11

Two examples of classical scattering off fixed sources

By
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1 Introduction

Penrose, in Penrose & MacCallum (1972, §3.2), describes the scattering of zero rest-mass particles by impulsive electromagnetic and gravitational waves by constructing hamiltonian equations on twistor space. The fixed source problem was also addressed in Penrose & MacCallum (1972, §5.2) and formal expressions for the hamiltonians, obtained by use of the twistor transform, were given. The approach to scattering off fixed sources described here differs in two ways from that given in Penrose & MacCallum. Firstly, we consider the scattering of zero rest-mass particles off fields generated by zero rest-mass particles, whereas Penrose & MacCallum considered zero rest-mass particles scattering off a massive fixed source. Secondly, because the fields produced by the zero rest-mass particles are impulsive waves, we can utilize the explicit 'scissors and paste' methods used in Penrose (1968) and Penrose & MacCallum (1972, §3.2). This avoids the use of the twistor transform. The fields off which the zero rest-mass particles scatter are treated as fixed although the sources are moving with the speed of light. In contrast to scattering off a fixed Coulomb field (with timelike source), the whole description is manifestly Lorentz covariant.

We construct the hamiltonian equations by applying the 'scissors and paste' method, in which the zero rest-mass particle undergoing scattering is described by a null twistor, to find a solution to the Lorentz force equation and to construct a space-time consisting of two regions of Minkowski space, $\mathcal{M}$, separated by an impulsive gravitational wave. The scattering is thus described in terms of the unfolding of a canonical transformation on the symplectic manifold of null twistors.

In the following sections we utilize the results of Bonnor (1969a,b) to construct fixed backgrounds generated by null fluids. These solutions to the Maxwell and Einstein equations have the form of plane-fronted waves which, upon dropping differentiability requirements, may be impulsive (i.e. with δ-function amplitude).

2 A twistor hamiltonian approach to scattering off fixed sources

2.1 *The Lorentz force problem*

Bonnor (1969a) showed that one can obtain plane wave solutions of Maxwell's equations where the source was taken to be a charge moving with the speed of light.

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Introduce a spin-frame α^A, β^A, with normalization $\alpha_A\beta^A = 1$, and the associated null tetrad $l^a = \alpha^A\overline{\alpha}^{A'}, n^a = \beta^A\overline{\beta}^{A'}, m^a = \alpha^A\overline{\beta}^{A'}, \overline{m}^a = \beta^A\overline{\alpha}^{A'}$. A position vector takes the form $x^a = vl^a + un^a + \zeta m^a + \overline{\zeta}\overline{m}^a$.

The field is generated by the vector potential

$$\Phi^a = (2\varphi, 0, 0, 0)\,, \tag{1}$$

where Bonnor takes the function φ as C^1 and piecewise C^2. This condition ensures the continuity of the Maxwell field F^{ab} and precludes surface charges and surface currents. (Later we will drop this requirement for the sake of considering fields with δ-function amplitude.) The function φ takes the form $\varphi = \varphi(u, \zeta, \overline{\zeta})$. The components of the Maxwell 2-form, F_{ab}, are given by $F_{ab} = \nabla_{[a}\Phi_{b]}$. The components of the 4-current $J^a = (4\pi)^{-1}\nabla_b F^{ab}$ are

$$J^a = (2\rho, 0, 0, 0)\,, \tag{2}$$

which implies $J^a \propto l^a$ and $4\pi\rho = -\nabla^2\varphi$. Thus in a charge-free region where $J^a = 0$, φ satisfies

$$\frac{\partial^2\varphi}{\partial\zeta\partial\overline{\zeta}} = 0\,. \tag{3}$$

The electromagnetic field is *null*, that is, it satisfies $F^{ab}F_{ab} = 0$ and $\varepsilon^{abcd}F_{ab}F_{cd} = 0$.

