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Some remarks on the stabilizer of the space of Hill operators and C. Neumann system

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Abstract

In this article we will show some close connections between the stabilizers of the coadjoint action of $Diff(S^1)/S^1$ on its dual i.e. the space of Hill operators and the Neumann system. The main point of this article is to show some interesting features of celebrated paper of Knörrer and of Kirillov's work.

1 Introduction

The Neumann system deals with the motion of a particle on a sphere under the influence of a quadratic potential. This system is completely integrable and given solutions by hyperelliptic theta function. Moser [Mo] observed that the integrals of the Hamiltonian system describing the motion of Neumann system have a very close similarity with the integrals of the Hamiltonian system describing the geodesics on a quadric. Knörrer [Kn] showed in his paper that the Neumann problem can be recast through the Gauss map as the geodesic motion problem on a quadric.

On the other hand we know from the work of Segal [Se] and Kirillov [Ki] that the KdV equation is the Euler equation for a central extension of the group $Diff(S^1)/S^1$. The centrally extended $\widehat{Diff(S^1)}/(S^1)\oplus R$ is described by the Gelfand-Fuks cocycle [Ki]

$$(\xi_1\frac{d}{dx},\xi_2\frac{d}{dx})\longmapsto\frac{1}{2}\int_{S^1}\xi_1'\xi_2''dx,\quad \xi_i\in Vect(S^1).$$

Let us recall that the dual space of $Diff(S^1)/S^1$ is the space of quadratic differentials $\Omega^{\otimes 2}$ and the dual of the $\widehat{Diff(S^1)}/S^1$ is the space of Hill operators $\{\lambda\frac{d^2}{dx^2}+q\}$.

¿From now we shall denote $\widehat{Diff(S^1)}/S^1$ by $\hat{\nu}$ and the space of Hill operators by $H(s)$.

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2 Coadjoint action and characterization

Let us consider the covariant transformation $L=\lambda\frac{d^2}{dx^2}+q(x)$ under a S^1 diffeomorphism.

$L\longrightarrow\tilde{L}$ induced by S^1 diffeomorphism.

$$x\longrightarrow s(x)=x+\epsilon f(x)$$

$$L=\lambda+q(x)\longmapsto\tilde{L}=s'(x)^{3/2}(\lambda(\frac{d}{ds(x)})^2+q(s(x)))s'^{1/2}$$

$$=\lambda\frac{d^2}{dx^2}+\tilde{q}(x)$$

where

$$\tilde{q(x)}=s'(x)^2q(s(x))+\frac{1}{2}(\frac{s'''}{s'}-\frac{3}{2}(\frac{s''}{s'})^2)$$

When $s(x)=x+\epsilon f(x)$, this becomes

$$\tilde{q}=\xi q'+2\xi'q+\frac{1}{2}\lambda\xi'''.$$

Let us confine our attention to a specific hyper-plane $\lambda=-1$ in the coadjoint orbit. The action in this hyperplane will be

$$\tilde{q}=\xi q'+2\xi'q-\frac{1}{2}\xi'''.$$

Now we seek to characterize the pairs $(q(x)(dx)^2,-1)$. We proceed by looking at the stabilizer of the action of $\xi(x)\frac{d}{dx}$ on the dual $(q(x)dx^2,-1)$ i.e. $(\xi,a)\in Stab(q,-1)$ if and only if

$$\xi'''=2q'\xi+4q\xi'.\qquad (*)$$

Proposition 2.1 *If* $\xi = <\chi, A^{-1}\chi>$ *and satisfies*

$$\xi''' = 2\xi q' + 4\xi' q$$

then χ *satisfies*

$$\ddot{\chi} = -A\chi + q\chi$$

where $<\chi, \chi>= 1$ *and* $q = <\chi, A\chi> - <\dot{\chi}, \dot{\chi}>$ *which is the system of Neumann equations.*

