The Geometry of Non-Intersecting Null Rays

The motivation of twistor theory is to replace spacetime points by light rays in Minkowski space as the fundamental physical objects, and to understand positive frequency as a holomorphic property of the spacetime fields. This requires one to complexify the light cone in Minkowski space. The space of complex null rays in **CM** is under the standard Klein correspondence K precisely projective *ambitwistor* space - this is the product space of **PT** with its dual; restricted to pairs of incident twistors, i.e.,

$$\mathbf{PA} := \{(Z^\alpha, W_\alpha) | Z^\alpha W_\alpha = 0\}.$$

Thus in **PT** a complex null ray is a point on a projective 2-plane, $\mathbf{CP}_2$. The point on the 2-plane determines a *plane pencil* which is the set of lines in the plane passing through the given point; the elements of this pencil in **PT** are themselves $\mathbf{CP}_1$'s and each corresponds under K to a single element of a $\mathbf{CP}_1$ in **CM**, which is the complex null ray.

The dimensionality of the space of complex null rays in **CM**, as a manifold, can be seen in two ways; in purely twistorial or purely spacetime terms. It is worth checking that these agree:

1) it is simply the (complex) dimension of **PA** - the 6 complex dimensional product space is subject to one complex equation and so **PA** has complex dimension 5.

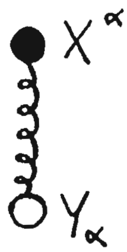
2) consider the space of all lines in a n-space - a generic element of this space can be given by a) the intersection point of the line with a fixed $(n-1)$-plane, and b) the direction of the line - this is simply given by a point on a $(n-1)$-sphere. This produces all lines apart from a set of measure zero - those lines parallel to or lying in the fixed plane. Thus the dimensionality of the space of lines in n-space is $2n-2$. When subject to the Lorentzian null constraint a) is unchanged but in b) the sphere loses one dimension to become a $(n-2)$-sphere. Thus the space of null lines in an n-space is $2n-3$ dimensional - these dimensions are complex if one begins with a n-complex dimensional space, and thus we have agreement with the twistor answer for our case $n = 4$.

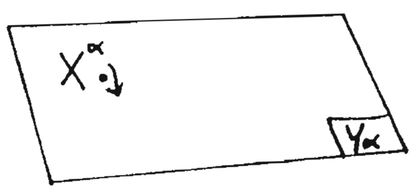
Consider now a pair of non-intersecting null rays. In **PT** there are as we shall see *two* natural conformally invariant classes of such pairs of rays. In **CM** there are again two natural classes of pairs: the set of all *parallel* pairs, and its complement. However, parallelism in **CM** is *not* conformally invariant.

In this article we give the interpretation in **CM** of the conformally invariant classes in **PT**, and also an interpretation in **PT** of the notion of parallelism in **CM**.

Conformally Invariant Classes in PT

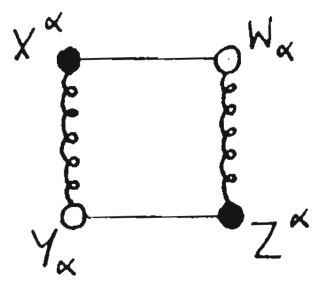
Let a connecting spring denote incident, and a connecting line denote non-incident twistors; let (un)shaded circles denote (dual) twistors. Then a complex null ray can be pictured in **PT** as,

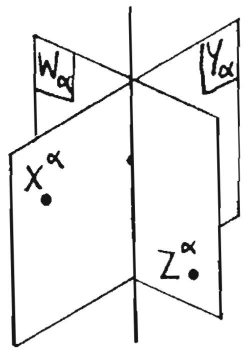




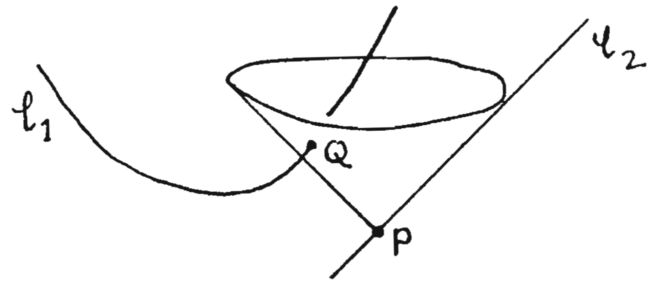
For a pair of null rays we have the following two conformally invariant classes.

