Spin 3/2 zero rest-mass fields in the Schwarzschild space-time

Jean-Philippe Nicolas
Département de Mathématiques
Université Bordeaux 1
351 cours de la Libération
33405 TALENCE Cedex
FRANCE
email: nicolas@math.u-bordeaux.fr

1 Introduction

Helicity 3/2 field equations seem to play a crucial role in general relativity as well as in twistor theory. Indeed, if we consider the Dirac equation for the first potential of a helicity 3/2 massless field,

$$\nabla^{AA'}\sigma_{A'B'}^{C} = 0 \ , \quad \sigma_{A'B'}^{C} = \sigma_{(A'B')}^{C}, \tag{1}$$

the vanishing of the Ricci curvature can be taken as a consistency condition for such an equation in curved space-time (see for example [1]). Such a close connection with Einstein's vacuum equations is quite remarquable. Moreover, twistors in flat space-time are interpreted as charges for spin 3/2 massless fields (see [3]) and it is therefore hoped that these fields in vacuum space-times might be used to define twistors.

The approach adopted here is purely analytical. The idea is to set up a technical basis that will (maybe) lead to a better understanding of the analytic or geometric obstacles to the definition of a "II-charge" in Ricci-flat space-times. We chose to study the case of the Schwarzschild metric

$$g_{\mu\nu}dx^{\mu}dx^{\nu} = Fdt^2 - F^{-1}dr^2 - r^2d\omega^2 \tag{2}$$

where $F = 1 - 1/r$, the Schwarzschild radius being here equal to 1.

Notations: Let (M, g) be a Riemannian manifold, $\mathcal{C}_0^\infty(M)$ denotes the set of $\mathcal{C}^\infty$functions with compact support in M, $H^k(M, g)$, $k \in I\!N$ is the Sobolev space, completion of $\mathcal{C}_0^\infty(M)$ for the norm

$$\|f\|^2_{H^k(M)} = \sum_{j=0}^{k} \int_M \left\langle \nabla^j f, \nabla^j f \right\rangle d\mu,$$

where ∇^j, $d\mu$ and $<,>$ are respectively the covariant derivatives, the measure of volume and the hermitian product associated with the metric g. We write $L^2(M, g) = H^0(M, g)$.

The 2-dimensional euclidian sphere S^2_ω is endowed with its usual metric

$$d\omega^2 = d\theta^2 + sin^2\theta d\varphi^2 \ , \ 0 \leq \theta \leq \pi \ , \ 0 \leq \varphi < 2\pi.$$

2 The global Cauchy problem

In this space-time, we consider the null tetrad $l^a, m^a, \overline{m}^a, n^a$, where

$$l^a \nabla_a = \frac{1}{\sqrt{2}} \left(F^{-1/2} \frac{\partial}{\partial t} + F^{1/2} \frac{\partial}{\partial r} \right), \tag{3}$$

$$n^a \nabla_a = \frac{1}{\sqrt{2}} \left(F^{-1/2} \frac{\partial}{\partial t} - F^{1/2} \frac{\partial}{\partial r} \right), \tag{4}$$

$$m^a \nabla_a = \frac{1}{r\sqrt{2}} \left(\frac{\partial}{\partial \theta} + \frac{i}{sin\theta} \frac{\partial}{\partial \varphi} \right). \tag{5}$$

It is chosen so that the "extent" of each vector is 1, in order not to emphasize the importance of any of the null directions with respect to the others. If we consider this tetrad as being associated with a spin-frame o^A, ι^A, i.e.