We now consider a specific choice for the function φ. For $r = (\zeta\overline{\zeta})^{\frac{1}{2}} \geq a$, $a \in \mathbb{R}$, a positive constant, take

$$\varphi = \delta(u)[2\log(r/a) + 1]\,.$$

(Bonnor chooses $\varphi = \psi(u)[2\log(r/a) + 1]$ for $r \geq a$, and $\varphi = \psi(u)r^2/a^2$, for $r \leq a$, with $\psi \in C^2$. So we are dropping the differentiability requirements, and we have no need of the solution for $r \leq a$. We note that Bonnor shows that such a solution can be obtained from an advanced potential, but the total energy of the field due to a single charge moving with the speed of light, diverges. When charges of both signs are present (so that the total charge is zero), he shows that the field has finite energy.)

We find that the electromagnetic spinor takes the form

$$\phi_{11} = \phi_{AB}\beta^A\beta^B = \delta(u)\frac{2a}{\overline{\zeta}}\,, \tag{4}$$

and remark

$$\phi_{AB} \propto \alpha_A\alpha_B\,.$$

All other components of the anti-self-dual part of the field vanish.

We wish to solve the Lorentz force equation in this background. The Lorentz force equation may be written in the form $P_b\nabla^b P^a = eF^{ab}P_b$, which gives well-defined equations of motion for null momenta $P_a = \overline{\pi}_A\pi_{A'}$. This equation reduces to

$$\overline{\pi}_B\pi_{B'}\nabla^{BB'}\overline{\pi}^A = e\phi^{AB}\overline{\pi}_B\,. \tag{5}$$

In order to solve (5) in the background (4) we make the ansatz

$$\overline{\pi}^A = \overline{\pi}^A_{\circ} + \Theta(u)[\overline{\pi}^A]\,. \tag{6}$$

Here, $\overline{\pi}_o^A$ is the initial momentum, $[\overline{\pi}_A]$ is the change in momentum and $\Theta(u)$ is the Heaviside step function. So we find upon substituting (6) into (5), and using (4),

$$[\overline{\pi}^A] = -\frac{2a\, e\, \alpha^A}{\zeta\, \pi_{oB'}\, \overline{\alpha}^{B'}}\,, \tag{7}$$

where we have inserted the initial momentum into $\overline{\pi}_A \pi_{A'} \nabla^{AA'}$. From here on we shall drop the constant $2a$.

We follow Penrose & MacCallum (1972, §3.2) in finding a twistor hamiltonian formulation of (7). The hamiltonian takes the form $\mathcal{H}(Z^\alpha, \overline{Z}_\alpha) = \mathcal{H}^+(Z^\alpha) + \mathcal{H}^-(\overline{Z}_\alpha)$, where

$$\mathcal{H}^+(Z^\alpha) = \overline{\mathcal{H}^-(\overline{Z}_\alpha)} = e \int^{\overline{\zeta}} f(x)\, \mathrm{d}x \tag{8}$$

and $f(x) = x^{-1}$. On the surface $u = 0$, the coordinate $\overline{\zeta}$ is given by

$$i\overline{\zeta} = \frac{\alpha_A \omega^A}{\overline{\alpha}^{A'} \pi_{A'}}\,. \tag{9}$$

Then

$$\delta\overline{\pi}_A = i\frac{\partial \mathcal{H}^+}{\partial \omega^A} = \frac{e\, \alpha_A}{\overline{\zeta}\overline{\alpha}^{A'} \pi_{A'}}\,, \tag{10}$$

and

$$\delta\overline{\omega}^{A'} = i\frac{\partial \mathcal{H}^+}{\partial \pi_{A'}} = -i\frac{e\overline{\alpha}^{A'}}{\overline{\alpha}^{B'} \pi_{B'}}\,, \tag{11}$$

where the hamiltonian (8) is

$$\mathcal{H}^+(Z^\alpha) = e \log\left(-i\frac{A_\alpha Z^\alpha}{\overline{B}_\alpha Z^\alpha}\right)\,, \tag{12}$$

with $A_\alpha \leftrightarrow (\alpha_A, 0)$ and $\overline{B}_\alpha \leftrightarrow (0, \overline{\alpha}^{A'})$. We observe that the hamiltonian is homogeneous of degree zero in Z^α. Equations (10) and (11) may be expressed in the manifestly twistorial form

$$\delta\overline{Z}_\alpha = i\frac{\partial \mathcal{H}^+}{\partial Z^\alpha}\,.$$

