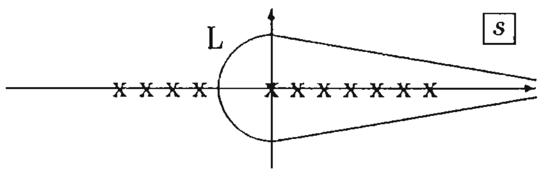
Some Calculational Results For The Massive Propagator In Twistor Diagrams

In [Hodges 91] a Barnes integral was derived for the massive scalar time-like propagator composed with massless *in* and *out* fields. This integral is

$$\mathcal{F}(m;(p-q)^2) = m^2 \int_L (\frac{1}{2})^{2s} \frac{\Gamma(1-s)\Gamma(-s)}{\sin \pi s} (m^2(p-q)^2)^{s-1} ds \qquad (1)$$

which we refer to as the Feynman function. The variable m is the mass of the scalar propagator, p and q are points in $\mathbb{CM}^{\#}$ defining the elementary states used for the *in* and *out* fields. The L path of integration is



The poles of the integrand are marked with Xs. The path of integration encloses the double pole at zero and the triple poles at all the positive integers (not the single poles at all the negative integers), so the Feynman function can be expressed as a sum of residues. We can write down a formula for the residue at the n^{th} pole

$$\frac{2(m^2)^n(y^2)^{n-1}}{\pi}\alpha(n)\{\frac{1}{2}\log^2(\frac{m^2y^2}{4}e^{2\gamma}) + \beta(n)\log(\frac{m^2y^2}{4}e^{2\gamma})$$
$$+\frac{1}{2}[\beta^2(n)+\sigma(n)] + \frac{\pi^2}{3}\} \qquad (2)$$

where

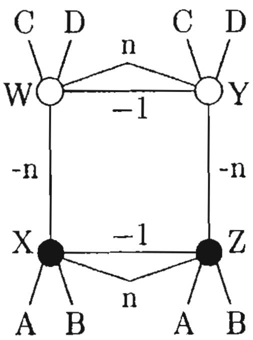
$$\alpha(n) := \frac{(-1)^{n+1}}{4^n n((n-1)!)^2}$$

$$\beta(n) := -\frac{1}{n} + 2\sum_{t=1}^{n-2}\frac{1}{t} + \sum_{r=1}^{n-1}\frac{n}{r(r-n)} - \sum_{r=1}^{n-2}\frac{1-n}{r(r+1-n)}$$

$$\sigma(n) := \frac{1}{n^2} + \frac{2}{(1-n)^2} + 2\sum_{r=1}^{n-2}\frac{1}{(r+1-n)^2}$$

and $y^2 := (p-q)^2$.

The central idea in [Hodges 91] was to regard this sum of residues as a power series in m^2 and that a twistor diagram should correspond to the coefficient of each power. The twistor diagram conjectured to correspond to the residue at n was

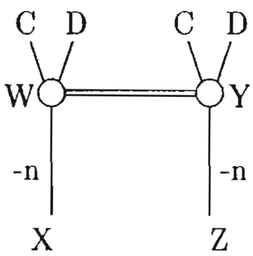


The bent lines labelled with n stand for terms like

$$\frac{1}{(W\,Y)^{n+1}}$$

The twistors A,B and C,D are related to the points p and q by the usual Klein correspondence. Different inhomogeneous parameters are used for the four boundary lines, going anticlockwise from the top: m_1, k_1, m_2, k_2. This enables us to apply differential operators to the lines individually.