$$l^a = o^A o^{A'} \quad , \quad n^a = \iota^A \iota^{A'} \quad , \quad m^a = o^A \iota^{A'} \tag{6}$$

we can calculate the Infeld-Van der Waerden symbols

$$g_{AA'}{}^a = \begin{pmatrix} l^a & m^a \\ \overline{m}^a & n^a \end{pmatrix} \quad , \quad g_a{}^{AA'} = \begin{pmatrix} n_a & -\overline{m}_a \\ -m_a & l_a \end{pmatrix} \tag{7}$$

and the non-zero spin-coefficients are

$$\rho = \mu = -\frac{F^{1/2}}{r\sqrt{2}} \quad , \quad \varepsilon = \gamma = \frac{F' F^{-1/2}}{4\sqrt{2}} \quad , \quad \beta = -\alpha = \frac{cotg\theta}{2r\sqrt{2}}. \tag{8}$$

We now have all we need to translate equation (1) in terms of partial derivatives in a coordinate basis. We obtain 8 scalar equations, two of which can be transformed into constraints involving only space-like partial derivatives. The system in its hamiltonian form can be written:

$$U = {}^t \left(\sigma^0_{0'0'} \, , \, \sigma^0_{0'1'} \, , \, \sigma^0_{1'1'} \, , \, \sigma^1_{0'0'} \, , \, \sigma^1_{0'1'} \, , \, \sigma^1_{1'1'} \right), \tag{9}$$

$$\frac{\partial U}{\partial t} = \frac{\partial}{\partial t} \begin{pmatrix} \sigma^0_{0'0'} \\ \sigma^0_{0'1'} \\ \sigma^0_{1'1'} \\ \sigma^1_{0'0'} \\ \sigma^1_{0'1'} \\ \sigma^1_{1'1'} \end{pmatrix} = \begin{pmatrix} h\sigma^0_{0'0'} + \frac{F^{1/2}}{r} \overline{L_1} \, \sigma^0_{0'1'} \\ -h\sigma^0_{0'1'} - \frac{F}{r} \left(\sigma^0_{0'1'} + \sigma^1_{0'0'} \right) + \frac{F^{1/2}}{r} L_3 \, \sigma^0_{0'0'} \\ -h\sigma^0_{1'1'} - \frac{F'}{2} \sigma^0_{1'1'} - \frac{F}{r} \sigma^1_{0'1'} + \frac{F^{1/2}}{r} L_2 \, \sigma^0_{0'1'} \\ h\sigma^1_{0'0'} + \frac{F'}{2} \sigma^1_{0'0'} + \frac{F}{r} \sigma^0_{0'1'} + \frac{F^{1/2}}{r} \overline{L_2} \, \sigma^1_{0'1'} \\ h\sigma^1_{0'1'} + \frac{F}{r} \left(\sigma^1_{0'1'} + \sigma^0_{1'1'} \right) + \frac{F^{1/2}}{r} \overline{L_3} \, \sigma^1_{1'1'} \\ -h\sigma^1_{1'1'} + \frac{F^{1/2}}{r} L_1 \, \sigma^1_{0'1'} \end{pmatrix} = HU \tag{10}$$

where

$$h = F \left(\frac{\partial}{\partial r} + \frac{F'}{4F} + \frac{1}{r} \right), \tag{11}$$

$$L_k = \frac{\partial}{\partial\theta} + \left(k - \frac{3}{2}\right) cotg\theta + \frac{i}{sin\theta}\frac{\partial}{\partial\varphi} \quad , \quad k = 1,2,3, \tag{12}$$

$$\overline{L_k} = \frac{\partial}{\partial\theta} + \left(k - \frac{3}{2}\right) cotg\theta - \frac{i}{sin\theta}\frac{\partial}{\partial\varphi} \quad , \quad k = 1,2,3, \tag{13}$$

together with the two constraints

$$2h\sigma^0_{0'1'} + \left(\frac{2F}{r} - \frac{F'}{2}\right)\sigma^0_{0'1'} + \frac{F}{r}\sigma^1_{0'0'} - \frac{F^{1/2}}{r}\left(L_3\ \sigma^0_{0'0'} - \overline{L_2}\ \sigma^0_{1'1'}\right) = 0, \tag{14}$$

$$2h\sigma^1_{0'1'} + \left(\frac{2F}{r} - \frac{F'}{2}\right)\sigma^1_{0'1'} + \frac{F}{r}\sigma^0_{1'1'} - \frac{F^{1/2}}{r}\left(L_2\ \sigma^1_{0'0'} - \overline{L_3}\ \sigma^1_{1'1'}\right) = 0. \tag{15}$$

