Concerning Space-Time Points for Spin 3/2 Twistor Spaces

In TN 38 (pp. 1-9) (and see that article for earlier related references) I discussed various sheaves, exact sequences, etc. relevant to the spin 3/2 approach to the long-standing programme of defining the appropriate twistor space $\mathcal{T}$ for a general vacuum (Ricci-flat) space-time $M$. Certain further ideas directed towards finding the twistorors for $M$ (i.e. points of $\mathcal{T}$) have arisen, but these remain inconclusive and will not be described here. Instead, I shall concern myself with a different question. Let us imagine that we have found the space $\mathcal{T}$. How would we go about re-constructing $M$ from $\mathcal{T}$?

In other words, what do we expect would be the definition of a space-time point (i.e. point of $M$) in terms of $\mathcal{T}$?

Let us recall a way in which (complex) space-time points have been viewed in relation to the "googly" programme and non-linear graviton construction (R.P. in TN 8, pp. 32-34). We first consider the standard (Poincaré-invariant) exact sequence for flat twistor space $\mathcal{T}$:

$$0 \rightarrow \mathcal{S}^A \rightarrow \mathcal{T}^A \rightarrow \mathcal{S}_A' \rightarrow 0$$

where the second map (i) takes $w^A$ to $(w^A, 0)$, and the third (p) takes $(w^A, \Pi_A')$ to $\Pi_A'$. A point $q$ in $CM$ can be associated with either of the two maps $q$, $\hat{q}$ indicated by the dotted lines in

$$0 \rightarrow \mathcal{S}^A \overset{\hat{q}}{\leftarrow} \mathcal{T}^A \overset{p}{\leftarrow} \mathcal{S}_A' \rightarrow 0$$

where $\hat{q}$ takes $(w^A, \Pi_A')$ to $w^A - i q^{AA'} \Pi_A'$.
(q being the position vector of q), and the map q takes \( T_A \) to \( i q A' \). The composition \( pq \) is the identity on \( S_A \), and \( qo i \) is the identity on \( S_A' \).

In the non-linear graviton construction for an anti-self-dual complex vacuum \( M \), the twistor space \( J \) still has a canonical projection \( J \to S_A' \), and space-time points (cross-sections of \( J \) in its capacity as a fibration over \( S_A' \)) can be identified with (holomorphic) maps \( q : S_A' \to J \) for which \( pq \) is the identity on \( S_A' \).

\[ J \to S_A' \to \]

(More correctly, \( S_A' \to \{0\} \) should replace \( S_A \) in the preceding assertion.) In the proposals for a "googly graviton" construction, the points of a self-dual complex vacuum \( M \) would be intended to arise correspondingly from the twistor space \( J \) in accordance with

\[ S_A \to J. \]

It is of interest to explore the way in which space-time points might come about in relation to a "spin3/2 definition" of the twistor space \( J \) for a general complex vacuum \( M \). Let us recall the exact commutative diagram (5) appearing on p. 7 of TN38, where I am also introducing additional maps \( q, \hat{q} \) (to be explained shortly) in the columns (upwards)
In accordance with the notation of that article (TN 38, pp. 1-9), I have now used \((\Sigma^A, \Pi^B)\) in place of \((w^A, \pi^A)\) as above, to denote constant twistors, therefore satisfying
\[
\nabla_{AA'} \Sigma_{B'} = -i \varepsilon_{AB} \Pi_{A'} \ , \ \nabla_{AA'} \Pi_{B'} = 0 ,
\]
with, also,
\[
\nabla_{AA'} \Sigma_B = 0 ;
\]
and the more general \((w^A, \pi^A)\) and \(\Sigma^A\) now merely satisfy
\[
\nabla^B \omega_{B'} = 2i \Pi_{A'} \ , \ \nabla_{B} \Pi_{A'} = 0
\]
with
\[
\nabla^B \omega_{B'} = 0 .
\]

The helicity \(\frac{3}{2}\) (Dirac-type) potential \(\Omega_{A'B'C}\) (symmetric in \(A'B'\)), and the second potential \(\Sigma_{A'B'C}\) (symmetric in \(BC\)) satisfy
\[
\nabla_{B'} \Sigma_{A'B'C} = 2i \Omega_{A'B'C} \ , \ \nabla_{B} \Omega_{A'B'C} = 0
\]
with
\[
\nabla_{B'} \Sigma_{A'B'C} = 0
\]
(symmetric in \(BC\)) the gauge freedoms being
\[
P_{A'B'C} \rightarrow P_{A'B'C} - i \varepsilon_{BC} \Pi_{A'} + \Pi_{CA} \omega_{B} , \ \nabla_{A'B'C} \rightarrow \nabla_{A'B'C} + \nabla_{CB} \Pi_{A'} , \ \Pi_{A'B'C} \rightarrow \Pi_{A'B'C} + \Pi_{CA} \omega_{B} .
\]