proof :: Let $\xi = <\chi, A^{-1}\chi>$ then $\xi' = 2 <\dot{\chi}, A^{-1}\chi>$. Then after using the condition of the Neumann equation, we obtain

$$\xi'' = -2 + 2q\xi + 2 <\dot{\chi}, A^{-1}\dot{\xi}> .$$

Taking one more derivative we get

$$\xi''' = 4q\xi' + 2q'\xi.$$

□

Karen Uhlenbeck [Uh] found the algebraic integrals for the Neumann problem. For $p, q \in \mathbf{R}^n$ let $\Phi_\lambda(p, q) \in \mathbf{C}(\lambda)$ be the rational function

$$\Phi_\lambda(p, q) := \sum_{i=1}^{n} \frac{q_i^2}{\alpha_i - \lambda} - \frac{1}{2} \sum_{i,j=1} \frac{(p_i q_j - p_j q_i)^2}{(\alpha_i - \lambda)(\alpha_j - \lambda)}$$

Moser [Mo] gave a nice geometrical interpretation of the zeros of these rational function. In particular

$$\Phi_0(\dot{\chi}, \chi) = 0$$

and Knörrer showed also

$$2\Phi_0(\dot{\chi}, \chi) = \dot{\xi}/2 - (\ddot{\xi} - 2q\xi)\xi.$$

Knörrer showed when ξ satisfies $\xi''' = 2q'\xi + 4q\xi'$ and $\xi := <\chi(x), A^{-1}\chi(x)>$ then the $-\frac{1}{2}$ density ι satisfies an auxiliary equation, Schrödinger equation

$$(-\frac{d^2}{dx^2} + q)\iota = 0. \qquad (**)$$

Recall that if $\xi \in Vect(S^1)$, then $\iota \in \Omega^{-1/2}$ i.e. the space of scalar densities of weight $-1/2$.

Let us define

$$\mathcal{G} = \mathcal{G}_0 \oplus \mathcal{G}_1$$

, where we denote $\mathcal{G}_0 \equiv Vect(S^1)$ and $\mathcal{G}_1 \equiv \Omega^{-1/2}(S^1)$. $\mathcal{G}_1$ is the $\mathcal{G}_0$ module and it is compatible with the structure of $\mathcal{G}_0$ module and satisfies

$\mathcal{G}_1 \times \mathcal{G}_1 \longrightarrow \mathcal{G}_0$. This is quite natural if we identify $Vect(S^1)$ and $\Omega^{-1}(S^1)$. A typical element of $\mathcal{G}$ would be

$$u(x)\frac{d}{dx} + v(x)\frac{d}{dx}^{1/2}.$$

Proposition 2.2 (Kirillov [Ki]) *$\mathcal{G}$ has the structure of a super Lie algebra.*

In this realization $\xi(x)\frac{d}{dx}\oplus\iota(x)\frac{d}{dx}^{1/2}$ i.e. (ξ,ι) forms a super Lie algebra. $(*)$ and $(**)$ give the stabilizer of a point in the dual space to a super Lie algebra. (ξ,ι) satisfies

$$\xi(x+2\pi) = \xi(x)$$

$$\iota(x+2\pi) = \pm\iota(x)$$

When it is '+ ' it is called Ramond sector super Lie algebra and for '−' it is known as Neveu-Schwarz sector.

We wish to know more about ι. We shall use Knörrer's construction. He made use of the usual Gauss mapping of the quadric onto the unit sphere which takes a point on the quadric into the exterior unit normal.

Knörrer showed that Jacobi field along this geodesic motion satisfies mKdV equation where $\xi \in Vect(S^1)$ and $\iota \in \Omega^{-1/2}$.

In the next section we will give a geometrical meaning of $\iota(x)$. We will show it is the tau function of the Jacobi field equation, in this case mKdV equation.

3 Geometrical meaning of ι

As we mentioned earlier that Knörrer showed the geodesics on quadrics problem is intimately related to the C. Neumann problem.