Until recently the evaluation of this diagram would have been extremely laborious. However a calculational 'shortcut' has been noticed by Hodges. We can begin with a diagram evaluated in [Hodges 85]



and integrate twice with respect to m_1 to turn it into the top half of our required diagram. This technique is limited by the ambiguity in the constants of integration, but for diagrams of this type Hodges has identified a condition which specifies the constants uniquely [Hodges 94b]. The bottom half of the diagram is completed by usual methods and yields

$$\frac{\Omega^{n-1}}{n((n-1)!)^2}\{\frac{1}{2}\log^2(-\frac{m_1m_2\Omega}{k_1k_2})+\mathcal{A}(n)\log(-\frac{m_1m_2\Omega}{k_1k_2})-\frac{\pi^2}{6}$$

$$-\frac{1}{n}\mathcal{A}(n)+n((n-1)!)^2\mathcal{B}(n)\}+\mathcal{D}(n) \tag{3}$$

where

$$\mathcal{A}(n):=-\frac{1}{n}-\frac{2\Gamma'(n)}{\Gamma(n)}-2\gamma$$

$$\mathcal{B}(n):=\sum_{s=0}^{n-2}\frac{1}{\Gamma(s+1)^2\Gamma(n-s)^2}\{\frac{1}{(s+1)(s-n+1)}[\frac{1}{s-n+1}+\frac{1}{s+1}-f(n,s)]$$

$$-\frac{1}{n(s+1)}[-\frac{1}{s+1}+f(n,s)]\}$$

$$f(n,s):=-\frac{2\Gamma'(s+1)}{\Gamma(s+1)}-\frac{2\Gamma'(n-s)}{\Gamma(n-s)}$$

$$\mathcal{D}(n):=\sum_{s=0}^{n-2}\frac{\Omega^{n-1}}{\Gamma(s+1)^2\Gamma(n-s)^2}(\frac{\Omega m_1m_2}{k_1k_2})^{s-n+1}\frac{1}{(s+1)(s-n+1)^2}$$

and

$$\Omega:=\begin{array}{c}\mathrm{A\ B}\\ \vdash\!\!\dashv\\ \mathrm{C\ D}\end{array}\ (\underbrace{\mathrm{A\ B}}\ \overbrace{\mathrm{C\ D}})^{-1}$$

We make the identifications

$$m_1=m_2=\frac{m}{\sqrt{2}}$$

$$k_1=k_2=e^{-\gamma}$$

$$\Omega=-\frac{1}{2}y^2$$

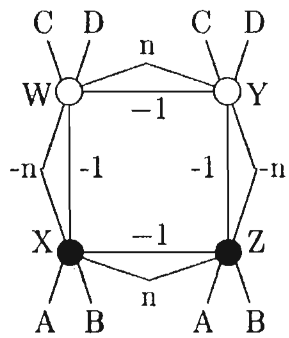
Furthermore, after some algebra we can show that

$$\mathcal{A}(n)=\beta(n)$$

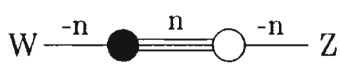
$$-\frac{1}{n}\mathcal{A}(n)+n((n-1)!)^2\mathcal{B}(n)=\frac{1}{2}[\beta^2(n)+\sigma(n)]$$

Thus the answer for the twistor diagram (3) agrees with the answer for the residue (2) apart from two terms. The co-efficients of π^2 are different and the answer for the diagram has the extra terms, $\mathcal{D}(n)$, which have become known as tail terms. These are the source of the extra term in the $n = 2$ diagram mentioned in [Hodges 94a].

The scheme for obtaining diagrams without the tail terms is outlined in [Hodges 94a]. We replace the twistor diagram above with



where the vertical bent lines are shorthand for



We calculate this diagram by integrating the previous diagram with respect to k_1 and k_2 and then taking the double period contour. The answer is

$$\frac{\Omega^{n-1}}{n((n-1)!)^2}\{\frac{1}{2}\log^2(-\frac{m_1 m_2 \Omega}{k_1 k_2}) + \mathcal{A}(n)\log(-\frac{m_1 m_2 \Omega}{k_1 k_2}) - \frac{\pi^2}{2}$$

$$-\frac{1}{n}\mathcal{A}(n) + n((n-1)!)^2 \mathcal{B}(n)\} \tag{4}$$

The tail terms have been successfully eliminated but the π^2 term has changed to $-\pi^2/2$. This happened as a result of taking the extra double period [Hodges 94b]. Thus the discrepancy between the answer for the diagram (4) and the residue (2) is $5\pi^2/6$.

References

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