The FRAUENHIELEN - SPÄHRING-type quantities \(P_{A'B'C}\) and \(\Omega_{A'B'C}\) (still symmetric in \(BC\) and \(A'B'\), respectively) generalize these equations to
\[
\nabla_{(B'} \Sigma_{A'B'C)} = 2i \Omega_{A'B'C} \ , \ \nabla_{(B} \Omega_{A'B'C)} = 0 , \ \nabla_{(B} \Pi_{A'B'C)} = 0
\]
(with the same gauge freedom as before) and the FRAUENHIELEN-type quantities \(\alpha_{A}, \beta_{A}, \Omega_{A}\) arise as
\[
\alpha_{C} = \frac{1}{2} \nabla^{BB'} \Omega_{B'B'C} , \ \beta_{A} = \nabla^{BB'} \Omega_{A'B'B} , \ \Omega_{A} = \frac{1}{2} \nabla^{BB'} \Omega_{B'B'C}
\]
and satisfy
\[
\nabla_{A} \alpha_{A} = 2i \beta_{A'} , \ \nabla_{A} \beta_{A'} = 0 , \ \nabla_{A} \Omega_{A} = 0
\]
All these relations hold consistently in Ricci-flat $M$, except for those involving $\omega^A$, $\Pi_A$, and $\bar{\omega}^A$, which require $M$ to be flat.

How are we to define the maps $\tilde{q}$, $\tilde{q}$ in each of the various columns? In the case of

$$0 \rightarrow \{\bar{\omega}^A\} \xrightarrow{\tilde{q}} \{\Pi_A\} \xrightarrow{\tilde{q}} \{\Pi_A\} \rightarrow 0,$$

these maps are just as defined earlier (for the flat case $\Pi^\alpha$), namely

$$\Pi_A \xrightarrow{\tilde{q}} (i\bar{q} A^A, \Pi_A, \Pi_A'),$$

and $$(\omega^A, \Pi_A) \xrightarrow{\tilde{q}} \omega^A - iq A^A \Pi_A'.$$

We need to generalize this to the remaining columns, and also in a way that makes sense in (Ricci-flat) $M$. What this amounts to, in the case of the map $\tilde{q}$, is finding a way of fixing the "second potential" $\omega_A$, $\omega_{A'B'C'}$ (mod $\omega_A$), and $\omega_A$, in terms of the "first potentials" and a given point $q \in M$.

This is achieved by requiring that the appropriate null-data for these second-potential quantities are zero on the light cone of $q$. In the case of the map $\tilde{q}$, we need a way of obtaining, as fixed by the point $q \in M$, a definite "free-field" quantity $\omega_A$, $\omega_{A'B'C'}$ (mod $\omega_A$), and $\omega_A$, given the respective "sourced" quantity $\omega_A$, $\omega_{A'B'C'}$ (mod $\omega_A$), and $\omega_A$, with its respective "source" $\Pi_A$, $\omega_{A'B'C'}$ (mod $\Pi_A$), and $\omega_A$. This is achieved, in each case, by requiring that the null-data of the required "free-field" be equal to those of its corresponding "sourced field" on the light cone of $q$. 
It may be recalled (cf. Penrose & Rindler, Vol. 1, §§5.11, 5.12; RP (1980) Gen. Rel. Grav. 12, 225-64) that the null-datum at a point \( r \) on the data light cone, for a field \( \Theta_{A...DP...'S'} = \Theta_{(A...)(P...'S')} \), is the quantity

\[
\Theta = \Sigma^A \Sigma^D \eta^P...' \eta^S' \Theta_{A...DP...'S'}
\]

where \( \Sigma^A \eta^A' \) is a (complex) tangent vector at \( r \) to the light cone. In order that the null-data for fields \( \Theta, ..., X, ... \) determine, freely, the fields themselves, we require that these fields constitute an exact set (i.e., that their totally symmetrized \( m \)-th derivatives \( (m = 0, 1, 2, ...) \) be independent and sufficient to determine all the unsymmetrized \( m \)-th derivatives \( (m = 0, 1, 2, ...) \). In the present context, we find that the appropriate null-data are, respectively,

\[
\Sigma^A \omega_A, \quad \left( \eta^A \Sigma^B \Sigma^C \rho_A^{B'C'} \right), \quad \left( \eta^A \Sigma^B \Sigma^C \nabla_A (\rho_A^{B'C'}) \right), \quad \left( \eta^A \Sigma^B \Sigma^C \nabla_A (\nabla_A (\rho_A^{B'C'})) \right), \quad \Sigma^A \rho_A
\]

except that \( \eta^A \Sigma^B \Sigma^C \rho_A^{B'C'} \) is mere "gauge," and so does not contribute to \( \Sigma^0 \Sigma^3 / \Sigma^0 \Sigma^3 \).

This enables all the \( \eta \) and \( \eta' \) maps to be defined.

Thanks to Robin Graham for a valuable discussion in which he brought up the issue addressed by this article. — Roger Penrose