Class 1 (measure 1)

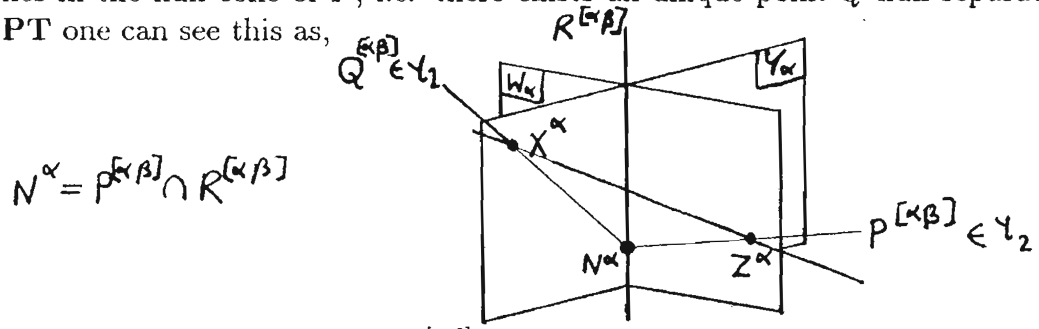




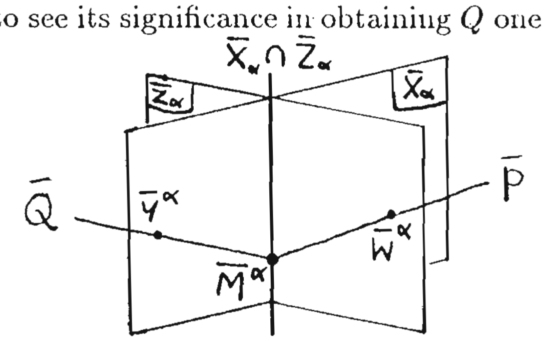
Firstly, note that there is no preferred point on either of the null rays. Also note that the rays are indistinguishable, from the picture in **PT** for class 1 shown. The interpretation in **CM** is,



Choose any point P on $l_2 = (Z^\alpha, W_\alpha)$. Then there exists an unique point Q on l_1 which lies in the null cone of P, i.e. there exists an unique point Q null separated from P. In **PT** one can see this as,

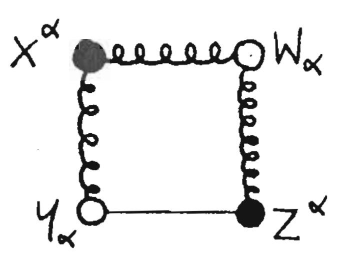


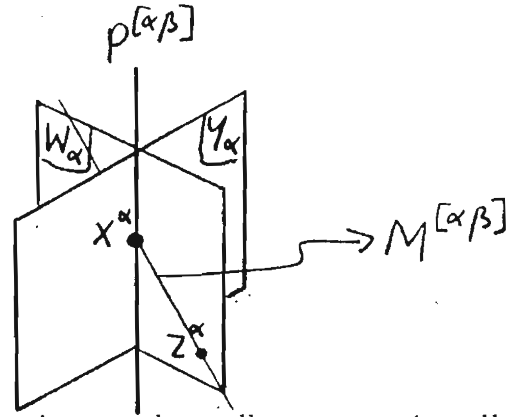
The significance of the line $R^{[\alpha\beta]}$ in obtaining the point Q is clear (note also that this line represents the unique point of intersection of the β-planes corresponding to W and Y). Similarly the line joining X and Z represents the unique point of intersection of the corresponding α-planes - to see its significance in obtaining Q one must complex conjugate the diagram above;



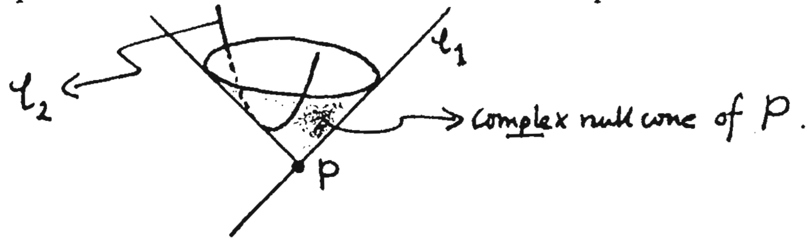
Note $\bar{M}^\alpha$ is the dual of the plane containing X^α, Z^α, and N^α.

