre curve $\mathbb{CP}^1$ in a complex contact 3-fold (Y, L) with $L_X = \mathcal{O}(3)$. Calculating $H^0(X, L_X \otimes (J^1L_X)^*)$, one easily concludes that the induced G-structure, $\mathcal{G}_{ind}$, on the associated 4-dimensional Legendre moduli space is exactly an exotic G_3-structure which has been studied by Bryant in his search for irreducibly acting holonomy groups of torsion-free affine connections which are missing in the Berger list [B]. Since $H^1(X, L_X \otimes S^2(J^1L_X)^*) = 0$, Theorem 1(v) says the induced G_3-structure is torsion-free in accordance with [Br3]. The cohomology class $\rho_t \in H^1(X, L_X \otimes S^3(J^1L_X)^*)$ from Theorem 1(v) is exactly the curvature tensor of the unique torsion-free affine connection with G_3-holonomy.

3. Much of the above theorem remains true when the assumption that the Legendre submanifold X is a generalized flag variety is replaced by the assumption that X is a compact complex manifold such that $H^1(X, L_X) = 0$ [Me1].

4. Any reductive non-semisimple irreducibly acting holonomy group must be of the form $G \times \mathbb{C}^*$ (cf. Theorem 1(vi) and (vii)), where G is semisimple [B].

5. Usually in the twistor theory one works with Kodaira [K] moduli spaces [P], that is with complete families $\{X_t \hookrightarrow Y \mid t \in M\}$ of compact complex submanifolds of a complex manifold Y obtained by all holomorphic deformations of a fixed submanifold $X \hookrightarrow Y$ *inside* Y. Any such a family can be canonically interpreted as a complete family $\{\hat{X}_t \hookrightarrow \hat{Y} \mid t \in M\}$ of compact Legendre submanifolds — take $\hat{Y} = \mathbb{P}_Y(\Omega^1 Y)$ with its natural contact structure and $\hat{X}_t = \mathbb{P}_{X_t}(N_t^*)$, where N_t^* is the conormal bundle of X_t. The point is that the map

$$\{X_t \hookrightarrow Y \mid t \in M\} \longrightarrow \{\hat{X}_t \hookrightarrow \hat{Y} \mid t \in M\}$$

preserves *completeness* while changing its meaning. This construction together with Theorem 1 imply that Kodaira moduli spaces often come equipped with induced geometric structures. Other (and more fine) results in this direction are discussed in [Me2, Me3, M-P].

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