We introduce the Hilbert space $\mathcal{H}$ defined by

$$\mathcal{H} = \left\{L^2\left(]1,+\infty[_r \times S^2_\omega; F^{-1}dr^2 + r^2 d\omega^2\right)\right\}^6 \tag{16}$$

and the successive domains of H in $\mathcal{H}$

$$D(H^k) = \left\{U \in \mathcal{H};\quad H^j U \in \mathcal{H},\ 1 \leq j \leq k\right\},\quad k \in I\!N^*. \tag{17}$$

We also consider the spaces $\mathcal{H}_c$ and $D(H^k)_c$, $k \in I\!N^*$, of the elements of $\mathcal{H}$ and $D(H^k)$, $k \in I\!N^*$, which satisfy the constraint equations (14), (15); i.e. if we write (14) in the following way

$$AU = 0,\quad A = \left(-\frac{F^{1/2}}{r}L_3\ ,\ 2h + \frac{2F}{r} - \frac{F'}{r}\ ,\ \frac{F^{1/2}}{r}\overline{L_2}\ ,\ \frac{F}{r}\ ,\ 0\ ,\ 0\right), \tag{18}$$

and in the same manner (15) could become

$$BU = 0,\quad B = \left(0\ ,\ 0\ ,\ \frac{F}{r}\ ,\ -\frac{F^{1/2}}{r}L_2\ ,\ 2h + \frac{2F}{r} - \frac{F'}{r}\ ,\ \frac{F^{1/2}}{r}\overline{L_3}\right), \tag{19}$$

then, we have simply

$$\mathcal{H}_c = KerA \bigcap KerB \tag{20}$$

where $KerA$ is the kernel of A in $\mathcal{H}$, and for $k \in I\!N^*$,

$$D(H^k)_c = (KerA)_{D(H^k)} \bigcap (KerB)_{D(H^k)} = KerA \bigcap KerB \bigcap D(H^k), \tag{21}$$

$(KerA)_{D(H^k)}$ being the kernel of A in $D(H^k)$. In these spaces with constraints, the following existence and uniqueness result holds:

Theorem 1 *For any initial data $U_0 \in \mathcal{H}_c$ (resp. $U_0 \in D(H^k)_c$, $k \in I\!N^*$), equation (10) admits a unique solution U such that*

$$U \in C\left(I\!R_t; \mathcal{H}_c\right) \quad \left(resp.\ U \in C\left(I\!R_t; D(H^k)_c\right)\right) \tag{22}$$

and

$$U|_{t=0} = U_0. \tag{23}$$

Note that if $U_0 \in D(H^k)_c$, $k \in I\!N^*$, the solution U has the following additional regularities which are straightforward consequences of (10)

$$U \in \bigcap_{j=0}^{k} C^j \left(I\!R_t ; D(H^{k-j})_c \right) . \qquad (24)$$

Moreover, the Rarita-Schwinger 3-form

$$\beta = i \sigma_{aC'} \, \overline{\sigma}_{bC} \, dx^a \wedge dx^b \wedge dx^c , \qquad (25)$$

is divergence-free, $\sigma^{A}_{B'C'}$ denoting the spinor, symmetric in B', C', whose components satisfy (10), (22) and (23). In other words, if we consider the explicit translation of (25)

$$< \xi, \eta >_\beta = \left(\xi^0_{0'0'}, \eta^0_{0'0'} \right)_{L^2} + \left(\xi^1_{1'1'}, \eta^1_{1'1'} \right)_{L^2} + \left(\xi^1_{0'1'}, \eta^1_{0'1'} \right)_{L^2} + \left(\xi^0_{0'1'}, \eta^0_{0'1'} \right)_{L^2}$$