Class 2 (measure 0)

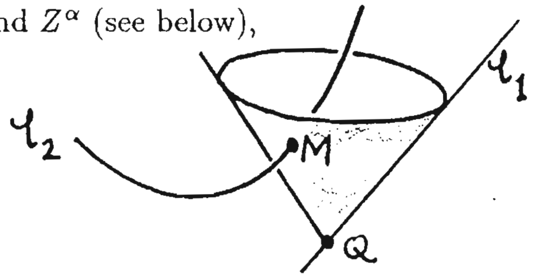




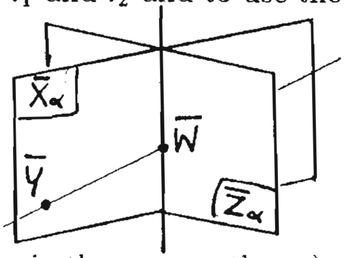
Property a): there exists a point P on l_1 such that its complex null cone contains *all* of l_2. This point is *unique* - under K it is the intersection of the planes W and Y (see below),



Property b): any other point Q on l_1, $Q \neq P$, has the property that its complex null cone contains only one point, M say of l_2, *independent* of Q, which under K is the line of intersection of X^α and Z^α (see below),



The situation is in fact symmetric with respect to interchange of l_1 and l_2. The most economical way to see this is to complex conjugate the diagram in **PT** above and to ask the same questions of the pair $\bar{l}_1$ and $\bar{l}_2$ and to use the result already obtained,



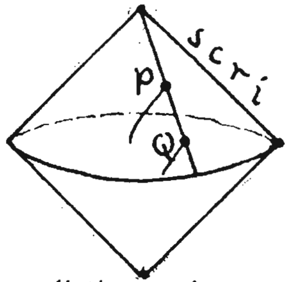
If we now complex conjugate again then we see that a) and b) hold with l_1, l_2 interchanged: for a) the point is represented by $X^{[\alpha}Z^{\beta]}$ and for b) the point of intersection corresponds under K to the intersection of the planes W_α, Y_β.

Thus a pair for this class 2 could be regarded as *ordered* (by saying for example that l_1 is the ray whose (upper indexed) twistor is connected by a spring to l_2), but this ordering is reversed under complex conjugation.

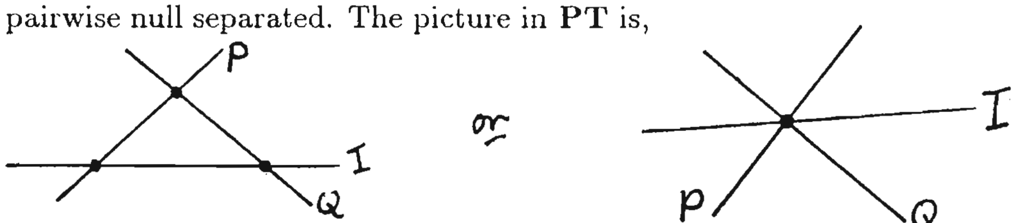
If we define the relation R by: $l_1 R l_2$ if and only if there exists a point on l_1 such that its null cone contains all of l_2; then this relation is both reflexive and symmetric, but fails to be transitive - the negation of R also fails transitivity.

Parallelism in CM

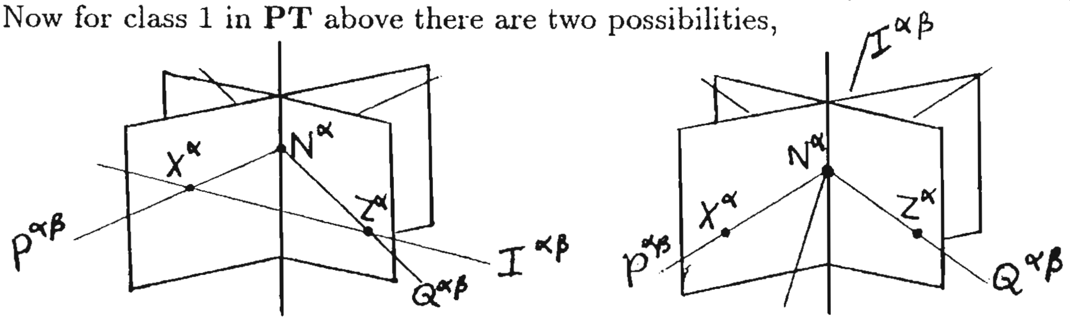
It is necessary here to consider the conformal compactification of **CM**, which we denote $\mathbf{CM}^\sharp$. Then two rays are parallel if and only if they meet *scri* (null infinity) in two points P, Q lying on a common generator - here we shall *exclude* the case that one of the rays itself be a generator of scri, i.e. we exclude null rays at infinity. In **PT** this is to say that the infinity twistor $I^{\alpha\beta}$ is *out* of incidence with all of X^α, Z^α, W_α, Y_α. In $\mathbf{CM}^\sharp$ the picture is,