Theorem 3.1 (Knörrer) *Let $\mathcal{Q} \subset \mathbf{R}^n$ be a quadric $\mathcal{Q} = \{t \in \mathbf{R}^n | \mho(t) = 0\}$ and $A := (\frac{\partial^2\mho}{\partial t_i \partial t_j})$. The geodesic $x(t)$ on $\mathcal{Q}$ is parametrized by*

$$\ddot{t}(x) = \aleph t(x) + w\dot{t}(x)$$

where $\aleph$ is the gradient of the function $\mho(x)$ in x. Let $\xi(x)$ be the unit normal vector of $\mathcal{Q}$ in the point $t(x)$

$$\xi = \iota.\aleph(t) \text{ where } \iota^2 = \frac{1}{<\aleph(t),\aleph(t)>}.$$

Then $\xi(t)$ satisfies Neumann equation

$$\ddot{\xi} = A\xi + q\xi \text{ where } q := \frac{1}{4}w^2 - \frac{1}{2}\dot{w}$$

where

$$w = -2\frac{< \aleph, At' >}{< t', At' >}$$

So there exist a one to one correspondence between the solutions of Neumann equation and geodesic on the quadric.

Knörrer also showed that the Jacobi-field along the geodesic $t(x)$ satisfies mKdV equation

$$\frac{\partial w}{\partial s} = \frac{3}{4}w^2w' - \frac{1}{2}w'''.$$

By a simple calculation one can show that

$$w = -2\frac{\partial}{\partial x}log\iota$$

The geometrical construction of solutions of the KdV hierarchy is based on an infinite dimensional grassmannian $Gr^{(2)}$ defined as follows. Let $L^2(S^1, \mathbf{C})$ be the Hilbert space H and multiplication by z is a unitary operator on the Hil bert space. Let H_+ be the Hilbert subspace of H consisting of boundary values of holomorphic function in the disc $|z| < 1$. Then Grassmannian is the closed subspace $W \subset H$, satisfies

$$(1)\ z^2W \subset W$$

$$(2)\ Pr_+ : W \longrightarrow H_+$$

is a Fredholm operator

$$(3)\ Pr_- : W \longrightarrow H_-$$

is a Hilbert Schmidt operator. Last two conditions mean that W is comparable with H_+.

To interpret the mKdV equation we recall Wilson's [Wi] construction. Let $W \in Gr^n(2)$ be a point in the Grassmannian, satisfying $z^2W \subset W$. Then W/z^2W has dimension n. Let $Fl^{(2)}$ be the periodic flag manifold consists of a pair (W_0, W_1) of closed subspaces $H = L^2(S^1, \mathbf{C})$ such that $W_0 \in Gr^{(2)}$ then

$$z^2W_0 \subset z^1W_1 \subset W_0$$

where z^1W^1 has codimension 1 in W. W_i is a point of $Fl^{(2)}$ and τ_i is the τ - function of W_i. Corresponding to this the mKdV solution is given by

$$v_i = \frac{\partial}{\partial x}log(\tau_i/\tau_{i+1}) \text{ for } 0 \leq i \leq 1$$

and $\tau_2 \equiv \tau_0$.

So it follows from our discussion that

Proposition 3.2 *ι can be interpreted as a τ function of the mKdV equation.*

4 Summary

In this paper we have shown that if $\xi(x)\frac{d}{dx} \in Stab(q,-1)$ then it satisfies (*) and auxiliary equation satisfies (**) where $\xi = \iota(x)^2$. Then (ξ, ι) satisfies super Lie algebra. We have shown that for a particular choice of $\xi =< \chi, A^{-1}\chi >$ in (*), χ satisfies Neumann equation. One can connect the Neumann system to geodesics on the quadric through Gauss mapping. Its Jacobi flow satisfies the mKdV equation and we interpret ι as the τ function of the mKdV equation.

5 References

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