2.2 *Scattering off an impulsive gravitational wave*

We now turn to the formulation of a twistor hamiltonian approach to scattering off a fixed gravitational source. Here we make use of the result of Bonnor (1969b, §7), to obtain an exact solution of Einstein's equations that represents the gravitational field of a zero rest-mass particle. The field has the structure of a plane-fronted impulsive gravitational wave and hence our twistor hamiltonian approach to scattering null geodesics off this field will follow the 'scissors and paste' technique of Penrose (1968).

The metric for plane-fronted gravitational waves can be written in the form

$$\mathrm{d}s^2 = 2\mathrm{d}u\, \mathrm{d}v - 2\mathrm{d}\zeta\, \mathrm{d}\overline{\zeta} + 2A(u, \zeta, \overline{\zeta})\mathrm{d}u^2\,. \tag{13}$$

The function $A(u, \zeta, \overline{\zeta})$ is taken by Bonnor to be piecewise C^1, however we will suspend this differentiability requirement to allow for a δ-function behaviour in u. For this metric l^a is a Killing vector. The Ricci tensor for (13) is given by

$$R^{ab} = -\frac{1}{2}\frac{\partial^2 A}{\partial\zeta\partial\overline{\zeta}}\, l^a l^b\,.$$

From the field equations

$$R^{ab} - \frac{1}{2}g^{ab}R = -8\pi T^{ab} ,$$

we have

$$T^{ab} = \frac{1}{16\pi}\frac{\partial^2 A}{\partial\zeta\partial\overline{\zeta}}\, l^a l^b$$

and thus we may interpret this as the energy tensor of a fluid of zero rest-mass particles. In (13) we choose

$$A = \delta(u)\log\left(\frac{r}{a}\right) ,$$

where a is a constant and $r^2 = \zeta\overline{\zeta} \geq a$.

We now outline the 'scissors and paste' method as applied to the impulsive gravitational wave. Consider two regions of Minkowski space, $\mathcal{M}$ and $\mathcal{M}^*$, separated by a hypersurface Σ given by $u = 0 = u^*$.

$$\begin{aligned} \mathcal{M} &: ds^2 = 2du\,dv - 2d\zeta\,d\overline{\zeta} , \\ \mathcal{M}^* &: ds^2 = 2du^*\,dv^* - 2d\zeta^*\,d\overline{\zeta}^* , \end{aligned}$$

and identify the two coordinate patches according to $\zeta = \zeta^*$ and $v = v^* + s(\zeta, \overline{\zeta})$, where $s(\zeta, \overline{\zeta}) = \log\left(\frac{r}{a}\right)$. This identification creates the curvature.

The deflection of a worldline passing through Σ is written in terms of the transformation of the spin frame α^A, β^A,

$$\beta^A \mapsto \beta^A + \frac{\partial s}{\partial\overline{\zeta}}\alpha^A , \tag{14}$$

$$\alpha^A \mapsto \alpha^A . \tag{15}$$

Whence

$$\delta\pi_{A'} = (\pi_{B'}\overline{\alpha}^{B'})\frac{\partial s}{\partial\zeta}\overline{\alpha}_{A'} , \tag{16}$$

and by construction

$$\delta X^{AA'} = -s(\zeta, \overline{\zeta})\,\alpha^A\overline{\alpha}^{A'} ,$$

so that for $\delta\omega^A$ we have

$$\begin{aligned} \delta\omega^A &= iX^{AA'}\delta\pi_{A'} + i\delta X^{AA'}\pi_{A'}, \\ &= iX^{AA'}(\pi_{B'}\overline{\alpha}^{B'})\frac{\partial s}{\partial\zeta}\overline{\alpha}_{A'} - is(\zeta,\overline{\zeta})\,\alpha^A\overline{\alpha}^{A'}\pi_{A'} . \end{aligned} \tag{17}$$

