Concerning Space-Time Points for Spin 3/2 Twistor Spaces

In $\mathbb{T}N$ 38 (pp. 1-9) (and see that article for earlier related references) I discussed various sheaves, exact sequences, etc. relevant to the spin 3/2 approach to the long-standing programme of defining the appropriate twistor space $\mathcal{T}$ for a general vacuum (Ricci-flat) space-time $\mathcal{M}$. Certain further ideas directed towards finding the twistors for $\mathcal{M}$ (i.e. points of $\mathcal{T}$) have arisen, but these remain inconclusive and will not be described here. Instead, I shall concern myself with a different question. Let us imagine that we have found the space $\mathcal{T}$. How would we go about re-constructing $\mathcal{M}$ from $\mathcal{T}$? In other words, what do we expect would be the definition of a space-time point (i.e. point of $\mathcal{M}$) in terms of $\mathcal{T}$?

Let us recall a way in which (complex) space-time points have been viewed in relation to the "googly" programme and non-linear graviton construction (R.P. in $\mathbb{T}N$ 8, pp. 32-34). We first consider the standard (Poincaré-invariant) exact sequence for flat twistor space $\mathbb{T}$:

$$0 \rightarrow \mathbb{S}^A \xrightarrow{i} \mathbb{T}^\alpha \xrightarrow{p} \mathbb{S}_{A'} \rightarrow 0$$

where the second map (i) takes ω^A to $(\omega^A, 0)$, and the third (p) takes $(\omega^A, \pi_{A'})$ to $\pi_{A'}$. A point q in $\mathbb{C}\mathbb{M}$ can be associated with either of the two maps $q, \hat{q}$ indicated by the dotted lines in

$$0 \rightarrow \mathbb{S}^A \underset{\hat{q}}{\overset{i}{\rightleftarrows}} \mathbb{T}^\alpha \underset{q}{\overset{p}{\rightleftarrows}} \mathbb{S}_{A'} \rightarrow 0$$

where $\hat{q}$ takes

$$(\omega^A, \pi_{A'}) \text{ to } \omega^A - iq^{AA'}\pi_{A'}$$

(q^a being the position vector of q), and the map q takes

$$\pi_{A'} \quad \text{to} \quad (i q^{AA'}\pi_{A'}, \pi_{A'}).$$

The composition $p \circ q$ is the identity on $\mathbb{S}_{A'}$, and $\hat{q} \circ i$ is the identity on $\mathbb{S}^A$.

In the non-linear graviton construction for an anti-self-dual complex vacuum $\mathcal{M}$, the twistor space $\mathcal{T}$ still has a canonical projection $\mathcal{T} \xrightarrow{p} \mathbb{S}_{A'}$, and space-time points (cross-sections of $\mathcal{T}$ in its capacity as a fibration over $\mathbb{S}_{A'}$) can be identified with (holomorphic) maps $q: \mathbb{S}_{A'} \rightarrow \mathcal{T}$ for which $p \circ q$ is the identity on $\mathbb{S}_{A'}$

$$\begin{tikzcd} \mathcal{T} \arrow[r, shift left, "p"] & \mathbb{S}_{A'} \arrow[l, shift left, dashed, "q"] \end{tikzcd} .$$

(More correctly, $\mathbb{S}_{A'} - \{0\}$ should replace $\mathbb{S}_{A'}$ in the preceding assertion.) In the proposals for a "googly graviton" construction, the points of a self-dual complex vacuum $\mathcal{M}$ would be intended to arise correspondingly from the twistor space $\mathcal{T}$ in accordance with

$$\begin{tikzcd} \mathbb{S}^{A} \arrow[r, shift left, "i"] & \mathcal{T} \arrow[l, shift left, dashed, "q"] \end{tikzcd} .$$

It is of interest to explore the way in which space-time points might come about in relation to a "spin 3/2 definition" of the twistor space $\mathcal{T}$ for a general complex vacuum $\mathcal{M}$. Let us recall the exact commutative diagram (S) appearing on p.7 of TN38, where I am also introducing additional maps $q, \hat{q}$ (to be explained shortly) in the columns (upwards)

