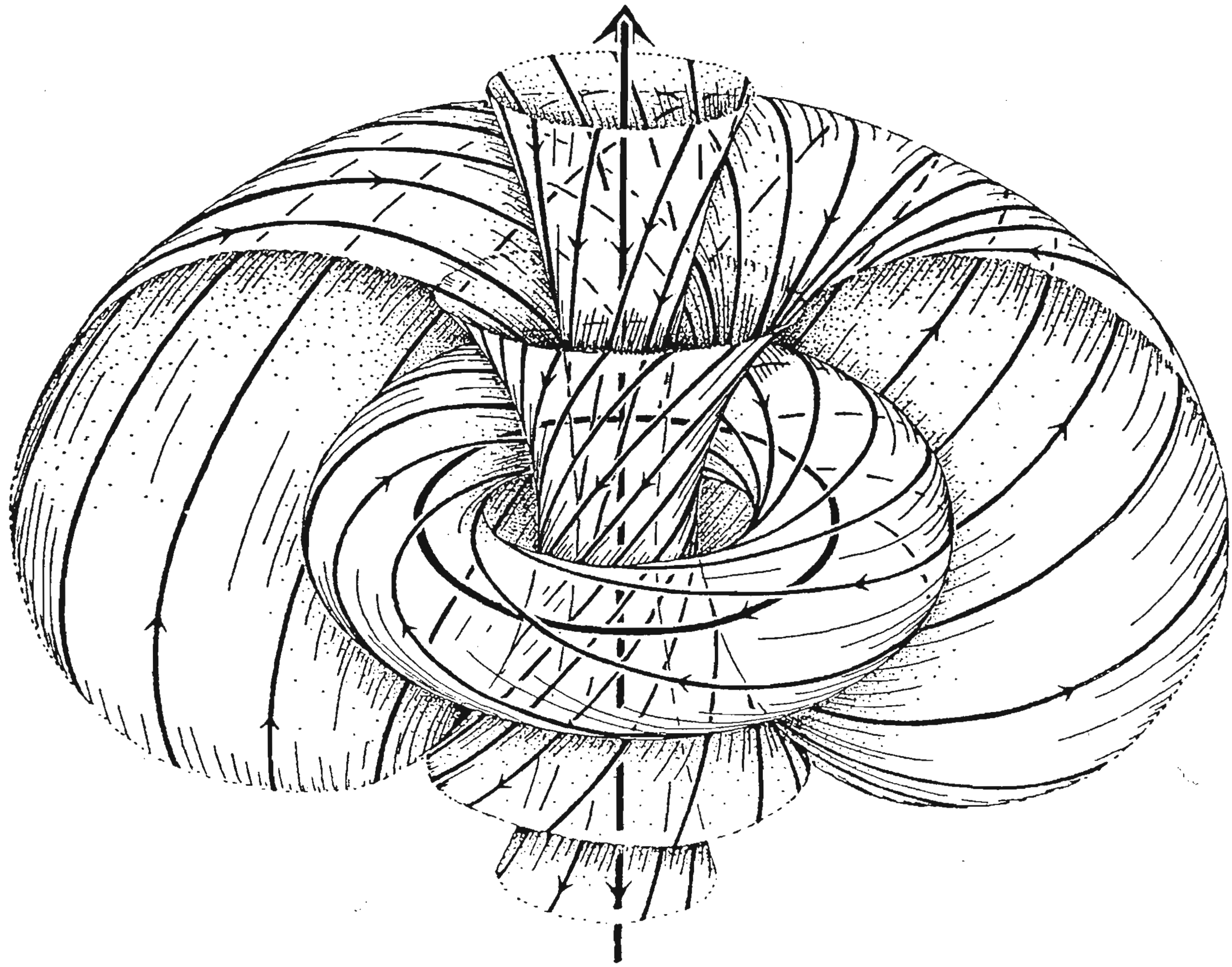
Twistor Newsletter (nº 38: 27 October, 1994)



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A Twistor-Topological Approach to the Einstein Equations

In my articles in TN 31 (pp. 6–8) and TN 32 (1–5) (cf. also RP(1992)) the suggestion was made that the appropriate twistor concept for vacuum space-times is as a *charge* for *helicity* 3/2 *fields*. This idea has been further persued by RP, TN 33 (1–6), RP(1994), LJM & RP, TN 37 (1–6), and JF, TN 37 (7–9). In particular, in RP(1992) and TN 32, I described how, in flat space-time $\mathbb{M}$, one may use a "chopping and pasting" method to work out the "twistor charge", where this charge appears as a "second kind" of gauge freedom for the potentials for the helicity 3/2 field, namely that freedom in the gauge quantities which does not affect the potentials. This is best exhibited in terms of an exact sequence

$$0 \to \begin{Bmatrix}\Omega\\ \Pi\end{Bmatrix} \to \begin{Bmatrix}\omega\\ \pi\end{Bmatrix} \to \begin{Bmatrix}\rho\\ \sigma\end{Bmatrix} \to \begin{Bmatrix}0\\ \psi\end{Bmatrix} \to 0 \qquad \text{(A)}$$

charges gauges potentials fields

(these actually being *local twistor* sheaves: LJM & RP, TN 37). I am here using the "Dirac gauge" for the potentials $\rho_{A'BC}$, $\sigma_{A'B'C}$, which implies that they are *symmetric*, as is the field $\psi_{A'B'C'}$:

$$\rho_{A'BC} = \rho_{A'(BC)}, \quad \sigma_{A'B'C} = \sigma_{(A'B')C}, \quad \psi_{A'B'C'} = \psi_{(A'B'C')},$$

and we have the equations

$$\nabla_A^{A'} \rho_{A'BC} = 0, \qquad \text{(B)}$$

$$\nabla_{B'}^{B} \rho_{A'BC} = 2i\,\sigma_{A'B'C}, \quad \nabla_B^{B'} \sigma_{A'B'C} = 0, \qquad \text{(C)}$$

$$\nabla_{C'}^{C} \sigma_{A'B'C'} = \psi_{A'B'C'}, \quad \nabla_A^{A'} \psi_{A'B'C'} = 0. \qquad \text{(D)}$$

(The factor $2i$ is chosen for later convenience.) The gauge freedom is given by

$$\rho_{A'BC} \mapsto \rho_{A'BC} - i\varepsilon_{BC}\pi_{A'} + \nabla_{CA'}\omega^{B}, \quad \sigma_{A'B'C} \mapsto \sigma_{A'B'C} + \nabla_{CB'}\pi_{A'} \qquad \text{(E)}$$

where

$$\nabla_{A'}^{B}\omega_B = 2i\,\pi_{A'}, \qquad \nabla_B^{A'}\pi_{A'} = 0, \qquad \text{(F)}$$

so $\pi_{A'}$ satisfies the Dirac-Weyl neutrino equation for helicity ½, and ω^A is a "potential" for π (or, equivalently, a "neutrino" field of helicity $-\frac{1}{2}$ which has π as a source).

The "second kind" of gauge freedom, leaving ρ and σ alone, is given by

$$\omega_A \longrightarrow \omega_A + \Omega_A\,, \quad \pi_{A'} \longrightarrow \pi_{A'} + \Pi_{A'} \qquad \text{(G)}$$

where

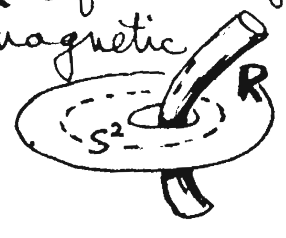
$$\nabla_{AA'}\Omega_B = -i\varepsilon_{AB}\Pi_{A'}\,, \quad \nabla_{AA'}\Pi_{B'} = 0\,, \qquad \text{(H)}$$

so $(\Omega^A, \Pi_{A'})$ are the spinor parts of a twistor (i.e. a constant local twistor).

For the "chopping and pasting" procedure, it is helpful to make a comparison with the electromagnetic case. Here, I refer to the <u>magnetic</u> charge that arises from Dirac's procedure which makes use of the non-global nature of the usual electromagnetic potential for a magnetic monopole. The relevant exact sequence is now (part of) the deRham sequence

$$0 \rightarrow \mathbb{C} \xrightarrow{i} \Omega^0 \xrightarrow{d} \Omega^1 \xrightarrow{d} \Omega^2 \qquad \text{(I)}$$

(Ω^r being the sheaf of **holomorphic** r-forms) and the procedure amounts to evaluating a closed 2-form (the Maxwell field) on a 2-cycle which lies in an open region **R** (of homotopy type S^2) surrounding a world-tube of magnetic charge. Breaking the sequence up as



$$0 \rightarrow d\Omega^0 \rightarrow \Omega^1 \rightarrow d\Omega^1 \rightarrow 0$$
$$0 \rightarrow \mathbb{C} \rightarrow \Omega^0 \rightarrow d\Omega^0 \rightarrow 0$$

(the relevant region being **R**, in each case), we get

$$\hookrightarrow H^0(d\Omega^1) \rightarrow H^1(d\Omega^0) \rightarrow H^1(\Omega^1)$$

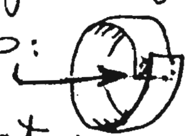
and

$$H^1(\Omega^0) \longrightarrow H^1(d\Omega^0) \rightarrow H^2(\mathbb{C}) \rightarrow H^2(\Omega^0).$$

Since $H^1(\Omega^1) = H^1(\Omega^0) = H^2(\Omega^0) = 0$, and $H^2(\mathbb{C}) = \mathbb{C}$, we have a map from $H^0(d\Omega^1)$ onto $\mathbb{C}$, this being the evaluation of the total magnetic charge of the world-tube. Although this could have been obtained as a Gauss integral of the field over S^2, the above procedure is seen more explicitly in terms of the following ("chopping and pasting") operation. The Maxwell field F is a closed (whence locally exact) 2-form, and is therefore in $H^0(d\Omega^1)$. Think of **R** being chopped into two (slightly overlapping) hemispherical, topologically trivial, regions $\mathbf{R}_1$, $\mathbf{R}_2$. In each region $\mathbf{R}_i$, we have $F = d\overset{i}{A}$, whence

$$d(\overset{2}{A}-\overset{1}{A}) = d\overset{2}{A} - d\overset{1}{A} = F - F = 0$$

on the overlap (homotopy type S^1).

We can then chop this overlap into two — or just cut it once, with an overlap:  — and try to write $\overset{2}{A}-\overset{1}{A}$ as $d\varphi$. We find that the scalar field φ jumps as we go around, the difference between the two φ-values $\overset{I}{\varphi}$, $\overset{II}{\varphi}$ at the overlap being a constant (since $d(\overset{II}{\varphi}-\overset{I}{\varphi}) = d\overset{II}{\varphi} - d\overset{I}{\varphi} = (\overset{2}{A}-\overset{1}{A}) - (\overset{2}{A}-\overset{1}{A}) = 0$), this constant being the required magnetic charge μ. In each case, we simply follow back along the de Rham sequence — which we can express graphically as the (Zeeman) diagram:

$$0 \to \{\mu\} \xrightarrow{i} \{\varphi\} \xrightarrow{d} \{A\} \xrightarrow{d} \{F\} \to 0 \qquad \text{(J)}$$

We wish to apply this to the helicity $3/2$ sequence (A)

$$0 \to \left\{\begin{matrix}\Omega\\ \Pi\end{matrix}\right\} \to \left\{\begin{matrix}\omega\\ \pi\end{matrix}\right\} \to \left\{\begin{matrix}\rho\\ \sigma\end{matrix}\right\} \to \left\{\begin{matrix}0\\ \psi\end{matrix}\right\} \to 0$$

with a view to using it in curved Ricci-flat space-time $\mathcal{M}$. In $\mathcal{M}$, however, the space $\left\{\begin{matrix}\Omega\\ \Pi\end{matrix}\right\}$ does not exist (or becomes trivial) since the equations (H) become inconsistent. Moreover, the equations (B) and (D) also become inconsistent, and must be abandoned. We can retain (C), (E), and (F), but in dropping (B), we can obtain an additional "field" quantity which lies in the extra gauge freedom

$$\rho_{A'BC} \mapsto \rho_{A'BC} + \overset{\bullet}{\rho}_{A'BC}$$

where

$$\nabla^{B}_{B'} \overset{\bullet}{\rho}_{A'BC} = 0 .$$

This "field" is a helicity $-3/2$ quantity $\tilde{\psi}_{ABC}\,(=\tilde{\psi}_{(ABC)})$, in flat space-time, defined by

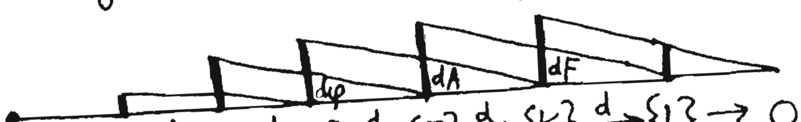
$$\nabla^{A'}_{A}\,\overset{\bullet}{\rho}_{A'BC} = \tilde{\psi}_{ABC}$$

(see LJM&RP, TN 37), the gauge freedom in $\overset{\bullet}{\rho}$ being part of that (the "π-part") which is already a freedom in ρ. In $\mathbb{M}$, we still have an exact sequence

$$0 \rightarrow \left\{ {\Omega \atop \Pi} \right\} \rightarrow \left\{ {\omega \atop \pi} \right\} \rightarrow \left\{ {\rho \atop \sigma} \right\} \rightarrow \left\{ {\tilde{\psi} \atop \psi} \right\} \rightarrow 0, \qquad \text{(K)}$$

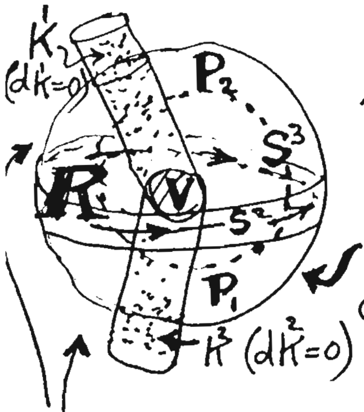
but now the twistor (Ω, Π) is not merely the charge Z^α for ψ, but contains a contribution from the projection part (i.e. secondary part) of the charge (dual twistor) $\mathbf{W}_\alpha$ for $\tilde{\psi}$. Thus, (Ω, Π) has the form "$Z^\alpha + I^{\alpha\beta} W_\beta$".