$$+ \left(\xi^1_{0'1'}, \eta^0_{1'1'} \right)_{L^2} + \left(\xi^0_{1'1'}, \eta^1_{0'1'} \right)_{L^2} + \left(\xi^1_{0'0'}, \eta^0_{0'1'} \right)_{L^2} + \left(\xi^0_{0'1'}, \eta^1_{0'0'} \right)_{L^2} , \quad \xi, \eta \in \mathcal{H} \qquad (26)$$

where $(\; , \;)_{L^2}$ denotes the standard scalar product on $L^2 \left( ]1, +\infty[_r \times S^2_\omega, F^{-1} dr^2 + r^2 d\omega^2 \right)$, then for any $U, V \in D(H)_c$

$$< HU, V >_\beta = - < U, HV >_\beta \qquad (27)$$

and if $U \in \mathcal{C} \left(I\!R_t ; \mathcal{H}_c \right)$ is a solution of (10), the quantity $< U, U >_\beta$ is conserved throughout time.

Hints of the proof: Firstly, we decompose equations (10), (14) and (15) into spin-weighted spherical harmonics. On each sub-space of given angular dependence, we prove a global existence and uniqueness result for solutions without constraints. This is done using a fixed point method: the evolution system (10) is expressed in the form of an integral equation, the fixed points of which are the solutions of (10). The next step is to prove that if the initial data have a given angular dependence and satisfy the constraints (14) and (15), then the solution associated with this initial data satisfies the constraints at each time t. To prove this, we show that the spaces with constraints are stable under H from which we infer their stability under the one parameter continuous group generated by H. At this point, we have proved the theorem for initial data with a fixed angular dependence. By linearity, the same result holds for initial data involving only a finite number of harmonics, i.e. belonging to a dense sub-space of $\mathcal{H}_c$ (or $D(H^k)_c$, $k \geq 1$). The last step is to extend the propagator to the entire space using an energy estimate. For more details about these analytical methods, see for example [2].

3 Where do we go from here?

Thus, the Cauchy problem is well-posed for a spin 3/2 potential $\sigma^{C}_{A'B'}$ in Schwarzschild's space-time and of course, it is also true for a pure gauge field which is simply given by a solution of Weyl's neutrino equation. In particular, a non global σ can be propagated into its domain of dependence and the same holds for the gauge. Therefore, at least in the case of the Schwarzschild metric, we now have some information about the nature of the obstacle to the construction of a II-charge. It was known that such an obstacle had to exists but it was not clear wether it resided in the propagation or in the patching of the potential modulo gauge (see [3]). The results obtained here seem to imply that there is nothing pathological about the propagation. Hence, the obstacle is more probably of a topological, rather than analytic, nature. More precisely, there might not exist at each time a proper covering of S^2.

Several directions of research can now be followed. Firstly, the same kind of study can be carried out in other Ricci-flat space-times. This would tell us wether the result obtained in Schwarzschild's space-time is an exception or if the propagator of the potential modulo gauge really is a "reliable" 1-parameter group in all vacuum space-times. Of course, one would also like to understand what features of the geometry cause the obstacle, be it topological or analytical, to arise. Studying the limit of a Schwarzschild black-hole when the mass goes to zero could shed some light on this conundrum. ¿From the viewpoint of analysis, it would be interesting to push further the study of spin 3/2 fields in black-hole backgrounds. An open problem is the construction of a time-dependent scattering theory in the Schwarzschild case. The technical difficulties are numerous, but it seems a fairly natural conjecture that the asymptotic behavior of linear spin 3/2 fields in the neighbourhood of horizons or in asymptotically flat regions can be described in terms of classical wave operators. Work is in progress.

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