Introducing the hamiltonian

$$\mathcal{H}(Z^\alpha, \overline{Z}_\alpha) = (\overline{\alpha}^{B'}\pi_{B'})(\alpha^B\overline{\pi}_B)\,s(\zeta, \overline{\zeta}) ,$$

we may write the equations (16) and (17) for the scattering of the null geodesics in twistor hamiltonian form

$$\delta Z^\alpha = -i\frac{\partial\mathcal{H}}{\partial\overline{Z}_\alpha} .$$

The spacelike coordinates on Σ may be written in terms of $\overline{Z}_\alpha$:

$$\zeta = i\frac{\overline{A}^\alpha\overline{Z}_\alpha}{B^\alpha\overline{Z}_\alpha}, \quad \text{where } A_\alpha \leftrightarrow (\alpha_A, 0),\ \overline{B}_\alpha \leftrightarrow (0, \overline{\alpha}^{A'}), \tag{18}$$

and therefore the twistor hamiltonian is given by

$$\mathcal{H}(Z^\alpha,\, \overline{Z}_\alpha) \;=\; \frac{1}{2}(\overline{B}_\alpha Z^\alpha)(B^\beta \overline{Z}_\beta)\,\log\left[\left(i\frac{\overline{A}^\alpha \overline{Z}_\alpha}{B^\beta \overline{Z}_\beta}\right)\left(-i\frac{A_\gamma Z^\gamma}{\overline{B}_\sigma Z^\sigma}\right)\right]\,. \tag{19}$$

This is homogeneous of degree 1 in Z^α and $\overline{Z}_\alpha$. We may write the function (19) as the sum of two expressions which involve functions that are homogeneous separately in Z^α and $\overline{Z}_\alpha$. To see this (cf. Penrose & MacCallum 1972, §3.2; Tod 1975), introduce two such functions g^+ and g^-, where

$$g^+ \;=\; \frac{i}{2}(\overline{B}_\alpha Z^\alpha)^2 \int^{\overline{\zeta}} \frac{\mathrm{d}x}{x}\,,$$

and $g^- \;=\; \overline{g^+}$. Clearly, g^+ and g^- are homogeneous of degree 2 in Z^α and $\overline{Z}_\alpha$ respectively. Then we see that the hamiltonian (19) can be written as $\mathcal{H}(Z^\alpha,\overline{Z}_\alpha) \;=\; \mathcal{H}^+ \,+\, \mathcal{H}^-$, with

$$\mathcal{H}^+ \;=\; \overline{Z}_\alpha I^{\alpha\beta}\frac{\partial g^+(Z^\alpha)}{\partial Z^\beta}$$

where $I^{\alpha\beta}$ is the infinity twistor, together with a similar expression for $\mathcal{H}^-$.

3 The scattering transformation

The results (10), (11), (16) and (17) for spin-1 and spin-2 monopole scattering may be analyzed both in terms of space-time and twistor space kinematics and dynamics. We begin with a brief analysis of the scattering dynamics for the case of the fixed electromagnetic source.

