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Geometric Aspects of Quantum Mechanics

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Abstract. A general theory of non-linear quantum mechanics is considered, for which the state space is a complex manifold with a compatible Riemannian structure: states are points, observables are smooth functions, and the value of an observable at a point is the expectation of the observable in that state. Such a state manifold has a natural symplectic structure that leads to the definition of a Poisson bracket for pairs of functions; the commutator of two observables is $-i$ times the Poisson bracket of the corresponding functions. Associated with each observable is a canonical vector field, obtained by taking its symplectic gradient. The magnitude of such a vector field with respect to the Riemannian metric is proportional to the squared uncertainty of the associated observable, and the usual Heisenberg relations can be seen to hold. The Schrödinger evolution of a state is then described by the special canonical vector field for which the generating function is the expectation of the physical Hamiltonian. The general framework of non-linear quantum mechanics is equivalent to a classical dynamical system on the quantum mechanical state manifold, a result that is *a fortiori* also valid for ordinary linear quantum mechanics. The rate of evolution of a quantum mechanical system along a Schrödinger trajectory in the non-linear theory is twice the uncertainty in the Hamiltonian; this generalises a result in the linear theory due to Anandan & Aharonov (1990). The relation of the non-linear theory to the linear theory is analysed, and in the case for which the state manifold is complex projective space and the Riemannian structure is the unitary-symmetric Fubini-Study metric the theory can be shown to reduce to a non-linear theory investigated by Weinberg (1989) and others. Ordinary linear quantum mechanics entails a further specialisation, for which an analysis is presented by use of projective algebraic geometry. The linear observables of ordinary quantum mechanics are functions for which the associated canonical vector fields are Killing vectors. In the general non-linear theory the scalar curvature of the complex manifold has the status of a preferred, geometrically determined observable, and it is suggested that this observable should be linear the Hamiltonian, with a relation of the form $\langle H \rangle = \lambda + \mu R$, where λ and μ are constants.

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Quantum Measurement and Stochastic Differential Geometry

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Abstract. The state space of a quantum mechanical system can be represented by a complex projective space, the space of rays in the associated Hilbert space. When regarded as a real manifold the state space comes naturally equipped with a Riemannian metric (the Fubini-Study metric) and a compatible symplectic structure. The familiar operations of ordinary quantum mechanics can thus be systematically reinterpreted in the language of differential geometry. It is especially interesting to pose some of the problems of quantum measurement in this spirit, with a view towards scrutinising the probabilistic assumptions that are brought in at various stages in the analysis of quantum dynamics, particularly in connection with state vector reduction. One of the most promising modern approaches to understanding reduction, studied recently by a number of authors, involves the use of non-linear stochastic dynamics to modify the ordinary linear Schrödinger evolution. Here we use methods of stochastic differential geometry to give a systematic geometric formulation for such stochastic models of state vector collapse. In this picture the conventional Schrödinger evolution, which corresponds to a Killing trajectory of the Fubini-Study metric, is replaced by a more general stochastic flow on the state manifold. In the simplest example of such a flow, the volatility term in the stochastic differential equation for the state trajectory is proportional to the gradient of the expectation of the Hamiltonian. The conservation of energy is represented by the requirement that the actual process followed by the expectation of the Hamiltonian, as the state evolves, should be a martingale. This requirement implies the existence of a non-linear term in the drift vector of the state process, which is always oriented opposite the direction of increasing energy uncertainty. As a consequence the state vector necessarily collapses to an energy eigenstate, and an elegant martingale argument can be used to show that the probability of collapse to a given eigenstate, from any particular initial state, is in fact given by precisely the usual quantum mechanical probability.

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