Of course, we require some sort of <u>non</u>-<u>linear</u> twistor space for $\mathcal{M}$, so it is just as well that a (Ω, Π) defined simply by (H) will not do. For various suggestive (but, as yet, inadequate) reasons, it appears that it would be greatly helpful if the flat-space sequence (K) could be extended to a longer one. (Roughly: each "twistor (Ω,Π)" exists, in some sense, "somewhere" — say at $\mathbb{C}\mathscr{I}^+$ — but different twistors exist at different places, so they cannot be added. The vacuum equations allow this twistor concept to be "spread" about the space-time. However, (K) really "looks at" regions of $\mathcal{M}$ that are essentially no more than two-dimensional, and we need at least three dimensions to "feel out" the space-time.) The idea is that we try to mirror what goes on with the de Rham sequence (I) extended further to the right, which I write as



$$0 \rightarrow \{\mu\} \xrightarrow{i} \{\varphi\} \xrightarrow{d} \{A\} \xrightarrow{d} \{F\} \xrightarrow{d} \{K\} \xrightarrow{d} \{L\} \rightarrow 0$$

Here, first, we allow the "Maxwell field" F to have a magnetic charge density K (a 3-form $K = dF$), so the standard Maxwell equations are violated. At this stage, this density is still conserved, however ($dK = 0$, i.e. $\nabla_a {}^*K^a = 0$), owing to the existence of F. But we can also allow some region **V** of violation of magnetic charge conservation, this charge violation being expressed by the 4-form $L=dK$, where K is not now of the form dF. Suppose that **V** is a small, topologically trivial, region of space-time,



where we can imagine world tubes of conserved magnetic charge density entering and leaving **V**. Now, surround **V** by an (open) region **P** of space-time (homotopy type S^3). We can imagine **P** to be built up from two topologically trivial "hemispherical" regions $\mathbf{P}_1$ ("past hemisphere") and $\mathbf{P}_2$ ("future hemisphere") which overlap in a region **R** (homotopy type S^2) containing an "equatorial" S^2. In $\mathbf{P}_i$, we can find $\overset{i}{F}$ such that $d\overset{i}{F}=\overset{i}{K}$, for each i, and in **R** we have $F=\overset{2}{F}-\overset{1}{F}$, with $dF=0$.

From there on, we proceed as before, to obtain the magnetic charge μ — which now measures the total charge creation in the region **V**. More abstractly, we extend the original evaluation procedure to obtain a map from $H^0(\mathbf{P}, d\Omega^2)$ onto $H^3(\mathbf{P}, \mathbb{C}) \cong \mathbb{C}$. The field K, being locally of the form dF, is an element of $H^0(\mathbf{P}, d\Omega^2)$.

To extend the sequence (K), we make use of various ideas, and, most specifically, one due to Jörge Frauendiner (JF in TN 37). It will simplify matters if we first consider merely the projection part of (K) suitably extended. In JF's scheme, instead of using a $\sigma_{A'B'C}$ subject to (C), we consider the weaker (conformally invariant) equation

$$\nabla^{B'}_{(B}\sigma_{C)A'B'} = 0 \qquad \text{(L)}$$

(with the symmetry $\sigma_{A'B'C} = \sigma_{(A'B')C}$ still holding), and then we define

$$\beta_{A'} = \nabla^{BB'}\sigma_{A'B'B} \qquad \text{(M)}$$

which, by virtue of the vacuum equations satisfies

$$\nabla^{AA'}\beta_{A'} = 0 \qquad \text{(N)}$$

with $\beta_{A'}$ invariant under the same gauge freedom as before:

$$\sigma_{A'B'C} \mapsto \sigma_{A'B'C} + \nabla_{B'C}\pi_{A'}\,,\quad \nabla^{AA'}\pi_{A'} = 0. \qquad \text{(O)}$$

However, the sequence is now not exact, and we have a Zeeman diagram looking like

$$0 \to \{\Pi\} \xrightarrow{i} \{\pi\} \xrightarrow{\nabla} \{\sigma\} \xrightarrow{\nabla\cdot} \{\beta\} \to 0 \,.$$

Exactness can be restored by introducing the somewhat odd-looking sheaf of "σ moddedout by π" to describe the "sourced spin 3/2 field":

$$0 \to \{\Pi\} \to \{\pi\} \to \{\sigma\} \to \{\sigma\}/\{\pi\} \to \{\beta\} \to 0 \qquad \text{(P)}$$

and now β describes a kind of charge density for this field.

We can try to understand this by resorting to the abstract procedures adopted previously, where all homology groups refer to the region **P** described earlier, and we imagine some sort of "twistor creation" taking place in the region **V** that **P** surrounds. We have (in an obvious notation)

$$H^0(\sigma/\pi) \to H^0(\beta) \to H^1(\nabla\sigma) \to H^1(\sigma/\pi) \to H^1(\beta) \qquad \text{(Q)}$$

from $\quad 0 \to \{\nabla\sigma\} \to \{\sigma\}/\{\pi\} \to \{\beta\} \to 0$.

In $\mathbb{M}$, the full "twistor charge" for helicity 3/2 should be found in $H^1(\nabla\sigma)$, since this contains the helicity raised "Gauss integral" for the field $\psi = \nabla\sigma$ defined by the potential σ according to (D). It can be seen that $H^1(\beta) = 0$, but we must examine $H^1(\sigma/\pi)$. Using

$$0 \to \{\pi\}/\{\Pi\} \to \{\sigma\} \to \{\sigma\}/\{\pi\} \to 0$$

we obtain

$$H^1(\sigma) \to H^1(\sigma/\pi) \to H^2(\pi/\Pi) \to H^2(\sigma).$$

By results due to JF and GAJS, it follows that $H^1(\sigma) = H^2(\sigma) = 0$, whence $H^1(\sigma/\pi) \cong H^2(\pi/\Pi)$. Using

$$0 \to \{\Pi\} \to \{\pi\} \to \{\pi\}/\{\Pi\} \to 0$$

we obtain
$$H^2(\pi) \to H^2(\pi/\Pi) \to H^3(\Pi) \to H^3(\pi).$$
We have $H^2(\pi) = H^3(\pi) = 0$, but $H^3(\Pi)$ is just the primed spin space $\mathbb{S}_{A'}$ ($\cong \mathbb{C}^2$). This gives us back the "charge integral" that we would have obtained by directly applying a "chop and paste" procedure to σ. It delivers (in the absence of the second potential ρ) merely the projection part $\Pi_{A'}$ of the twistor $(\Omega^A, \Pi_{A'})$. However, the <u>full</u> twistor must be in $H^1(\nabla\sigma)$, whence it follows that the primary part lies in $H^0(\beta)$ (when $\Pi_{A'} = 0$). Somehow, in the sequence (P), the twistor is "split" between the two ends of the sequence, Ω^A being found in H^0 of the right-hand end and $\Pi_{A'}$ in H^3 of the left-hand end.

To proceed further, we must re-introduce the second potential ρ. This gives the whole scheme another (short) exact sequence structure, involving
$$0 \to \{\mathring{\rho}\} \to \{\dot{\rho}\} \to \{\dot{\sigma}\} \to 0,$$
where (L) must be filled out to
$$\nabla^{B'}_{(B}\dot{\sigma}_{C)A'B'} = 0, \quad \nabla^{B}_{(B'}\dot{\rho}_{A')BC} = 2i\dot{\sigma}_{A'B'C} \qquad \text{(R)}$$
$\mathring{\rho}_{A'BC}$ subject to $\nabla^{B}_{(B'}\mathring{\rho}_{A')BC} = 0$ acting as freedom in $\dot{\rho}$ (each spinor symmetric in the relevant index pair). All this works in Ricci-flat M, with gauge freedom for $\dot{\rho}, \dot{\sigma}$ the same as with ρ, σ, as before, in (E), (F). In $\mathbb{M}$, we get

$$\begin{array}{ccccccccccccc}
 & & 0 & & 0 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \{\mathring{\Omega}\} & \to & \{\mathring{\omega}\} & \to & \{\mathring{\rho}\} & \to & \{\mathring{\dot{\rho}}\}/\{\mathring{\omega}\} & \to & \{\mathring{\alpha}\} & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \left\{\begin{matrix}\Omega \\ \Pi\end{matrix}\right\} & \to & \left\{\begin{matrix}\omega \\ \pi\end{matrix}\right\} & \to & \left\{\begin{matrix}\rho \\ \sigma\end{matrix}\right\} & \to & \left\{\begin{matrix}\dot{\rho} \\ \dot{\sigma}\end{matrix}\right\}/\left\{\begin{matrix}\omega \\ \pi\end{matrix}\right\} & \to & \left\{\begin{matrix}\alpha \\ \beta\end{matrix}\right\} & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \{\Pi\} & \to & \{\pi\} & \to & \{\sigma\} & \to & \{\dot{\sigma}\}/\{\pi\} & \to & \{\beta\} & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & & 0 & & 0 & &
\end{array} \qquad \text{(S)}$$

as a commutative exact diagram of sheaves. Here, we have

$$\alpha_C = \tfrac{1}{2}\nabla^{BB'}\rho_{B'BC}\,, \quad \beta_{A'} = \nabla^{BB'}\sigma_{A'B'B} \qquad (T)$$

(the "freedom" in α and ρ being denoted $\mathring{\alpha}$, $\mathring{\rho}$, etc.) and the equations

$$\nabla^{A}_{A'}\alpha_A = 2i\,\beta_{A'}\,, \quad \nabla^{A'}_{A}\beta_{A'} = 0$$

(compare (F)). As sheaf sequences, everything seems to work consistently from the third ($\{\omega\}$, $\{{\omega \atop \pi}\}$, $\{\pi\}$) column out to the right, provided that $\mathcal{M}$ is Ricci-flat. In $\mathbb{M}$, we have, for the second column, the standard Poincaré invariant twistor short exact sequence

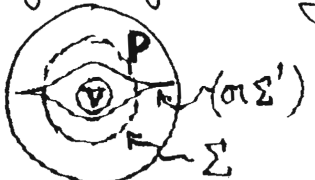
$$0 \to S^A \to \mathbb{T}^\alpha \to S_{A'} \to 0.$$

In fact, we find four versions of this sequence (two in dual form) running round the outside of the square

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \leftarrow & S_A & \leftarrow & \mathbb{T}_\alpha & \leftarrow & S^A & \leftarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \leftarrow & \mathbb{T}^\alpha & \leftarrow & \boxed{\text{charges for } \psi \text{ and } \tilde{\varphi}} & \leftarrow & \mathbb{T}_\alpha & \leftarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \leftarrow & S_{A'} & \leftarrow & \mathbb{T}^\alpha & \leftarrow & S^A & \leftarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

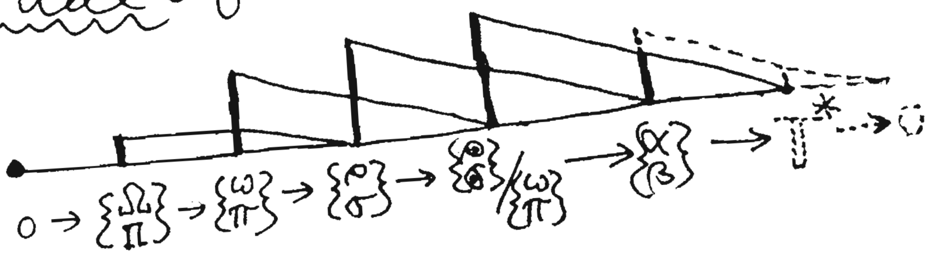
There is a certain "twist" in this diagram owing to the "$Z^\alpha + I^{\alpha\beta}W_\beta$" nature of (Ω, Π), as mentioned earlier (after (K)). The second column of this sequence comes from H^3 of the second column of (S), and the fourth column comes from H^0 of the sixth column of (S).