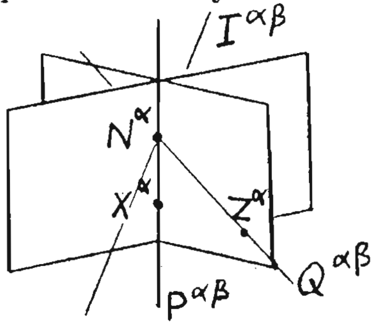
Suppose first that P, Q are *distinct* points on *scri*. In **PT** there are two cases. Let I denote the point at infinity in $\mathbf{CM}^\sharp$. Then the rays are parallel if and only if the points P, Q, I are pairwise null separated. The picture in **PT** is,



Now for class 1 in **PT** above there are two possibilities,

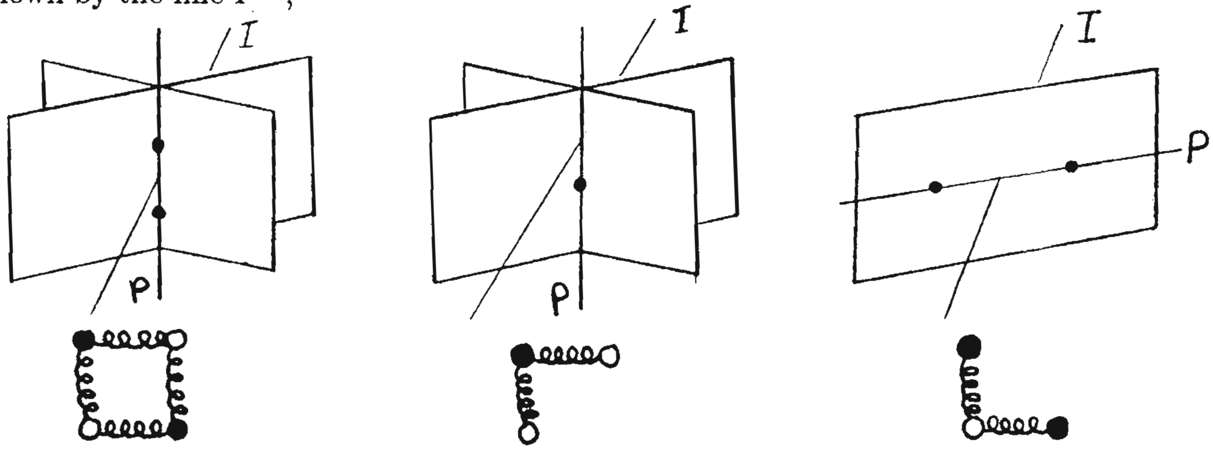


In the left hand picture, I^a lies in the β-plane defined by N^α, X^α, Z^α, and in the right hand picture, I^a lies in the α-plane defined by N^α. For class 2 above there is in fact only one possibility,



The line $I^{\alpha\beta}$ must contain the point N^α and not lie in either of the planes W_α, Y_α. Note in particular that N^α is distinct from X^α, since otherwise $I^{\alpha\beta}$ is forced to lie in the plane W_α or to contain X^α, which gives a null ray at infinity.

Suppose now that $P = Q$. Then the rays intersect at a common *point* of *scri*, and neither of classes 1 or 2 in **PT** described earlier apply - these two classes exclude all points of intersection in *compactified* **CM**. If two complex null rays intersect at a point P of $\mathbf{CM}^\sharp$ then this point is unique. In **PT** there are three cases, and the situation for parallel rays is shown by the line $I^{\alpha\beta}$,



Thanks especially to Franz Muller for hospitality at I.H.E.S., to S.A. Huggett for suggesting this for TN, and to R.P. for helpful comments.

Tim Field.

Conference Proceedings: Twistor Theory

edited by S.Huggett

The proceedings of the 1993 Twistor Conference, edited by S.Huggett, are being published as Volume 169 of the Marcel Dekker, Inc. Series of Pure and Applied Mathematics books.

The contents are as follows:

(continued on page 42)