Let X_o^a be the position vector with respect to the origin O, of the point where the worldline of the zero rest-mass particle meets the wave. Having such a vector, we construct the position vectors for the particle before (with subscript 'b') and after (with subscript 'a') scattering in the following manner

$$x_b^{AA'} \;=\; X_o^{AA'} \,+\, \lambda\overline{\pi}_o^A\pi_o^{A'}\,,$$
$$x_a^{AA'} \;=\; X_o^{AA'} \,+\, \lambda\overline{\pi}_*^A\pi_*^{A'}\,,$$

for varying $\lambda \in \mathbb{R}$. The initial momenta are $\overline{\pi}_o^A\pi_o^{A'}$, and the final momenta $\overline{\pi}_*^A\pi_*^{A'}$ are obtained from Hamilton's equations. In the case of scattering off an electromagnetic wave $\delta X_o^{AA'} \;=\; 0$, so the change in $x^{AA'}$ given by

$$\delta x^{AA'} \;=\; x_a^{AA'} \;-\; x_b^{AA'}\,,$$

may be obtained by the substitution $\pi_*^{A'} \;=\; \pi_o^{A'} \,+\, [\pi^{A'}]$, as follows

$$\delta x^{AA'} \;=\; \lambda(\overline{\pi}_o^A[\pi^{A'}] \,+\, \pi_o^{A'}[\overline{\pi}^A] \,+\, [\pi^{A'}][\overline{\pi}^A])\,. \tag{20}$$

We may describe some of the salient features of the dynamics, in terms of the magnitudes of 'time delay' and 'deflection' vectors, obtained by projecting (20) into timelike and spacelike 2-surfaces respectively.

If we denote the spacelike 2-surface by S then the projection operator taking a vector at some point of S into the 2-surface is (see e.g. Penrose & Rindler 1984, p. 271)

$$S_{BB'}^{AA'} \;=\; -\alpha^A\overline{\beta}^{A'}\beta_B\overline{\alpha}_{B'} - \beta^A\overline{\alpha}^{A'}\alpha_B\overline{\beta}_{B'}\,,$$

and we can project the connecting vector between the scattered and unscattered geodesics (in a spacelike direction) into the 2-surface

$$\delta\tilde{x}^{AA'} = S^{AA'}_{BB'}\delta x^{BB'} .$$

We find that the magnitude of $\delta\tilde{x}^{AA'}$ is

$$\left|\delta\tilde{x}^{AA'}\delta\tilde{x}_{AA'}\right| \propto \sqrt{\frac{e^2}{\zeta\overline{\zeta}}} .$$

Thus the deflection decreases as r^{-1}. We may examine the magnitude of the time delay in a similar manner. In this case we use a timelike 2-surface T, with the projection operator T^a_b that takes a 4-vector at some point of T into the 2-surface

$$T^{AA'}_{BB'} = \alpha^A\overline{\alpha}^{A'}\beta_B\overline{\beta}_{B'} + \beta^A\overline{\beta}^{A'}\alpha_B\overline{\alpha}_{B'} ,$$

which, together with the expression for the change in $x^{AA'}$, gives

$$T^{AA'}_{BB'}\delta x^{BB'} \equiv \delta\hat{x}^{AA'} = -\frac{e\beta_B\overline{\pi}^B\alpha^A\overline{\alpha}^{A'}}{\zeta\alpha^B\overline{\pi}_B} - \frac{e\overline{\beta}_{B'}\pi^{B'}\alpha^A\overline{\alpha}^{A'}}{\overline{\zeta}\overline{\alpha}^{B'}\pi_{B'}} .$$

The vector $\delta\hat{x}^{AA'}$ is *null* and therefore the magnitude of the time delay is a constant (zero) and independent of the impact parameter.