We can also introduce duals for the spaces of these various types of "field" in the region **P**, constructing a divergence-free vector J^a, to be integrated over some topologically non-trivial compact 3-surface Σ, ($\cong S^3$) within **P**. Remarkably, we find that the relevant dual sheaf sequence is simply the same sequence as (P) all

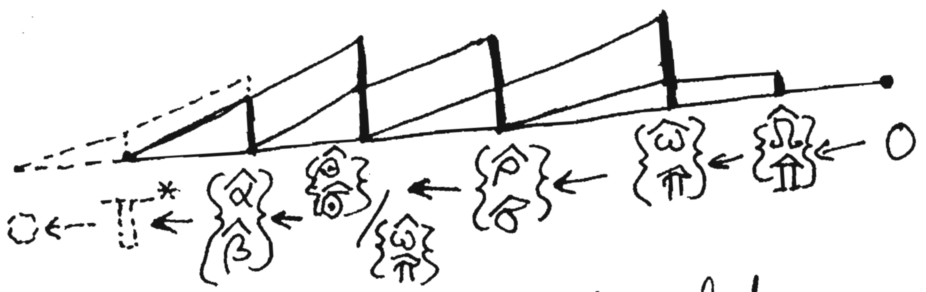


over again, but running in the opposite direction:

Dual of



is



where to express this duality we use

$$J^{AA'} = \omega^{A}\hat{\beta}^{A'} + \pi^{A'}\hat{\alpha}^{A} \quad \text{or} \quad \rho^{AB}_{B'}\hat{\odot}^{A'B'}_{B} + \sigma^{A'B'}_{B}\hat{\odot}^{AB}_{B'}$$

$$\text{or} \quad \odot^{AB}_{B'}\hat{\sigma}^{A'B'}_{B} + \odot^{A'B'}_{B}\hat{\rho}^{AB}_{B'} \quad \text{or} \quad \alpha^{A}\hat{\pi}^{A'} + \beta^{A'}\hat{\omega}^{A},$$

as the case may be (possibly modulo factors). The dotted part at the end of each sequence is there if we consider fields throughout **P** (i.e. $H^0(\mathbf{P},\ )$), whereas the dotted part is absent if we merely consider sheaves. The existence of this duality serves to confirm some of the statements concerning $H^0(\mathbf{R}, \beta)$, etc., made earlier.

There is still a good deal that remains mysterious about what happens in a general Ricci-flat M. Work in progress.

Thanks to LJM and JF (and also GAJS) especially.

Roger P.

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Deformed Twistor spaces and the KP equation

While twistor theory continues to come to terms with the geometry of the KP equation, one closely related $(2+1)$-dimensional integrable system that does possess such a description is the dispersionless KP (or dKP) equation:

$$(u_{2,t} - u_2 u_{2,x})_x = u_{2,yy}\,. \tag{1}$$

This arises from the following equations for the 2-form $\omega(\lambda)$:

$$\begin{aligned} \omega(\lambda) \wedge \omega(\lambda) &= 0\,, \\ d\omega(\lambda) &= 0\,, \end{aligned} \tag{2}$$

where $\omega = d\mathcal{B}_1 \wedge dx + d\mathcal{B}_2 \wedge dy + d\mathcal{B}_3 \wedge dt$, and

$$\begin{aligned} \mathcal{B}_1 &= \lambda\,, \\ \mathcal{B}_2 &= \frac{\lambda^2}{2} + u_2(x,y,t)\,, \\ \mathcal{B}_3 &= \frac{\lambda^3}{3} + \lambda u_2(x,y,t) + u_3(x,y,t)\,. \end{aligned}$$

Equation (2) is equivalent to the zero-curvature equation

$$\frac{\partial \mathcal{B}_2}{\partial t} - \frac{\partial \mathcal{B}_3}{\partial y} + \{\mathcal{B}_2, \mathcal{B}_3\} = 0 \tag{3}$$

with $\{\ ,\ \}$ being the Poisson bracket

$$\{f,g\} = \frac{\partial f}{\partial \lambda}\frac{\partial g}{\partial x} - \frac{\partial g}{\partial \lambda}\frac{\partial f}{\partial x}\,.$$

Equating powers of λ in (3) gives, on eliminating $u_3(x,y,t)$ the dKP equation (1). Equation (2) imply (locally) the existence of functions $\mathcal{P}(\lambda)$ and $\mathcal{Q}(\lambda)$ such that $\omega = d\mathcal{P} \wedge d\mathcal{Q}$, and any two such set of coordinates are connected by a Riemann-Hilbert problem.

Conventional approaches to the KP equation use the algebra of pseudo-differential operators. However, an alternative approach which is closer to the above derivation of the dKP equation may be obtained by replacing the Poisson bracket in (3) with the Moyal bracket [1]:

$$\{f,g\}_\kappa = \sum_{s=0}^{\infty} \frac{(-1)^s \kappa^{2s}}{(2s+1)!} \sum_{j=0}^{2s+1} (-1)^j \begin{pmatrix} 2s+1 \\ j \end{pmatrix} (\partial_x^j \partial_\lambda^{2s+1-j} f)(\partial_x^{2s+1-j} \partial_\lambda^j g)\,. \tag{4}$$

With the same $\mathcal{B}_2$ and $\mathcal{B}_3$ as before equation (3) gives (on replacing the Poisson bracket with the Moyal bracket and on eliminating u_3) the KP equation

$$\left(\tfrac{1}{3}\kappa^2 u_{2,xxx} + u_{2,t} - u_2 u_{2,x} \right)_x = u_{2,yy}\,. \tag{5}$$

In the $\kappa \to 0$ limit the Moyal bracket collapses to the Poisson bracket and one recovers the dKP equation. Thus we have a description of the KP equation which avoids the use of pseudo-differential operators. A similar Moyal-algebraic deformation of the self-dual vacuum equation was introduced in [2] and shown by Takasaki [3] to be integrable via a Riemann-Hilbert problem in the corresponding Moyal loop group.

At first sight the definition (4) looks somewhat unwieldy, but it is, in many ways, very natural. If one wants to deform the Poisson bracket by introducing higher-order derivative terms, the Jacobi identity turns out to be highly restrictive, and one is automatically lead to the Moyal bracket. Moreover, the bracket may be written in terms of an associative $\star$-product

$$\{f,g\}_\kappa = f\star g - g\star f\,.$$

Such $\star$-products have a long history, having been studied by both Moyal [1] and Weyl [4].

Thus, in conclusion, one possible approach to the understanding of the geometry of the KP equation might be to try to formulate a version of twistor theory which makes use of this deformed Poisson bracket.

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Ian Strachan

Reduced Hypersurface Twistor Spaces — M. Dunn.

In Woodhouse & Mason (1988) twistor space **PT** was factored out by the Killing vector symmetries ∂_t and ∂_θ to obtain a reduced twistor space R with non-Hausdorff structure as shown in Figure 1.

In this article it will be shown that for a *curved* vacuum space-time M which is cylindrically symmetric (i.e. has spacelike Killing vectors ∂_z and ∂_θ), its hypersurface twistor space can be reduced by those symmetries to give a reduced hypersurface twistor space $\mathcal{R}$ whose structure is identical to that of R. It will also be shown how the initial data of the metric on the hypersurface can be encoded into this structure.

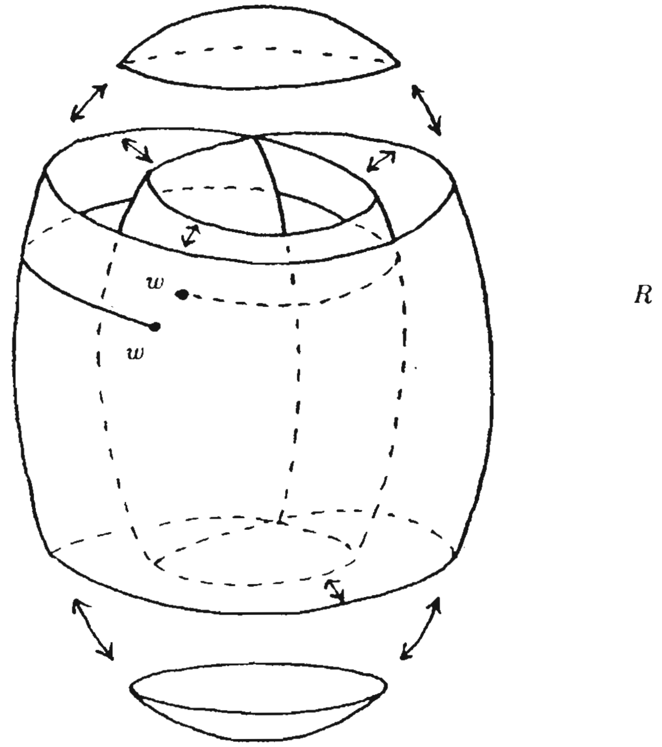

Figure 1

The metric of a cylindrically symmetric space-time can be written in *Weyl's canonical coordinates*:

$$\mathrm{d}s^2 = \Omega^2(\mathrm{d}t^2 - \mathrm{d}r^2) - f(\mathrm{d}z + \omega\mathrm{d}\theta)^2 - \frac{r^2}{f}\mathrm{d}\theta^2, \tag{1}$$

where Ω, f and ω are functions of t, r only (see Kramer *et al.* (1980)). The vacuum field equations $R_{ab} = 0$ imply that there exists a potential ψ such that

$$\psi_t = \frac{f^2}{r}\omega_r, \quad \psi_r = \frac{f^2}{r}\omega_t. \tag{2}$$

Defining the *Ernst potential* $\mathcal{E} = f + i\psi$, the field equations take the form

$$\mathcal{E}_{rr} + \tfrac{1}{r}\mathcal{E}_r - \mathcal{E}_{tt} = \tfrac{2}{\mathcal{E}+\overline{\mathcal{E}}}(\mathcal{E}_r^2 - \mathcal{E}_t^2); \tag{3a}$$

$$\left.\begin{aligned} k_r &= \tfrac{r}{(\mathcal{E}+\overline{\mathcal{E}})^2}(\mathcal{E}_r\overline{\mathcal{E}}_r + \mathcal{E}_t\overline{\mathcal{E}}_t), \\ k_t &= \tfrac{r}{(\mathcal{E}+\overline{\mathcal{E}})^2}(\mathcal{E}_r\overline{\mathcal{E}}_t + \overline{\mathcal{E}}_r\mathcal{E}_t). \end{aligned}\right\} \tag{3b}$$

The equation (3a), the *evolution equation*, is the integrability condition for equations (3b), the *constraint equations*, so all the information of the space-time is contained in $\mathcal{E}$. The values of $\mathcal{E}$ and $\mathcal{E}_t$ on a spacelike hypersurface $\mathcal{H}$ constitute the initial data for the space-time.

Recall that hypersurface twistors (strictly speaking, *dual* hypersurface twistors, which I am using for convenience of notation, avoiding a lot of primes!) correspond to curves in the complex "thickening" of $\mathcal{H}$, $\mathbf{C}\mathcal{H}$, called β-curves. These are the analogue of β-planes in flat space, which do not in general exist in curved space-times, since the curvature causes integrability problems. However, the restriction of the β-plane distribution (the vector fields spanned by $\{\eta^A\nabla_{AA'}\}$) to $\mathbf{C}\mathcal{H}$ is just a single vector field

$$\mathbf{V}_{\mathbf{C}\mathcal{H}} = \eta^A N^{BA'}\eta_B\nabla_{AA'}, \tag{4}$$

where N^a is a normal vector field to $\mathcal{H}$. The integral curves of $\mathbf{V}_{\mathbf{C}\mathcal{H}}$ are the β-curves.

The spinor η^A is parallelly propagated along the β-curve, which tells us that the lift of $\mathbf{V}_{\mathbf{C}\mathcal{H}}$ to the spin bundle S^A is

$$\mathbf{V}_{S^A} = \eta^{\mathbf{A}}N^{\mathbf{DA'}}\eta_{\mathbf{D}}\left(\nabla_{\mathbf{AA'}} - \gamma_{\mathbf{AA'B}}{}^{\mathbf{C}}\eta^{\mathbf{B}}\frac{\partial}{\partial\eta^{\mathbf{C}}}\right), \tag{5}$$

where $\gamma_{\mathbf{AA'B}}{}^{\mathbf{C}}$ are the *spin-coefficients* (see Penrose & Rindler (1984)).

We next factor out by the Euler vector field $\Upsilon = \eta^{\mathbf{A}}\partial/\partial\eta^{\mathbf{A}}$ to obtain a vector field $\mathbf{V}_{\mathbf{P}S^A}$ on the projective spin bundle.

Hypersurface twistor space $\mathbf{P}\mathcal{T}^*$ is the space of β-curves, i.e. the quotient of $\mathbf{P}S^A$ by $\mathbf{V}_{\mathbf{P}S^A}$. We factor out this space by the symmetries induced by ∂_z and ∂_θ to obtain $\mathcal{R}$.

Alternatively, we could first reduce $\mathbf{P}S^A$ by the symmetries to obtain a space $\mathcal{F}$ (and an associated vector field $\mathbf{V}_{\mathcal{F}}$, tangent to the β-curves on $\mathcal{F}$), and then take the quotient by $\mathbf{V}_{\mathcal{F}}$ to get $\mathcal{R}$. This route will give us the structure of $\mathcal{R}$.

The natural null tetrad to use for a cylindrically symmetric space-time is:

$$\begin{aligned} D &= \tfrac{1}{\sqrt{2\Omega}}(\partial_t + \partial_r), & D' &= \tfrac{1}{\sqrt{2\Omega}}(\partial_t - \partial_r), \\ \delta &= \lambda\left(\left(\tfrac{r}{f} + i\omega\right)\partial_z - i\partial_\theta\right), & \delta' &= \lambda\left(\left(\tfrac{r}{f} - i\omega\right)\partial_z + i\partial_\theta\right), \end{aligned} \tag{6}$$

where $\lambda = r^{-1}\sqrt{f/2}$. We take our hypersurface to be one of constant time, so that the normal vector field to $\mathcal{H}$ is $N^a = o^A o^{A'} + \iota^A\iota^{A'}$. The tangent vector field to the β-curves on $\mathbf{C}\mathcal{H}$ is

$$\mathbf{V}_{\mathbf{C}\mathcal{H}} = -uvD - v^2\delta' + u^2\delta + uvD', \tag{7}$$

where $u = \eta^0, v = \eta^1$. The only non-zero spin-coefficients are

$$\begin{aligned}
\varepsilon = -\gamma' &= \tfrac{1}{2\Omega} D\Omega + \tfrac{if}{4r} D\omega, \\
\varepsilon' = -\gamma &= \tfrac{1}{2\Omega} D'\Omega - \tfrac{if}{4r} D'\omega, \\
\rho &= -\tfrac{1}{2r} Dr, \\
\rho' &= -\tfrac{1}{2r} D'r, \\
\sigma &= \tfrac{f}{2r} D\left(\tfrac{r}{f} + i\omega\right), \\
\sigma' &= \tfrac{f}{2r} D'\left(\tfrac{r}{f} - i\omega\right),
\end{aligned} \tag{8}$$

where $\varepsilon = \gamma_{00'0}{}^0$, etc., as in Penrose & Rindler (1984).