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \{\mathring{\Omega}\} \arrow[r] \arrow[d, shift right, "i"'] & \{\mathring{\omega}\} \arrow[r] \arrow[d, shift right, "i"'] & \{\mathring{\rho}\} \arrow[r] \arrow[d, shift right, "i"'] & \{\mathring{\rho}\}/\{\mathring{\omega}\} \arrow[r] \arrow[d, shift right, "i"'] & \{\mathring{\alpha}\} \arrow[r] \arrow[d, shift right, "i"'] & 0 \\
0 \arrow[r] & \left\{ \begin{matrix} \Omega \\ \Pi \end{matrix} \right\} \arrow[r] \arrow[u, shift right, dashed, "\hat{q}"'] \arrow[d, shift right, "p"'] & \left\{ \begin{matrix} \omega \\ \pi \end{matrix} \right\} \arrow[r] \arrow[u, shift right, dashed, "\hat{q}"'] \arrow[d, shift right, "p"'] & \left\{ \begin{matrix} \rho \\ \sigma \end{matrix} \right\} \arrow[r] \arrow[u, shift right, dashed, "\hat{q}"'] \arrow[d, shift right, "p"'] & \left\{ \begin{matrix} \rho \\ \sigma \end{matrix} \right\} / \left\{ \begin{matrix} \omega \\ \pi \end{matrix} \right\} \arrow[r] \arrow[u, shift right, dashed, "\hat{q}"'] \arrow[d, shift right, "p"'] & \left\{ \begin{matrix} \alpha \\ \beta \end{matrix} \right\} \arrow[r] \arrow[u, shift right, dashed, "\hat{q}"'] \arrow[d, shift right, "p"'] & 0 \\
0 \arrow[r] & \{\Pi\} \arrow[r] \arrow[u, shift right, dashed, "q"'] \arrow[d] & \{\pi\} \arrow[r] \arrow[u, shift right, dashed, "q"'] \arrow[d] & \{\sigma\} \arrow[r] \arrow[u, shift right, dashed, "q"'] \arrow[d] & \{\sigma\}/\{\pi\} \arrow[r] \arrow[u, shift right, dashed, "q"'] \arrow[d] & \{\beta\} \arrow[r] \arrow[u, shift right, dashed, "q"'] \arrow[d] & 0 \\
 & 0 & 0 & 0 & 0 & 0 &
\end{tikzcd}$$

In accordance with the notation of that article (TN 38, pp. 1-9), I have now used $(\Omega^A, \Pi_{A'})$, in place of $(\omega^A, \pi_{A'})$ as above, to denote <u>constant</u> twistors, therefore satisfying

$$\nabla_{AA'}\Omega_B = -i\varepsilon_{AB}\Pi_{A'}, \quad \nabla_{AA'}\Pi_{B'} = 0,$$

with, also,

$$\nabla_{AA'}\mathring{\Omega}_B = 0;$$

and the more general $(\omega^A, \pi_{A'})$ and $\mathring{\omega}^A$ now merely satisfy

$$\nabla_{A'}^{B}\omega_B = 2i\pi_{A'}, \quad \nabla_B^{A'}\pi_{A'} = 0$$

with

$$\nabla_{A'}^{B}\mathring{\omega}_B = 0.$$

The helicity 3/2 (Dirac-type) potential $\sigma_{A'B'C}$ (symmetric in $A'B'$), and the second potential $\rho_{A'BC}$ (symmetric in BC) satisfy

$$\nabla_{B'}^{B}\rho_{A'BC} = 2i\sigma_{A'B'C}, \quad \nabla_B^{B'}\sigma_{A'B'C} = 0$$

with

$$\nabla_{B'}^{B}\mathring{\rho}_{A'BC} = 0$$

(symmetric in BC) the gauge freedoms being

$$\rho_{A'BC} \mapsto \rho_{A'BC} - i\varepsilon_{BC}\pi_{A'} + \nabla_{CA'}\omega_B, \quad \sigma_{A'B'C} \mapsto \sigma_{A'B'C} + \nabla_{CB'}\pi_{A'}, \quad \mathring{\rho}_{A'BC} \mapsto \mathring{\rho}_{A'B'C} + \nabla_{CA'}\mathring{\omega}^B$$

The Frauendiener–Sparling-type quantities $\rho_{A'BC}$ and $\sigma_{A'B'C}$ (still symmetric in BC and $A'B'$, respectively) generalize these equations to

$$\nabla_{(B'}^{B}\rho_{A')BC} = 2i\sigma_{A'B'C}, \quad \nabla_{(B}^{B'}\sigma_{C)A'B'} = 0, \quad \nabla_{(B'}^{B}\mathring{\rho}_{A')BC} = 0$$

(with the same gauge freedom as before) and the Frauendiener-type quantities $\alpha_A, \beta_{A'}, \mathring{\alpha}_A$ arise as

$$\alpha_C = \tfrac{1}{2}\nabla^{BB'}\rho_{B'BC}, \quad \beta_{A'} = \nabla^{BB'}\sigma_{A'B'B}, \quad \mathring{\alpha}_A = \tfrac{1}{2}\nabla^{BB'}\mathring{\rho}_{B'BC}$$

and satisfy

$$\nabla_{A'}^{A}\alpha_A = 2i\beta_{A'}, \quad \nabla_A^{A'}\beta_{A'} = 0, \quad \nabla_{A'}^{A}\mathring{\alpha}_A = 0$$

All these relations hold consistently in Ricci-flat $\mathcal{M}$ — except for those involving Ω^A, $\Pi_{A'}$, and $\mathring{\Omega}^A$, which require $\mathcal{M}$ to be flat.