We may consider the time delay for the scattering of a null geodesic off the impulsive gravitational wave. For the space-time given by (13) t^a is not a Killing vector. The transformation for t^a, $t^a \mapsto t^{a*}$, is given by, from (14) and (15),

$$\frac{1}{2}\left(\alpha^A\overline{\alpha}^{A'} + \beta^A\overline{\beta}^{A'}\right) \mapsto \frac{1}{2}\left(t^{AA'} + \beta^A\overline{\alpha}^{A'}\frac{\partial s}{\partial\overline{\zeta}} + \alpha^A\overline{\beta}^{A'}\frac{\partial s}{\partial\zeta} + \alpha^A\overline{\alpha}^{A'}\frac{\partial s}{\partial\zeta}\frac{\partial s}{\partial\overline{\zeta}}\right) .$$

Thus, for instance, a 2-surface with projection operator T^a_b, will have no invariant meaning. However, as Penrose & MacCallum (1972) remarked, there is no 'absolute' concept of time delay in general relativity and thus the time delay that we shall calculate here is not inconsistent with that obtained from examination of orbits in the Schwarzschild solution.

The deflection may be calculated in a straightforward manner. Write down an infinitesimal change in the coordinates of some position vector as

$$\begin{aligned} \delta x^{AA'} &= \delta X^{AA'} + \lambda\delta\overline{\pi}^A\pi^{A'} + \lambda\overline{\pi}^A\delta\pi^{A'} , \\ &= -s(\zeta,\overline{\zeta})\alpha^A\overline{\alpha}^{A'} + \lambda(\overline{\pi}^B\alpha_B)\frac{\partial s}{\partial\zeta}\alpha^A\pi^{A'} + \lambda(\overline{\alpha}_{B'}\pi^{B'})\frac{\partial s}{\partial\overline{\zeta}}\overline{\pi}^A\overline{\alpha}^{A'} . \end{aligned} \quad (21)$$

Then using (14) and (15), for the spacelike surface we calculate $S^{*BB'}_{AA'}$ and for the timelike surface we calculate $T^{*BB'}_{AA'}$. Thus the 'time delay' and 'deflection' may be computed from $\delta x^{AA'}T^{*BB'}_{AA'}$ and $\delta x^{AA'}S^{*BB'}_{AA'}$ and from the norms we obtain the r^{-1} dependence for the deflection and zero for the time delay (the vector being null). (Remark: if we had chosen to transvect (21) with $\beta_B\overline{\beta}_{B'}$ then the function $s(\zeta,\overline{\zeta})$ would contribute to the 'null delay'.)

We have shown that the behaviour of null geodesics in flat space-time with an impulsive wave background may be understood in terms of the unfolding of a canonical transformation of null twistors. Penrose (1968) discusses the transformation properties of the canonical equations on twistor space and in particular refers to the invariant structures of the system. Following Penrose *op. cit.*, we shall examine the invariance of the twistor norm $Z^\alpha \overline{Z}_\alpha$ and the inner product $Y^\alpha \overline{X}_\alpha$, under the action of the transformation δ, generated by the hamiltonians (12) and (19). To examine the symplectic invariance, we consider certain differential forms: $\vartheta = iZ^\alpha \mathbf{d}\overline{Z}_\alpha$ and the symplectic two-form $\Omega = \mathbf{d}\vartheta = i\mathbf{d}Z^\alpha \wedge \mathbf{d}\overline{Z}_\alpha$.

If $\mathcal{H}$ is homogeneous of degree m in Z^α and $\overline{Z}_\alpha$, then $\delta(Z^\alpha \overline{Z}_\alpha) = 0$ and the norm is part of the invariant structure. This holds for hamiltonians (12) and (19). It follows that this may be interpreted as the invariance of the helicity under δ. The symplectic two-form is the natural integral invariant associated with the hamiltonian system. It is invariant under the action of δ. Examination of $\delta\vartheta$ shows that it will vanish iff $\mathcal{H}$ is homogeneous of degree one in Z^α. Thus ϑ is an invariant for the hamiltonian (19) but not for the electromagnetic case (12).