This gives us the tangent vector field to the β-curves on $\mathcal{F}$ as:

$$\begin{aligned}
\mathbf{V}_{\mathcal{F}} &= -2\partial_r + \sqrt{2}\Omega\left(\sigma'\zeta^3 - 2(\varepsilon+\varepsilon')\zeta + \sigma\zeta^{-1}\right)\partial_\zeta \\
&= \tfrac{\mathrm{d}r}{\mathrm{d}s}\partial_r + \tfrac{\mathrm{d}\zeta}{\mathrm{d}s}\partial_\zeta,
\end{aligned} \tag{9}$$

where s is a parameter along the β-curves and $\zeta = v/u$. The equation of the β-curves on $\mathcal{F}$ is therefore

$$\frac{\mathrm{d}}{\mathrm{d}r}(\zeta^2) = f_2\zeta^4 + f_1\zeta^2 + f_0, \tag{10}$$

where

$$\begin{aligned}
f_0 &= \tfrac{1}{\mathcal{E}+\overline{\mathcal{E}}}\mathcal{E}_t + \left(\tfrac{1}{\mathcal{E}+\overline{\mathcal{E}}}\mathcal{E}_r - \tfrac{1}{2r}\right), \\
f_1 &= \tfrac{2r}{(\mathcal{E}+\overline{\mathcal{E}})^2}\left(\mathcal{E}_r\overline{\mathcal{E}}_t + \overline{\mathcal{E}}_r\mathcal{E}_t\right) - \tfrac{2}{\mathcal{E}+\overline{\mathcal{E}}}\overline{\mathcal{E}}_t, \\
f_2 &= \tfrac{1}{\mathcal{E}+\overline{\mathcal{E}}}\overline{\mathcal{E}}_t - \left(\tfrac{1}{\mathcal{E}+\overline{\mathcal{E}}}\overline{\mathcal{E}}_r - \tfrac{1}{2r}\right),
\end{aligned} \tag{11}$$

and where the constraint equations (3b) were used to obtain f_1 in terms of the initial data.

Equation (10) is a Riccati equation, with solutions of the form

$$\zeta^2 = -\frac{1}{f_2}\left(\frac{g_1' + wg_2'}{g_1 + wg_2}\right), \tag{12}$$

where w is a constant and g_1, g_2 are linearly independent solutions of

$$g'' - \left(\frac{f_2'}{f_2} + f_1\right)g' + f_0 f_2 g = 0, \tag{13}$$

the prime denoting differentiation with respect to r (see Hille (1969)). We can use w as a coordinate on $\mathcal{R}$.

This give us a picture of the β-curves on $\mathcal{F}$ as in Figure 2. We are interested in solving the field equations on one connected coordinate patch (in general not intersecting the $r = 0$ axis), corresponding to some interval V on the r-axis in $\mathcal{F}$. A value of w corresponds to one point of $\mathcal{R}$ if, for any $r \in V$, one can move continuously along the β-curve Q_w, staying within V, from one root of (12) to the other. For example, in Figure 2, w_1 corresponds to one point of $\mathcal{R}$, whereas w_2 corresponds to two, since Q_{w_2} has two disconnected leaves in V. The curve Q_∞ is degenerate and always corresponds to two points.

This gives a structure for $\mathcal{R}$ identical to that of the reduced twistor space in Woodhouse & Mason (1988).

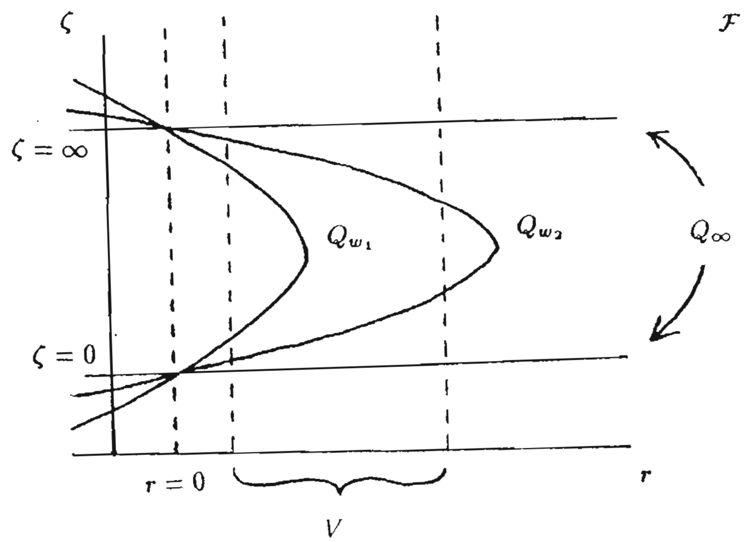

Figure 2

Now, the initial data correspond to a pair of curves on $\mathcal{R}$. Since, for a particular w, ζ has a double root when $w = -g_1'(r)/g_2'(r)$ ($\zeta = 0$) and when $w = -g_1(r)/g_2(r)$ ($\zeta = \infty$), we define the curves, parametrised by real values of r, on $\mathcal{R}$, to be

$$\begin{aligned} w &= -g_1'/g_2', \\ w &= -g_1/g_2. \end{aligned} \tag{14}$$

For each (real) value r_0 of r, the points of the curves on $\mathcal{R}$ are those corresponding to the two β-curves which *touch* the line $r = r_0$ in $\mathcal{F}$. There is one curve in each of the "glued down" caps in $\mathcal{R}$ (see Figure 1). In the axis-regular case, where the point $r = 0$ lies in V, the space $\mathcal{R}$ consists of two Riemann spheres glued down over one connected region. In this case, both curves lie in this region and intersect at $w = 0$, which corresponds to the line $r = 0$ in $\mathcal{F}$.

Indeed, any two functions which define such curves on $\mathcal{R}$ (in either case) contain the information of an initial data set for a cylindrically symmetric space-time M. For, given two arbitrary independent functions g_1, g_2 of r, we can obtain a second order differential equation

$$g'' + ag' + bg = 0 \tag{15}$$

by solving the simultaneous equations

$$\begin{aligned} g_1'' + ag_1' + bg_1 &= 0 \\ g_2'' + ag_2' + bg_2 &= 0 \end{aligned} \tag{16}$$

for a and b. Putting

$$-\left(\frac{f_2'}{f_2} + f_1\right) = a, \quad f_0 f_2 = b, \tag{17}$$

gives us two further simultaneous equations for $\mathcal{E}$ and $\mathcal{E}_t$.

These equations are, of course, rather hard to solve in practise!

It is hoped that the conformal scale of the space-time and the location of $\mathcal{H}$ within the space-time may be encoded as cohomology classes of twistor functions on $\mathcal{R}$, as these structures will generalise to a non-symmetric space-time. So far, the conformal scale has been encoded, but the "time function" has proved elusive.

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A twistor construction of irreducible torsion-free G-structures

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0. Introduction. One of the most useful characteristics of an affine connection on a manifold M is its (restricted) holonomy group which is defined, up to a conjugation, as a subgroup of $GL(T_tM)$ consisting of all automorphisms of the tangent space T_tM at a point $t \in M$ induced by parallel translations along the t-based contractable loops in M. Which groups can occur as holonomies of affine connections? By Hano and Ozeki [H-O], any closed subgroup of a general linear group can be realized as a holonomy of some affine connection (which in general has non-vanishing torsion tensor). The same question, when restricted to the class of *torsion-free* affine connections only, is not yet answered. According to Berger [B], the list of all possible irreducibly acting holonomies of torsion-free affine connections is very restricted. How much is known about this list? In his seminal paper [B], Berger found a list of groups which embraces all possible holonomies of torsion-free *metric* connections, though his approach provides no method to distinguish which entries can indeed be realized as holonomies and which are superfluous. Later much work has been done to refine this list and to prove existence of Riemannian metrics with special holonomies [Al, Br1, Br2, S]. In the same paper Berger presented also a list of all but a finite number of possible candidates to irreducible holonomies of "non-metric" torsion-free affine connections. How many holonomies are missing from this second list is not known, but, as was recently shown by Bryant [Br3], the set of missing, or *exotic*, holonomies is non-empty. As usual in the representation theory, in order to get a deeper understanding of all irreducible real holonomies one should first try to address a complex version of the problem. The main result announced in this paper asserts that any torsion-free holomorphic affine connection with irreducibly acting holonomy group can be generated by twistor methods.

1. Complex contact structures. Let Y be a complex $(2n+1)$-dimensional manifold. A complex contact structure on Y is a rank $2n$ holomorphic subbundle $D \subset TY$ of the holomorphic tangent bundle to Y such that the Frobenius form

$$\begin{aligned} \Phi : D \times D &\longrightarrow TY/D \\ (v,w) &\longrightarrow [v,w] \bmod D \end{aligned}$$

is non-degenerate. A complex n-dimensional submanifold X of the complex contact manifold Y is called a *Legendre submanifold* if $TX \subset D$. The normal bundle of a Legendre submanifold $X \hookrightarrow Y$ is isomorphic to J^1L_X [L2], where $L_X = L|_X$ and L is the contact line bundle on Y defined by the exact sequence

$$0 \longrightarrow D \longrightarrow TY \longrightarrow L \longrightarrow 0.$$

Given a Legendre subnamifold $X \hookrightarrow Y$, there is a naturally associated "flat" model, $X \hookrightarrow J^1 L_X$, consisting of the total space of the vector bundle $J^1 L_X$ together with its canonical contact structure and the Legendre submanifold X realized as a zero section of $J^1 L_X \to X$. The Legendre submanifold $X \hookrightarrow Y$ is called *k-flat* if the kth-order Legendre jet [L2] of X in Y is isomorphic to the kth-order Legendre jet of X in $J^1 L_X$. Every complex Legendre submanifold is 1-flat, while the obstruction to be 2-flat is a cohomology class in $H^1(X, L_X \otimes S^2(J^1 L_X)^*)$.

2. Irreducible G-structures. Let M be an m-dimensional complex manifold and $\mathcal{L}^* M$ the holomorphic coframe bundle $\pi : \mathcal{L}^* M \to M$ whose fibers $\mathcal{L}^*_t M = \pi^{-1}(t)$ consist of all $\mathbb{C}$-linear isomorphisms $e : \mathbb{C}^m \to \Omega^1_t M$. The space $\mathcal{L}^* M$ is a principle right $GL(m, \mathbb{C})$-bundle with the right action given by $R_g(e) = e \circ g$. If G is a closed subgroup of $GL(m, \mathbb{C})$, then a (holomorphic) G-structure on M is a principle subbundle $\mathcal{G}$ of $\mathcal{L}^* M$ with the group G. It is clear that there is a one-to-one correspondence between the set of G-structures on M and holomorphic sections σ of the quotient bundle $\tilde{\pi} : \mathcal{L}^* M/G \to M$ whose typical fibre is isomorphic to $GL(m, \mathbb{C})/G$. A G-structure on M is called *locally flat* if $\mathcal{L}^* M/G$ can be trivialized over a sufficiently small neighbourhood, U, of each point $t \in M$ in such a way that the associated section σ of $\mathcal{L}^* M/G$ is represented over U by a constant $GL(m, \mathbb{C})/G$-valued function. A G-structure is called *1-flat* if, for each $t \in M$, the first jet of the associated section σ of $\mathcal{L}^* M/G$ at t is isomorphic to the first jet of some locally flat section of $\mathcal{L}^* M/G$. It is easy to show that a G-structure admits a torsion-free affine connection if and only if it is 1-flat (cf. [Br2]). A G-structure on M is called *irreducible* if the action of G on $\mathbb{C}^m$ leaves no non zero invariant subspaces.

3. Main theorem. Recall that a generalized flag variety X is a compact simply connected homogeneous Kähler manifold [B-E]). Any such a manifold is of the form $X = H/P$, where H is a complex semisimple Lie group and $P \subset H$ a fixed parabolic subgroup.

Theorem 1 *Let X be a generalised flag varity embedded as a Legendre submanifold into a complex contact manifold Y with contact line bundle L such that $h^0(X, L_X) = m > 0$. Then*

(i) There exists a complete family $\{X_t \hookrightarrow Y \mid t \in M\}$ of compact complex Legendre submanifolds obtained by holomorphic deformations of X inside Y. Each submanifold X_t is isomorphic to X. The moduli space M, called a Legendre moduli space, is an m-dimensional complex manifold.

(ii) The Legendre submanifold $X \hookrightarrow Y$ is stable under holomorphic deformations of the contact structure on (the tubular neighbourhood of X in) Y.