How are we to define the maps q, $\hat{q}$ in each of the various columns? In the case of

$$0 \to \{\mathring{\Omega}^A\} \underset{\hat{q}}{\overset{i}{\rightleftarrows}} \left\{\begin{matrix}\Omega^A\\ \Pi_{A'}\end{matrix}\right\} \underset{q}{\overset{p}{\rightleftarrows}} \{\Pi_{A'}\} \to 0,$$

these maps are just as defined earlier (for the flat case $\mathbb{T}^\alpha$), namely

$$\Pi_{A'} \overset{q}{\longmapsto} (iq^{AA'}\Pi_{A'}, \Pi_{A'}) \quad\text{and}\quad (\Omega^A, \Pi_{A'}) \overset{\hat{q}}{\longmapsto} \Omega^A - iq^{AA'}\Pi_{A'}.$$

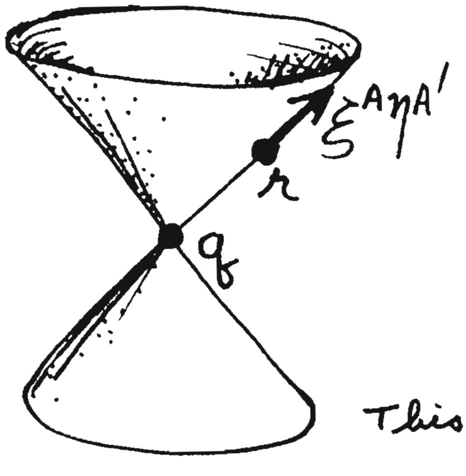
We need to generalize this to the remaining columns, and also in a way that makes sense in (Ricci-flat) $\mathcal{M}$. What this amounts to, in the case of the map q, is finding a way of <u>fixing</u> the "second potentials" ω_A, $\rho_{A'BC}$, $\mathring{\rho}_{A'BC}$ (mod ω_A), and α_A, in terms of the "first potentials" and a given point $q \in \mathcal{M}$. This is achieved by requiring that the appropriate <u>null-data</u> for these second-potential quantities are <u>zero</u> on the light cone of q. In the case of the map $\hat{q}$, we need a way of obtaining, as fixed by the point $q \in \mathcal{M}$, a definite "free-field" quantity $\mathring{\omega}_A$, $\mathring{\rho}_{A'BC}$, $\mathring{\rho}_{A'BC}$ (mod $\mathring{\omega}_A$), and $\mathring{\alpha}_A$, given the respective "sourced" quantity ω_A, $\rho_{A'BC}$, $\rho_{A'BC}$ (mod ω_A), and α_A, with its **respective** "source" $\pi_{A'}$, $\sigma_{A'B'C}$, $\sigma_{A'BC}$ (mod $\pi_{A'}$), and $\beta_{A'}$. This is achieved, in each case, by requiring that the null-data of the required "free fields" be equal to those of its corresponding "sourced fields" on the light cone of q.

It may be recalled (cf. Penrose & Rindler, Vol. 1, §§5.11, 5.12; R.P. (1980) Gen. Rel. Grav. 12, 225–64) that the null-datum at a point r on the data light cone, for a field $\theta_{A\ldots DP'\ldots S'}$ $(=\theta_{(A\ldots D)(P'\ldots S')})$, is the quantity

$$\theta = \xi^A \ldots \xi^D \eta^{P'} \ldots \eta^{S'} \theta_{A\ldots DP'\ldots S'}$$

where $\xi^A \eta^{A'}$ is a (complex) tangent vector at r to the light cone. In order that the null-data for fields $\theta_{\ldots}, \ldots, \chi_{\ldots}$ determine, freely, the fields themselves, we require that these fields constitute an exact set (i.e. that their totally symmetrized n^{th} derivatives $(n=0,1,2,\ldots)$ be independent and sufficient to determine all the unsymmetrized m^{th} derivatives $(m=0,1,2,\ldots)$. In the present context, we find that the appropriate null-data are, respectively

$$\xi^A \omega_A\,,\quad \begin{pmatrix} \eta^{A'}\xi^B\xi^C \rho_{A'BC} \\ \xi^A\xi^B\xi^C \nabla^{A'}_{A} \rho_{A'BC} \end{pmatrix},\quad \begin{pmatrix} \eta^{A'}\xi^B\xi^C \mathring{\rho}_{A'BC} \\ \xi^A\xi^B\xi^C \nabla^{A'}_{A} \mathring{\rho}_{A'BC} \\ \xi^C \nabla^{BA'} \mathring{\rho}_{A'BC} \end{pmatrix},\quad \xi^A \alpha_A$$



except that $\eta^{A'}\xi^B\xi^C \mathring{\rho}_{A'BC}$ is mere "gauge", and so does not contribute to $\{\mathring{\rho}\}/\{\omega\}$

This enables all the q and $\hat{q}$ maps to be defined.

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~ Roger Penrose