(iii) For each $t \in M$, there is a canonical isomorphism $s : T_t M \to H^0(X_t, L_{X_t})$ representing a tangent vector at t as a global holomorphic section of the line bundle $L_{X_t} = L|_{X_t}$.

(iv) The Legendre moduli space M comes equipped with an induced irreducible G-structure, $\mathcal{G}_{ind} \to M$, with G isomorphic to the group of all global biholomorphisms $\phi : L_X \to L_X$ which commute with the projection $\pi : L_X \to X$.

(v) *The induced G-structure on M is 1-flat (i.e. torsion-free) if and only if the complete family $\{X_t \hookrightarrow Y \mid t \in M\}$ consists of 2-flat Legendre submanifolds. The obstruction for the induced torsion-free G-structure to be locally flat is given by a tensor field on M whose value at each point $t \in M$ is represented by a cohomology class ρ_t in $H^1(X_t, L_{X_t} \otimes S^3(J^1 L_{X_t})^*)$.*

(vi) *Let $G \subset GL(m, \mathbb{C})$ be one of the following groups: (a) $SO(2n+1, \mathbb{C})$ when $m = 2n+2 \geq 8$; (b) $Sp(2n+2, \mathbb{C})$ when $m = 2n+2 \geq 4$; (c) G_2 when $m = 7$. If $\mathcal{G}$ is any torsion-free $G \times \mathbb{C}^*$-structure on an m-dimensional manifold M, then there exists a complex contact manifold (Y, L) and a generalized flag variety X embedded into Y as a Legendre submanifold such that, at least locally, M is canonically isomorphic to the associated Legendre moduli space and $\mathcal{G} \subset \mathcal{G}_{ind}$. In the case (a) $X = SO(2n+2, \mathbb{C})/U(n+1)$ and $\mathcal{G}_{ind}$ is a $CO(2n+2, \mathbb{C})$-structure; in the case (b) $X = \mathbb{CP}^{2n+1}$ and $\mathcal{G}_{ind}$ is a $GL(2n+2, \mathbb{C})$-structure; and in the case (c) $X = Q_5$ and $\mathcal{G}_{ind}$ is a $CO(7, \mathbb{C})$-structure.*

(vii) *Let $G \subset GL(m, \mathbb{C})$ be an aribtrary semisimple Lie subgroup except the ones considered in (vi). If $\mathcal{G}$ is any torsion-free $G \times \mathbb{C}^*$-structure on an m-dimensional manifold M, then there exists a complex contact manifold (Y, L) and a Legendre submanifold $X \hookrightarrow Y$ with $X = G/P$ for some parabolic subgroup $P \subset G$ such that, at least locally, M is canonically isomorphic to the associated Legendre moduli space and $\mathcal{G} = \mathcal{G}_{ind}$.*

Remarks:

1. The Lie algebra of the group G of all global biholomorphisms $L_X \to L_X$ which commute with the projection $\pi : L_X \to X$ is exactly the vector space $H^0(X, L_X \otimes (J^1 L_X)^*)$ with its natural Lie algebra structure [Me1]. If $X = H/P$, then the induced G-structure on the associated Legendre moduli space is often isomorphic to $H \times \mathbb{C}^*$, but there are exceptions [A] which are considered in Theorem 1(vi). In these exceptional cases the originial G-structure may not be equal to the induced one, and one might try to identify some additional structures on the associated twistor spaces (Y, L) which ensure that $\mathcal{G}_{ind}$ admits a necessary reduction. However, in the context of problems discussed in the introduction there is no need in such a study, because these "exceptional" G-structures are fairly well understood by now [B, Br1, Br2, S]. If there is an exotic torsion-free G-structure other than Bryant's G_3 [Br3], it must be covered, up to a $\mathbb{C}^*$ action, by the "generic" clause (vii) in Theorem 1.

2. Two particular examples of this general construction have been considered earlier [L1, Br3]. The first example is a pair $X \hookrightarrow Y$ consisting of an n-quadric Q_n embedded into a $(2n+1)$-dimensional contact manifold (Y, L) with $L|_X \simeq i^* \mathcal{O}_{\mathbb{CP}^{n+1}}(1)$, $i : Q_n \hookrightarrow \mathbb{CP}^{n+1}$ being a standard projective realization of Q_n. It is easy to check that in this case $H^0(X, L_X \otimes (J^1 L_X)^*)$ is precisely the conformal algebra implying that the associated $(n+2)$-dimensional Legendre moduli space M comes equipped canonically with a conformal structure. This is in accord with LeBrun's paper [L1], where it has been shown how a conformal Weyl connection can be encoded into complex contact structure on the space of complex null geodesics. Since $H^1(X, L_X \otimes S^2(J^1 L_X)^*) = 0$, the induced conformal structure must be torsion-free in agreement with the classical result of differential geometry. Easy calculations show that the vector space

$H^1(X, L_X \otimes S^3(J^1 L_X)^*)$ is exactly the subspace of $TM \otimes \Omega^1 M \otimes \Omega^2 M$ consisting of tensors with Weyl curvature symmetries. Thus Theorem 1(v) implies the well-known Schouten conformal flatness criterion.

The second example, which also was among motivations behind the present work, is Bryant's [Br3] relative deformation problem $X \hookrightarrow Y$ with X being a rational Legendre curve $\mathbb{CP}^1$ in a complex contact 3-fold (Y, L) with $L_X = \mathcal{O}(3)$. Calculating $H^0(X, L_X \otimes (J^1 L_X)^*)$, one easily concludes that the induced G-structure, $\mathcal{G}_{ind}$, on the associated 4-dimensional Legendre moduli space is exactly an exotic G_3-structure which has been studied by Bryant in his search for irreducibly acting holonomy groups of torsion-free affine connections which are missing in the Berger list [B]. Since $H^1(X, L_X \otimes S^2(J^1 L_X)^*) = 0$, Theorem 1(v) says the induced G_3-structure is torsion-free in accordance with [Br3]. The cohomology class $\rho_t \in H^1(X, L_X \otimes S^3(J^1 L_X)^*)$ from Theorem 1(v) is exactly the curvature tensor of the unique torsion-free affine connection with G_3-holonomy.

3. Much of the above theorem remains true when the assumption that the Legendre submanifold X is a generalized flag variety is replaced by the assumption that X is a compact complex manifold such that $H^1(X, L_X) = 0$ [Me1].

4. Any reductive non-semisimple irreducibly acting holonomy group must be of the form $G \times \mathbb{C}^*$ (cf. Theorem 1(vi) and (vii)), where G is semisimple [B].

5. Usually in the twistor theory one works with Kodaira [K] moduli spaces [P], that is with complete families $\{X_t \hookrightarrow Y \mid t \in M\}$ of compact complex submanifolds of a complex manifold Y obtained by all holomorphic deformations of a fixed submanifold $X \hookrightarrow Y$ *inside* Y. Any such a family can be canonically interpreted as a complete family $\{\hat{X}_t \hookrightarrow \hat{Y} \mid t \in M\}$ of compact Legendre submanifolds — take $\hat{Y} = \mathbb{P}_Y(\Omega^1 Y)$ with its natural contact structure and $\hat{X}_t = \mathbb{P}_{X_t}(N_t^*)$, where N_t^* is the conormal bundle of X_t. The point is that the map

$$\{X_t \hookrightarrow Y \mid t \in M\} \longrightarrow \{\hat{X}_t \hookrightarrow \hat{Y} \mid t \in M\}$$

preserves *completeness* while changing its meaning. This construction together with Theorem 1 imply that Kodaira moduli spaces often come equipped with induced geometric structures. Other (and more fine) results in this direction are discussed in [Me2, Me3, M-P].

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Some remarks on the stabilizer of the space of Hill operators and C. Neumann system

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Abstract

In this article we will show some close connections between the stabilizers of the coadjoint action of $Diff(S^1)/S^1$ on its dual i.e. the space of Hill operators and the Neumann system. The main point of this article is to show some interesting features of celebrated paper of Knörrer and of Kirillov's work.

1 Introduction

The Neumann system deals with the motion of a particle on a sphere under the influence of a quadratic potential. This system is completely integrable and given solutions by hyperelliptic theta function. Moser [Mo] observed that the integrals of the Hamiltonian system describing the motion of Neumann system have a very close similarity with the integrals of the Hamiltonian system describing the geodesics on a quadric. Knörrer [Kn] showed in his paper that the Neumann problem can be recast through the Gauss map as the geodesic motion problem on a quadric.

On the other hand we know from the work of Segal [Se] and Kirillov [Ki] that the KdV equation is the Euler equation for a central extension of the group $Diff(S^1)/S^1$. The centrally extended $\widehat{Diff(S^1)}/(S^1) \oplus R$ is described by the Gelfand-Fuks cocycle [Ki]

$$(\xi_1 \frac{d}{dx}, \xi_2 \frac{d}{dx}) \longmapsto \frac{1}{2} \int_{S^1} \xi_1' \xi_2'' dx, \quad \xi_i \in Vect(S^1).$$

Let us recall that the dual space of $Diff(S^1)/S^1$ is the space of quadratic differentials $\Omega^{\otimes 2}$ and the dual of the $\widehat{Diff(S^1)}/S^1$ is the space of Hill operators $\{\lambda \frac{d^2}{dx^2} + q\}$.

¿From now we shall denote $\widehat{Diff(S^1)}/S^1$ by $\hat{\nu}$ and the space of Hill operators by $H(s)$.

Acknowledgement : I wish to thank Jack Evans, Pablo Gastesi and Jyoti Sengupta for their interest, support and discussions.

2 Coadjoint action and characterization

Let us consider the covariant transformation $L = \lambda \frac{d^2}{dx^2} + q(x)$ under a S^1 diffeomorphism.

$L \longrightarrow \tilde{L}$ induced by S^1 diffeomorphism.

$$x \longrightarrow s(x) = x + \epsilon f(x)$$

$$L = \lambda + q(x) \longmapsto \tilde{L} = s'(x)^{3/2} (\lambda (\frac{d}{ds(x)})^2 + q(s(x))) s'^{1/2}$$

$$= \lambda \frac{d^2}{dx^2} + \tilde{q}(x)$$

where

$$\tilde{q(x)} = s'(x)^2 q(s(x)) + \frac{1}{2} (\frac{s'''}{s'} - \frac{3}{2} (\frac{s''}{s'})^2)$$

When $s(x) = x + \epsilon f(x)$, this becomes

$$\tilde{q} = \xi q' + 2\xi' q + \frac{1}{2} \lambda \xi'''.$$

Let us confine our attention to a specific hyper-plane $\lambda = -1$ in the coadjoint orbit. The action in this hyperplane will be

$$\tilde{q} = \xi q' + 2\xi' q - \frac{1}{2} \xi'''.$$

Now we seek to characterize the pairs $(q(x)(dx)^2, -1)$. We proceed by looking at the stabilizer of the action of $\xi(x) \frac{d}{dx}$ on the dual $(q(x) dx^2, -1)$ i.e. $(\xi, a) \in Stab(q, -1)$ if and only if

$$\xi''' = 2q'\xi + 4q\xi'. \qquad (*)$$

Proposition 2.1 *If* $\xi = <\chi, A^{-1}\chi>$ *and satisfies*

$$\xi''' = 2\xi q' + 4\xi' q$$

then χ *satisfies*

$$\ddot{\chi} = -A\chi + q\chi$$

where $<\chi, \chi>= 1$ *and* $q = <\chi, A\chi> - <\dot{\chi}, \dot{\chi}>$ *which is the system of Neumann equations.*

proof :: Let $\xi = <\chi, A^{-1}\chi>$ then $\xi' = 2 <\dot{\chi}, A^{-1}\chi>$. Then after using the condition of the Neumann equation, we obtain

$$\xi'' = -2 + 2q\xi + 2 <\dot{\chi}, A^{-1}\dot{\xi}> .$$

Taking one more derivative we get

$$\xi''' = 4q\xi' + 2q'\xi.$$

□

Karen Uhlenbeck [Uh] found the algebraic integrals for the Neumann problem. For $p, q \in \mathbf{R}^n$ let $\Phi_\lambda(p, q) \in \mathbf{C}(\lambda)$ be the rational function

$$\Phi_\lambda(p, q) := \sum_{i=1}^{n} \frac{q_i^2}{\alpha_i - \lambda} - \frac{1}{2} \sum_{i,j=1} \frac{(p_i q_j - p_j q_i)^2}{(\alpha_i - \lambda)(\alpha_j - \lambda)}$$

Moser [Mo] gave a nice geometrical interpretation of the zeros of these rational function. In particular

$$\Phi_0(\dot{\chi}, \chi) = 0$$

and Knörrer showed also

$$2\Phi_0(\dot{\chi}, \chi) = \dot{\xi}/2 - (\ddot{\xi} - 2q\xi)\xi.$$

Knörrer showed when ξ satisfies $\xi''' = 2q'\xi + 4q\xi'$ and $\xi := <\chi(x), A^{-1}\chi(x)>$ then the $-\frac{1}{2}$ density ι satisfies an auxiliary equation, Schrödinger equation

$$(-\frac{d^2}{dx^2} + q)\iota = 0. \qquad (**)$$

Recall that if $\xi \in Vect(S^1)$, then $\iota \in \Omega^{-1/2}$ i.e. the space of scalar densities of weight $-1/2$.

Let us define

$$\mathcal{G} = \mathcal{G}_0 \oplus \mathcal{G}_1$$

, where we denote $\mathcal{G}_0 \equiv Vect(S^1)$ and $\mathcal{G}_1 \equiv \Omega^{-1/2}(S^1)$. $\mathcal{G}_1$ is the $\mathcal{G}_0$ module and it is compatible with the structure of $\mathcal{G}_0$ module and satisfies

$\mathcal{G}_1 \times \mathcal{G}_1 \longrightarrow \mathcal{G}_0$. This is quite natural if we identify $Vect(S^1)$ and $\Omega^{-1}(S^1)$. A typical element of $\mathcal{G}$ would be

$$u(x)\frac{d}{dx} + v(x)\frac{d}{dx}^{1/2}.$$

Proposition 2.2 (Kirillov [Ki]) *$\mathcal{G}$ has the structure of a super Lie algebra.*

In this realization $\xi(x)\frac{d}{dx} \oplus \iota(x)\frac{d}{dx}^{1/2}$ i.e. (ξ, ι) forms a super Lie algebra. $(*)$ and $(**)$ give the stabilizer of a point in the dual space to a super Lie algebra. (ξ, ι) satisfies

$$\xi(x + 2\pi) = \xi(x)$$

$$\iota(x + 2\pi) = \pm\iota(x)$$

When it is '+ ' it is called Ramond sector super Lie algebra and for '−' it is known as Neveu-Schwarz sector.

We wish to know more about ι. We shall use Knörrer's construction. He made use of the usual Gauss mapping of the quadric onto the unit sphere which takes a point on the quadric into the exterior unit normal.

Knörrer showed that Jacobi field along this geodesic motion satisfies mKdV equation where $\xi \in Vect(S^1)$ and $\iota \in \Omega^{-1/2}$.

In the next section we will give a geometrical meaning of $\iota(x)$. We will show it is the tau function of the Jacobi field equation, in this case mKdV equation.

3 Geometrical meaning of ι

As we mentioned earlier that Knörrer showed the geodesics on quadrics problem is intimately related to the C. Neumann problem.

Theorem 3.1 (Knörrer) *Let $Q \subset \mathbf{R}^n$ be a quadric $Q = \{t \in \mathbf{R}^n | \mho(t) = 0\}$ and $A := (\frac{\partial^2 \mho}{\partial t_i \partial t_j})$. The geodesic $x(t)$ on Q is parametrized by*

$$\ddot{t}(x) = \aleph t(x) + w\dot{t}(x)$$

where $\aleph$ is the gradient of the function $\mho(x)$ in x. Let $\xi(x)$ be the unit normal vector of Q in the point $t(x)$

$$\xi = \iota.\aleph(t) \text{ where } \iota^2 = \frac{1}{< \aleph(t), \aleph(t) >}.$$

Then $\xi(t)$ satisfies Neumann equation

$$\ddot{\xi} = A\xi + q\xi \text{ where } q := \frac{1}{4}w^2 - \frac{1}{2}\dot{w}$$

where

$$w = -2\frac{< \aleph, At' >}{< t', At' >}$$

So there exist a one to one correspondence between the solutions of Neumann equation and geodesic on the quadric.

Knörrer also showed that the Jacobi-field along the geodesic $t(x)$ satisfies mKdV equation

$$\frac{\partial w}{\partial s} = \frac{3}{4}w^2 w' - \frac{1}{2}w'''.$$

By a simple calculation one can show that

$$w = -2\frac{\partial}{\partial x} log\iota$$

The geometrical construction of solutions of the KdV hierarchy is based on an infinite dimensional grassmannian $Gr^{(2)}$ defined as follows. Let $L^2(S^1, \mathbf{C})$ be the Hilbert space H and multiplication by z is a unitary operator on the Hil bert space. Let H_+ be the Hilbert subspace of H consisting of boundary values of holomorphic function in the disc $|z| < 1$. Then Grassmannian is the closed subspace $W \subset H$, satisfies

$$(1)\ z^2 W \subset W$$

$$(2)\ Pr_+ : W \longrightarrow H_+$$

is a Fredholm operator

$$(3)\ Pr_- : W \longrightarrow H_-$$

is a Hilbert Schmidt operator. Last two conditions mean that W is comparable with H_+.

To interpret the mKdV equation we recall Wilson's [Wi] construction. Let $W \in Gr^n(2)$ be a point in the Grassmannian, satisfying $z^2 W \subset W$. Then W/z^2W has dimension n. Let $Fl^{(2)}$ be the periodic flag manifold consists of a pair (W_0, W_1) of closed subspaces $H = L^2(S^1, \mathbf{C})$ such that $W_0 \in Gr^{(2)}$ then

$$z^2 W_0 \subset z^1 W_1 \subset W_0$$

where $z^1 W^1$ has codimension 1 in W. W_i is a point of $Fl^{(2)}$ and τ_i is the τ - function of W_i. Corresponding to this the mKdV solution is given by

$$v_i = \frac{\partial}{\partial x} log(\tau_i/\tau_{i+1}) \text{ for } 0 \leq i \leq 1$$

and $\tau_2 \equiv \tau_0$.

So it follows from our discussion that

Proposition 3.2 *ι can be interpreted as a τ function of the mKdV equation.*

4 Summary

In this paper we have shown that if $\xi(x)\frac{d}{dx} \in Stab(q,-1)$ then it satisfies (*) and auxiliary equation satisfies (**) where $\xi = \iota(x)^2$. Then (ξ, ι) satisfies super Lie algebra. We have shown that for a particular choice of $\xi =< \chi, A^{-1}\chi >$ in (*), χ satisfies Neumann equation. One can connect the Neumann system to geodesics on the quadric through Gauss mapping. Its Jacobi flow satisfies the mKdV equation and we interpret ι as the τ function of the mKdV equation.

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Geometric Aspects of Quantum Mechanics

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Abstract. A general theory of non-linear quantum mechanics is considered, for which the state space is a complex manifold with a compatible Riemannian structure: states are points, observables are smooth functions, and the value of an observable at a point is the expectation of the observable in that state. Such a state manifold has a natural symplectic structure that leads to the definition of a Poisson bracket for pairs of functions; the commutator of two observables is $-i$ times the Poisson bracket of the corresponding functions. Associated with each observable is a canonical vector field, obtained by taking its symplectic gradient. The magnitude of such a vector field with respect to the Riemannian metric is proportional to the squared uncertainty of the associated observable, and the usual Heisenberg relations can be seen to hold. The Schrödinger evolution of a state is then described by the special canonical vector field for which the generating function is the expectation of the physical Hamiltonian. The general framework of non-linear quantum mechanics is equivalent to a classical dynamical system on the quantum mechanical state manifold, a result that is *a fortiori* also valid for ordinary linear quantum mechanics. The rate of evolution of a quantum mechanical system along a Schrödinger trajectory in the non-linear theory is twice the uncertainty in the Hamiltonian; this generalises a result in the linear theory due to Anandan & Aharonov (1990). The relation of the non-linear theory to the linear theory is analysed, and in the case for which the state manifold is complex projective space and the Riemannian structure is the unitary-symmetric Fubini-Study metric the theory can be shown to reduce to a non-linear theory investigated by Weinberg (1989) and others. Ordinary linear quantum mechanics entails a further specialisation, for which an analysis is presented by use of projective algebraic geometry. The linear observables of ordinary quantum mechanics are functions for which the associated canonical vector fields are Killing vectors. In the general non-linear theory the scalar curvature of the complex manifold has the status of a preferred, geometrically determined observable, and it is suggested that this observable should be linear the Hamiltonian, with a relation of the form $\langle H\rangle = \lambda + \mu R$, where λ and μ are constants.

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Quantum Measurement and Stochastic Differential Geometry

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Abstract. The state space of a quantum mechanical system can be represented by a complex projective space, the space of rays in the associated Hilbert space. When regarded as a real manifold the state space comes naturally equipped with a Riemannian metric (the Fubini-Study metric) and a compatible symplectic structure. The familiar operations of ordinary quantum mechanics can thus be systematically reinterpreted in the language of differential geometry. It is especially interesting to pose some of the problems of quantum measurement in this spirit, with a view towards scrutinising the probabilistic assumptions that are brought in at various stages in the analysis of quantum dynamics, particularly in connection with state vector reduction. One of the most promising modern approaches to understanding reduction, studied recently by a number of authors, involves the use of non-linear stochastic dynamics to modify the ordinary linear Schrödinger evolution. Here we use methods of stochastic differential geometry to give a systematic geometric formulation for such stochastic models of state vector collapse. In this picture the conventional Schrödinger evolution, which corresponds to a Killing trajectory of the Fubini-Study metric, is replaced by a more general stochastic flow on the state manifold. In the simplest example of such a flow, the volatility term in the stochastic differential equation for the state trajectory is proportional to the gradient of the expectation of the Hamiltonian. The conservation of energy is represented by the requirement that the actual process followed by the expectation of the Hamiltonian, as the state evolves, should be a martingale. This requirement implies the existence of a non-linear term in the drift vector of the state process, which is always oriented opposite the direction of increasing energy uncertainty. As a consequence the state vector necessarily collapses to an energy eigenstate, and an elegant martingale argument can be used to show that the probability of collapse to a given eigenstate, from any particular initial state, is in fact given by precisely the usual quantum mechanical probability.

Presented at the ESI conference on Spinors, Twistors, and Conformal Invariants, September 1994, Vienna.

The Geometry of Non-Intersecting Null Rays

The motivation of twistor theory is to replace spacetime points by light rays in Minkowski space as the fundamental physical objects, and to understand positive frequency as a holomorphic property of the spacetime fields. This requires one to complexify the light cone in Minkowski space. The space of complex null rays in **CM** is under the standard Klein correspondence K precisely projective *ambitwistor* space - this is the product space of **PT** with its dual; restricted to pairs of incident twistors, i.e.,

$$\mathbf{PA} := \{(Z^\alpha, W_\alpha) | Z^\alpha W_\alpha = 0\}.$$

Thus in **PT** a complex null ray is a point on a projective 2-plane, $\mathbf{CP}_2$. The point on the 2-plane determines a *plane pencil* which is the set of lines in the plane passing through the given point; the elements of this pencil in **PT** are themselves $\mathbf{CP}_1$'s and each corresponds under K to a single element of a $\mathbf{CP}_1$ in **CM**, which is the complex null ray.

The dimensionality of the space of complex null rays in **CM**, as a manifold, can be seen in two ways; in purely twistorial or purely spacetime terms. It is worth checking that these agree:

1) it is simply the (complex) dimension of **PA** - the 6 complex dimensional product space is subject to one complex equation and so **PA** has complex dimension 5.

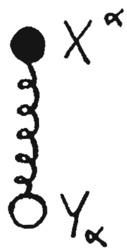
2) consider the space of all lines in a n-space - a generic element of this space can be given by a) the intersection point of the line with a fixed $(n-1)$-plane, and b) the direction of the line - this is simply given by a point on a $(n-1)$-sphere. This produces all lines apart from a set of measure zero - those lines parallel to or lying in the fixed plane. Thus the dimensionality of the space of lines in n-space is $2n-2$. When subject to the Lorentzian null constraint a) is unchanged but in b) the sphere loses one dimension to become a $(n-2)$-sphere. Thus the space of null lines in an n-space is $2n-3$ dimensional - these dimensions are complex if one begins with a n-complex dimensional space, and thus we have agreement with the twistor answer for our case $n = 4$.

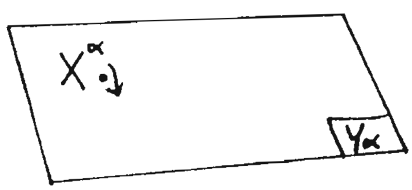
Consider now a pair of non-intersecting null rays. In **PT** there are as we shall see *two* natural conformally invariant classes of such pairs of rays. In **CM** there are again two natural classes of pairs: the set of all *parallel* pairs, and its complement. However, parallelism in **CM** is *not* conformally invariant.

In this article we give the interpretation in **CM** of the conformally invariant classes in **PT**, and also an interpretation in **PT** of the notion of parallelism in **CM**.

Conformally Invariant Classes in PT

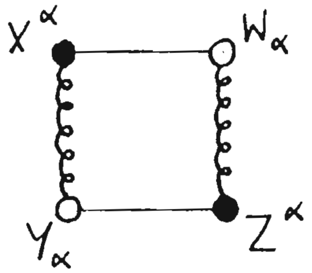
Let a connecting spring denote incident, and a connecting line denote non-incident twistors; let (un)shaded circles denote (dual) twistors. Then a complex null ray can be pictured in **PT** as,

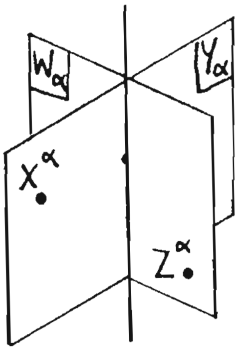




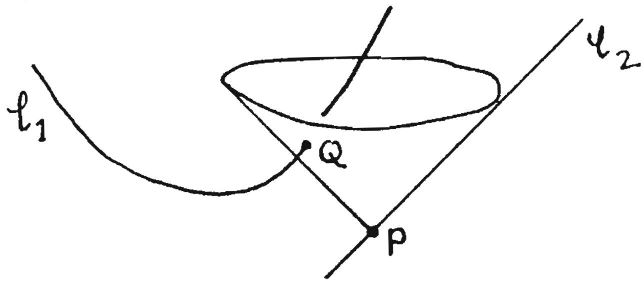
For a pair of null rays we have the following two conformally invariant classes.

Class 1 (measure 1)



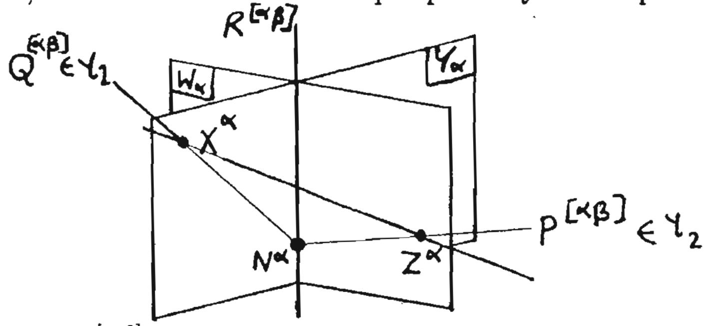


Firstly, note that there is no preferred point on either of the null rays. Also note that the rays are indistinguishable, from the picture in **PT** for class 1 shown. The interpretation in **CM** is,

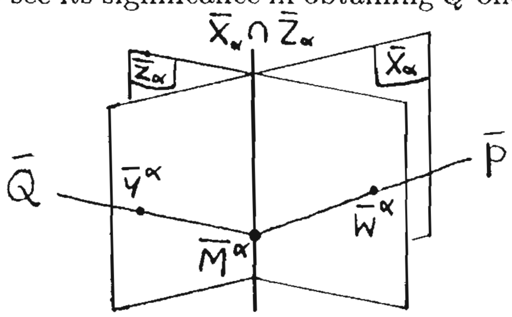


Choose any point P on $l_2 = (Z^\alpha, W_\alpha)$. Then there exists an unique point Q on l_1 which lies in the null cone of P, i.e. there exists an unique point Q null separated from P. In **PT** one can see this as,

$N^\alpha = P^{[\alpha\beta]} \cap R^{[\alpha\beta]}$

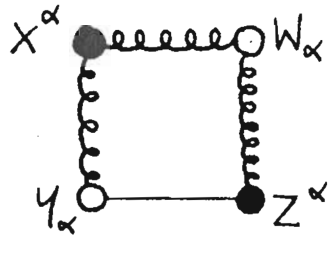


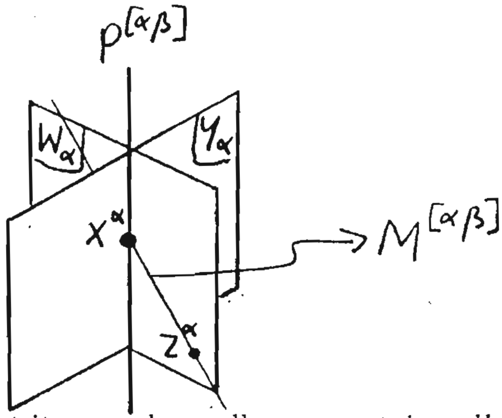
The significance of the line $R^{[\alpha\beta]}$ in obtaining the point Q is clear (note also that this line represents the unique point of intersection of the β-planes corresponding to W and Y). Similarly the line joining X and Z represents the unique point of intersection of the corresponding α-planes - to see its significance in obtaining Q one must complex conjugate the diagram above;



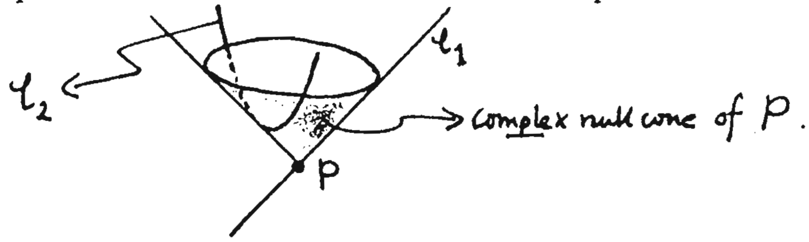
Note $\bar{M}^\alpha$ is the dual of the plane containing X^α, Z^α, and N^α.

Class 2 (measure 0)

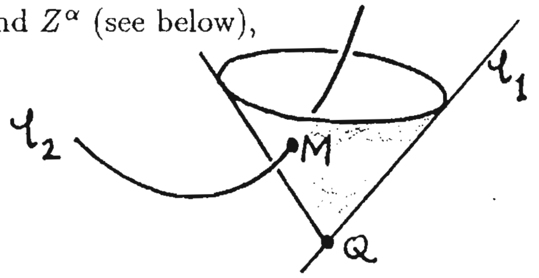




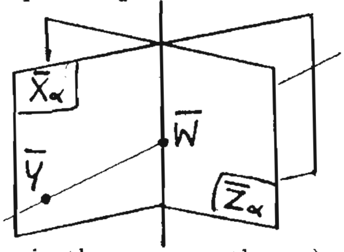
Property a): there exists a point P on l_1 such that its complex null cone contains *all* of l_2. This point is *unique* - under K it is the intersection of the planes W and Y (see below),



Property b): any other point Q on l_1, $Q \neq P$, has the property that its complex null cone contains only one point, M say of l_2, *independent* of Q, which under K is the line of intersection of X^α and Z^α (see below),



The situation is in fact symmetric with respect to interchange of l_1 and l_2. The most economical way to see this is to complex conjugate the diagram in **PT** above and to ask the same questions of the pair $\bar{l}_1$ and $\bar{l}_2$ and to use the result already obtained,



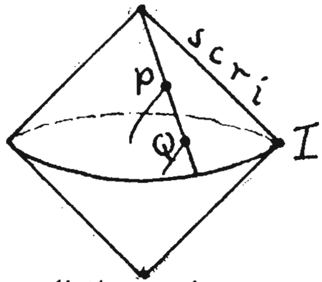
If we now complex conjugate again then we see that a) and b) hold with l_1, l_2 interchanged: for a) the point is represented by $X^{[\alpha}Z^{\beta]}$ and for b) the point of intersection corresponds under K to the intersection of the planes W_α, Y_β.

Thus a pair for this class 2 could be regarded as *ordered* (by saying for example that l_1 is the ray whose (upper indexed) twistor is connected by a spring to l_2), but this ordering is reversed under complex conjugation.

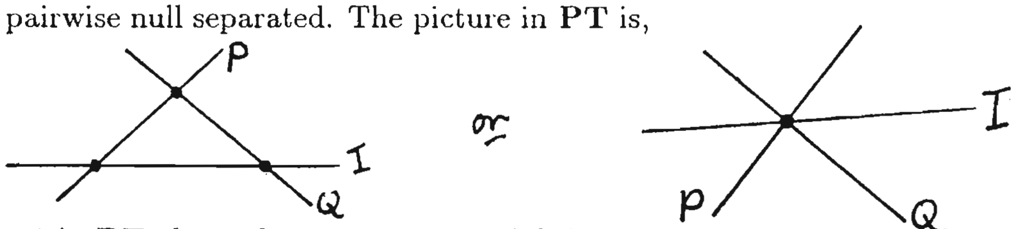
If we define the relation R by: $l_1 R l_2$ if and only if there exists a point on l_1 such that its null cone contains all of l_2; then this relation is both reflexive and symmetric, but fails to be transitive - the negation of R also fails transitivity.

Parallelism in CM

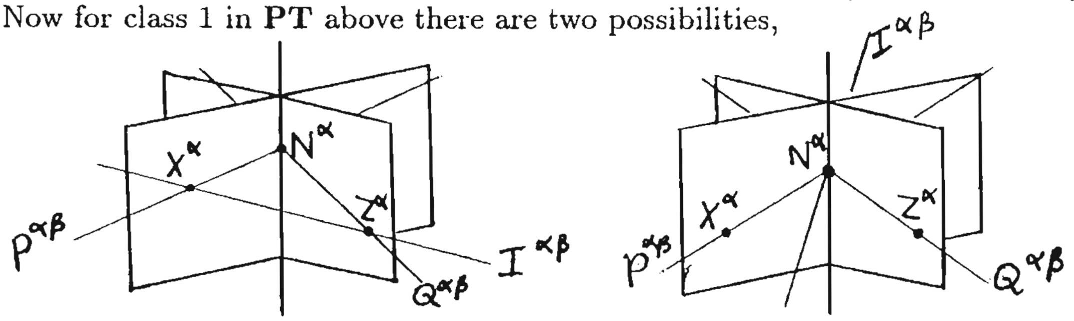
It is necessary here to consider the conformal compactification of **CM**, which we denote $\mathbf{CM}^\sharp$. Then two rays are parallel if and only if they meet *scri* (null infinity) in two points P, Q lying on a common generator - here we shall *exclude* the case that one of the rays itself be a generator of scri, i.e. we exclude null rays at infinity. In **PT** this is to say that the infinity twistor $I^{\alpha\beta}$ is *out* of incidence with all of X^α, Z^α, W_α, Y_α. In $\mathbf{CM}^\sharp$ the picture is,



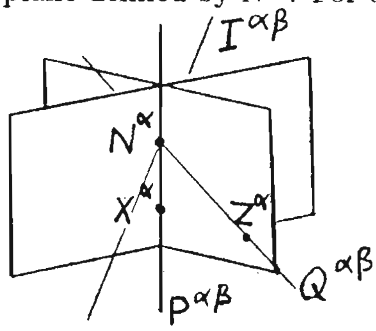
Suppose first that P, Q are *distinct* points on *scri*. In **PT** there are two cases. Let I denote the point at infinity in $\mathbf{CM}^\sharp$. Then the rays are parallel if and only if the points P, Q, I are pairwise null separated. The picture in **PT** is,



Now for class 1 in **PT** above there are two possibilities,

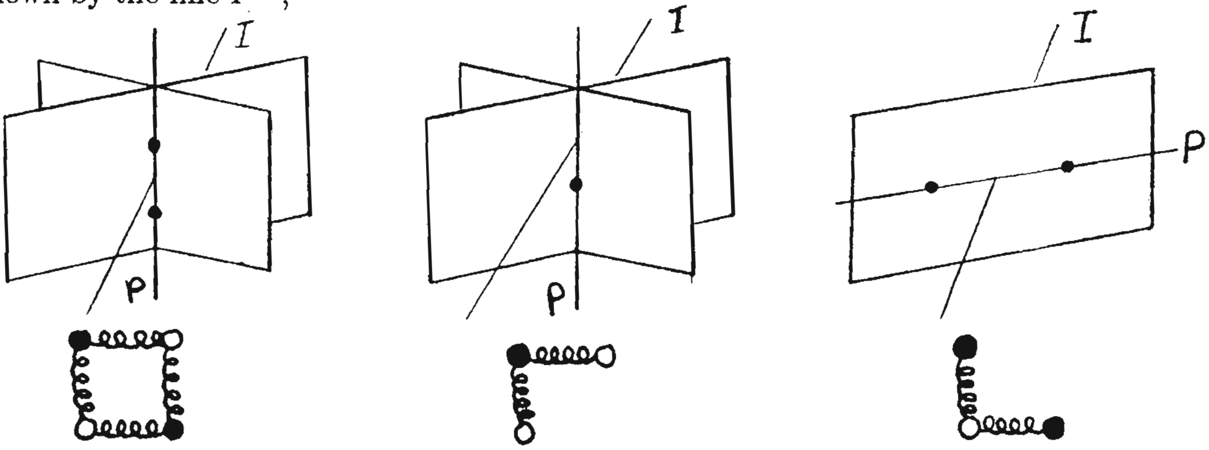


In the left hand picture, I^a lies in the β-plane defined by N^α, X^α, Z^α, and in the right hand picture, I^a lies in the α-plane defined by N^α. For class 2 above there is in fact only one possibility,



The line $I^{\alpha\beta}$ must contain the point N^α and not lie in either of the planes W_α, Y_α. Note in particular that N^α is distinct from X^α, since otherwise $I^{\alpha\beta}$ is forced to lie in the plane W_α or to contain X^α, which gives a null ray at infinity.

Suppose now that $P = Q$. Then the rays intersect at a common *point* of *scri*, and neither of classes 1 or 2 in **PT** described earlier apply - these two classes exclude all points of intersection in *compactified* **CM**. If two complex null rays intersect at a point P of $\mathbf{CM}^\sharp$ then this point is unique. In **PT** there are three cases, and the situation for parallel rays is shown by the line $I^{\alpha\beta}$,



Thanks especially to Franz Muller for hospitality at I.H.E.S., to S.A. Huggett for suggesting this for TN, and to R.P. for helpful comments.

Tim Field.

Conference Proceedings: Twistor Theory

edited by S.Huggett

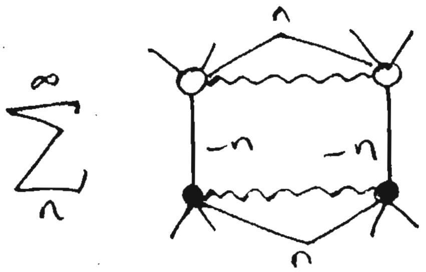
The proceedings of the 1993 Twistor Conference, edited by S.Huggett, are being published as Volume 169 of the Marcel Dekker, Inc. Series of Pure and Applied Mathematics books.

The contents are as follows:

(continued on page 42)

Progress with the massive propagator in twistor diagrams

In TN37 I outlined some new ideas on defining twistor diagrams for the massive Feynman propagator. The first key idea was that of replacing the original (1990) conjecture



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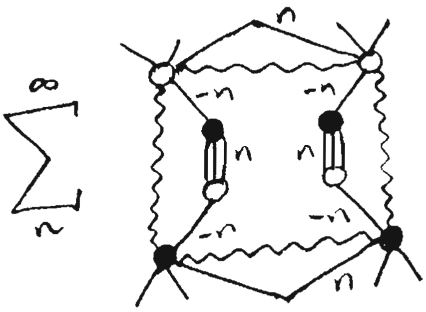


Some formulas were given which, it was stated, show that the new version must yield a solution of the inhomogeneous Klein-Gordon equation. But actually it is also necessary that a particular contour choice is made, namely a *double period* contour. At first sight one might think that introducing two boundaries, then taking two periods, will just get us back to where we began. But in fact we get back to something *almost* the same, but differing by lacking just the unwanted terms which arise (as shown by S.T.S.) in the originally conjectured integrals. But why should this be? A further observation helps give some intuitive feel, viz. that:

Roughly, this means that making the replacement is like performing two operations of $(-k\frac{\partial}{\partial k})^{-1}$, which will transform log(W.Z / k) into $^1/_6\log^3$(W.Z / k). On taking the double period, we regain log(W.Z / k), so that the 'wanted' term is left intact. However the 'unwanted' terms vanish under this sequence of operations. So the replacement acts almost like a projection operator for the wanted terms. One must say 'almost' because $^1/_2\log^2$ ((W.Z / k) is transformed not to itself but to $^1/_2\log^2$ ((W.Z / k) $-$ $^1/_3\,\pi^2$. We return to this point later.

This rough idea can be made exact. In doing so we are greatly helped by an observation about integrals where a boundary on W.Z = k goes with an integrand of form log(W.Z / k). Hitherto, our methods (as in Lewis O'Donald's work) have required the expansion of the integrand as a power series in (W.Z – k). In the present case these methods would lead to the summation of a formidable triple power series. However it turns out that it's quite unnecessary actually to do such a summation. Recall first that

has the property of vanishing when W.Z=k. Analogously, the spinor integral

$$\oint_{x.a=k,\ w.c=k} \frac{(w.c-k)^n (x.a-k)^n}{(w.x-k)^2}\, d^2w \wedge d^2x$$

vanishes when a.c=k; and it can be shown (by using power series) that this feature also extends to the more general

$$\oint_{\substack{x.a=k \\ w.c=k}} \frac{\left(\frac{x.a}{k}\right)^p \left(\frac{w.c}{k}\right)^q}{(w.x-k)^2}\, d^2w \wedge d^2x \qquad (*)$$

where p and q are any complex numbers. It follows also that

$$\oint_{\substack{x.a=k_1 \\ w.c=k_2}} \frac{\left[\log\left(\frac{x.a}{k_1}\right)\right]^m \left[\log\left(\frac{w.c}{k_2}\right)\right]^n}{m!\ (w.x-\mu)^2\ n!}\, d^2w \wedge d^2x$$

has the property of vanishing when μ a.c = k_1k_2, for each m, n.

Moreover, the value of this integral must be

$$\frac{\left[\log\left(\frac{\mu\, a.c}{k_1 k_2}\right)\right]^{m+n+1}}{(m+n+1)!}$$

(Proof: double induction on m and n, using repeated integration with respect to k_1, k_2, starting from the known result for m=n=0). Now write

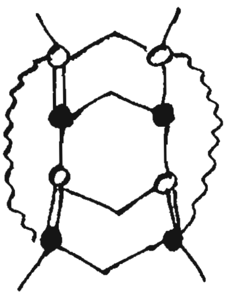
$$\left(\frac{x.a}{k}\right)^p = \sum_{m=0}^{\infty} \frac{p^m}{m!} \left[\log\left(\frac{x.a}{k}\right)\right]^m$$

and by summing the resulting double series we deduce

$$(*) = \frac{\left(\frac{a.c}{k}\right)^p - \left(\frac{a.c}{k}\right)^q}{p-q} = \frac{1}{2\pi i}\oint \frac{\left(\frac{a.c}{k}\right)^s ds}{(s-p)(s-q)}$$

By inserting a factor $(2i \sin\pi s)^2$ into the integrand we obtain the result of the double period contour. By straightfoward application and extension of this formula we can obtain all the results we need. See S.T.S's article in this TN for a statement of the conclusion, using the idea outlined above in a different way.

The same method ('integration with respect to parameters') can be used to simplify the calculations for the second of the key ideas introduced in TN37, that of replacing the (–n)-lines by 'ladders' with n rungs of the form



The overall picture is now that this line of development is firmly established and justified, thus taking us much closer to a scheme in which mass is generated by interaction with a Higgs field; i.e. the nth twistor diagram, with a ladder of n rungs, corresponds to n successive interaction with Higgs fields. However a new problem, at first overlooked, prevents us claiming that this is achieved. This is the $\frac{1}{3}\pi^2$ term that arises when the $\frac{1}{2}\log^2$ term is transformed to $\frac{1}{2}\log^2 - \frac{1}{3}\pi^2$. This is not fatal to the general programme, because we know that that the twistor diagrams so far arrived at are not actually quite of the form we want. They do not project out spin eigenstates, and they do not quite fit into the correct 'skeleton' pattern as required by my general dogma for the correspondence of twistor and Feynman diagrams. There is a reason for supposing that when the diagrams are yet again modified to meet these criteria, the $\frac{1}{3}\pi^2$ discrepancy will be eliminated.

Meanwhile there is no problem with getting the 'on-shell' (Hankel) functions from this general scheme, thus completing and superseding the formulation of my 1985 paper. In view of this I suggest that the two-twistor 'functions' of form

$$f(Z^\alpha)g(X^\alpha)(\overline{X}Z - \mu)^{-1}$$

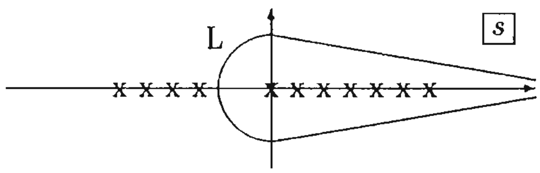
should now be considered as primitive elements of the theory, corresponding to the measurement of a single free massive field rather than as two free massless fields, and should have a new cohomological interpretation. *Andrew Hodges*

Some Calculational Results For The Massive Propagator In Twistor Diagrams

In [Hodges 91] a Barnes integral was derived for the massive scalar time-like propagator composed with massless *in* and *out* fields. This integral is

$$\mathcal{F}(m;(p-q)^2) = m^2 \int_L (\frac{1}{2})^{2s} \frac{\Gamma(1-s)\Gamma(-s)}{\sin \pi s} (m^2(p-q)^2)^{s-1} ds \qquad (1)$$

which we refer to as the Feynman function. The variable m is the mass of the scalar propagator, p and q are points in $\mathbb{CM}^{\#}$ defining the elementary states used for the *in* and *out* fields. The L path of integration is



The poles of the integrand are marked with Xs. The path of integration encloses the double pole at zero and the triple poles at all the positive integers (not the single poles at all the negative integers), so the Feynman function can be expressed as a sum of residues. We can write down a formula for the residue at the n^{th} pole

$$\frac{2(m^2)^n(y^2)^{n-1}}{\pi} \alpha(n) \{ \frac{1}{2} \log^2(\frac{m^2y^2}{4} e^{2\gamma}) + \beta(n) \log(\frac{m^2y^2}{4} e^{2\gamma}) + \frac{1}{2}[\beta^2(n) + \sigma(n)] + \frac{\pi^2}{3} \} \qquad (2)$$

where

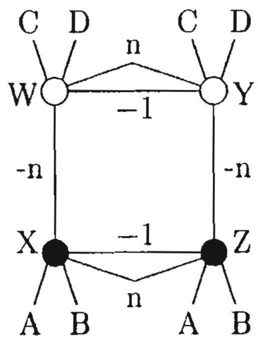
$$\alpha(n) := \frac{(-1)^{n+1}}{4^n n((n-1)!)^2}$$

$$\beta(n) := -\frac{1}{n} + 2\sum_{t=1}^{n-2} \frac{1}{t} + \sum_{r=1}^{n-1} \frac{n}{r(r-n)} - \sum_{r=1}^{n-2} \frac{1-n}{r(r+1-n)}$$

$$\sigma(n) := \frac{1}{n^2} + \frac{2}{(1-n)^2} + 2\sum_{r=1}^{n-2} \frac{1}{(r+1-n)^2}$$

and $y^2 := (p-q)^2$.

The central idea in [Hodges 91] was to regard this sum of residues as a power series in m^2 and that a twistor diagram should correspond to the coefficient of each power. The twistor diagram conjectured to correspond to the residue at n was

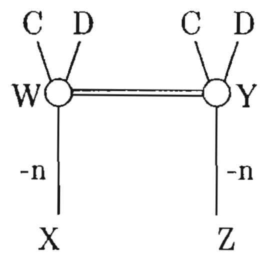


The bent lines labelled with n stand for terms like

$$\frac{1}{(W\,Y)^{n+1}}$$

The twistors A,B and C,D are related to the points p and q by the usual Klein correspondence. Different inhomogeneous parameters are used for the four boundary lines, going anticlockwise from the top: m_1, k_1, m_2, k_2. This enables us to apply differential operators to the lines individually.

Until recently the evaluation of this diagram would have been extremely laborious. However a calculational 'shortcut' has been noticed by Hodges. We can begin with a diagram evaluated in [Hodges 85]



and integrate twice with respect to m_1 to turn it into the top half of our required diagram. This technique is limited by the ambiguity in the constants of integration, but for diagrams of this type Hodges has identified a condition which specifies the constants uniquely [Hodges 94b]. The bottom half of the diagram is completed by usual methods and yields

$$\frac{\Omega^{n-1}}{n((n-1)!)^2}\{\frac{1}{2}\log^2(-\frac{m_1m_2\Omega}{k_1k_2}) + \mathcal{A}(n)\log(-\frac{m_1m_2\Omega}{k_1k_2}) - \frac{\pi^2}{6}$$

$$-\frac{1}{n}\mathcal{A}(n) + n((n-1)!)^2\mathcal{B}(n)\} + \mathcal{D}(n) \qquad (3)$$

where

$$\mathcal{A}(n) := -\frac{1}{n} - \frac{2\Gamma'(n)}{\Gamma(n)} - 2\gamma$$

$$\mathcal{B}(n) := \sum_{s=0}^{n-2}\frac{1}{\Gamma(s+1)^2\Gamma(n-s)^2}\{\frac{1}{(s+1)(s-n+1)}[\frac{1}{s-n+1}+\frac{1}{s+1}-f(n,s)]$$

$$-\frac{1}{n(s+1)}[-\frac{1}{s+1}+f(n,s)]\}$$

$$f(n,s) := -\frac{2\Gamma'(s+1)}{\Gamma(s+1)} - \frac{2\Gamma'(n-s)}{\Gamma(n-s)}$$

$$\mathcal{D}(n) := \sum_{s=0}^{n-2}\frac{\Omega^{n-1}}{\Gamma(s+1)^2\Gamma(n-s)^2}(\frac{\Omega m_1m_2}{k_1k_2})^{s-n+1}\frac{1}{(s+1)(s-n+1)^2}$$

and

$$\Omega := \begin{smallmatrix}\mathrm{A}\ \mathrm{B}\\ \vdash\!\dashv\\ \mathrm{C}\ \mathrm{D}\end{smallmatrix}\ (\underset{\sqcup}{\mathrm{A}\ \mathrm{B}}\ \overset{\sqcap}{\mathrm{C}\ \mathrm{D}})^{-1}$$

We make the identifications

$$m_1 = m_2 = \frac{m}{\sqrt{2}}$$

$$k_1 = k_2 = e^{-\gamma}$$

$$\Omega = -\frac{1}{2}y^2$$

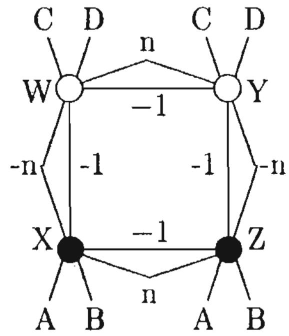
Furthermore, after some algebra we can show that

$$\mathcal{A}(n) = \beta(n)$$

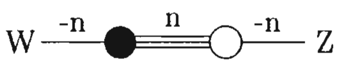
$$-\frac{1}{n}\mathcal{A}(n) + n((n-1)!)^2\mathcal{B}(n) = \frac{1}{2}[\beta^2(n) + \sigma(n)]$$

Thus the answer for the twistor diagram (3) agrees with the answer for the residue (2) apart from two terms. The co-efficients of π^2 are different and the answer for the diagram has the extra terms, $\mathcal{D}(n)$, which have become known as tail terms. These are the source of the extra term in the $n = 2$ diagram mentioned in [Hodges 94a].

The scheme for obtaining diagrams without the tail terms is outlined in [Hodges 94a]. We replace the twistor diagram above with



where the vertical bent lines are shorthand for



We calculate this diagram by integrating the previous diagram with respect to k_1 and k_2 and then taking the double period contour. The answer is

$$\frac{\Omega^{n-1}}{n((n-1)!)^2}\{\frac{1}{2}\log^2(-\frac{m_1 m_2 \Omega}{k_1 k_2}) + \mathcal{A}(n)\log(-\frac{m_1 m_2 \Omega}{k_1 k_2}) - \frac{\pi^2}{2}$$

$$-\frac{1}{n}\mathcal{A}(n) + n((n-1)!)^2\mathcal{B}(n)\} \qquad (4)$$

The tail terms have been successfully eliminated but the π^2 term has changed to $-\pi^2/2$. This happened as a result of taking the extra double period [Hodges 94b]. Thus the discrepancy between the answer for the diagram (4) and the residue (2) is $5\pi^2/6$.

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