Of particular interest here is the effect of the transformations on the twistor inner product $Y^\alpha \overline{X}_\alpha$, which, in general, will not be preserved under the action of the transformation. We construct a measure of the shift induced by the transformation generated by (12) on a point of $\mathbb{P}\mathbb{N}$. Let two rays μ and μ' be described by the two null twistors X^α and Y^α

$$X^\alpha \longleftrightarrow (ix^{AA'}\eta_{A'}, \eta_{A'}) \text{ and } Y^\alpha \longleftrightarrow (iy^{AA'}\xi_{A'}, \xi_{A'}) \ .$$

Choose the two rays to be abreast and focused at infinity. That is

$$y^{AA'} - x^{AA'} \equiv \Delta^{AA'} = m\,\alpha^A \overline{\beta}^{A'} - \overline{m}\,\beta^A \overline{\alpha}^{A'} ,$$

where $m, \overline{m} \in \mathbb{C}$, $\Delta^a \nabla_a \overline{\beta}^{A'} = 0$, and $Y^\alpha \overline{X}_\alpha = 0$. We then scatter the two rays using our hamiltonian prescription and then consider the product $X^{*\alpha}\overline{Y}^*_\alpha$. We find

$$\delta(Y^\alpha \overline{X}_\alpha) = ie\left(\frac{(\zeta\overline{\zeta})_x - (\zeta\overline{\zeta})_y}{\zeta_y \overline{\zeta}_x}\right) \tag{22}$$

(where $\zeta_{x/y}$ denotes the ζ-coordinate of the vector $x^{AA'}$ or $y^{AA'}$). This will vanish only if X and Y are coincident. In other words, even at large values of ζ, the expression will only become small if the values of the x and y coordinates are close together. Hence the scattering transformation induces a shift in viewpoint from $\mathbb{P}\mathbb{N}$ to $\mathbb{N}$. This is a realization of the fact that, in general, the canonical transformation will induce shear on a bundle of shear-free rays. The asymptotic behaviour is typical of the Coulomb problem, where at large distances from the source the field has an effect on the dynamics unless the interaction is switched off (for further discussion see Hodges 1983, 1985; Roulstone 1994).

A similar calculation for the scattering of two parallel rays off an impulsive gravitational wave gives, from (14) *et seq.*, the following for the change in the inner product

$$\begin{aligned}
\delta(Y^\alpha \overline{X}_\alpha) &= -i\overline{X}_\alpha \frac{\partial \mathcal{H}}{\partial \overline{Y}_\alpha} + iY^\alpha \frac{\partial \mathcal{H}}{\partial X^\alpha} , \\
&= -\frac{i}{2}\overline{B}_\alpha Y^\alpha B^\beta \overline{X}_\beta \log(\zeta\overline{\zeta})_y - \frac{i}{2}\overline{B}_\alpha Y^\alpha B^\beta \overline{Y}_\beta \left(\frac{\overline{A}^\gamma \overline{X}_\gamma}{\overline{A}^\delta \overline{Y}_\delta} - \frac{B^\gamma \overline{X}_\gamma}{B^\delta \overline{Y}_\delta}\right) \\
&\quad + \frac{i}{2}\overline{B}_\alpha Y^\alpha B^\beta \overline{X}_\beta \log(\zeta\overline{\zeta})_x + \frac{i}{2}\overline{B}_\alpha X^\alpha B^\beta \overline{X}_\beta \left(\frac{A_\gamma Y^\gamma}{A_\delta X^\delta} - \frac{\overline{B}_\gamma Y^\gamma}{\overline{B}_\delta X^\delta}\right) , \\
&\neq 0 ,
\end{aligned} \tag{23}$$

as $\zeta \to \infty$. Thus we see that the s_x and s_y terms now contribute directly to the displacement.

4 Summary

We conclude that the unfolding of a canonical transformation on the symplectic manifold of null twistors can describe the scattering off a fixed background and preserves the usual integral invariants. In contrast to the formal inhomogeneous expressions for the hamiltonians given in Penrose & MacCallum (1972, §5.2), and by virtue of their construction, the hamiltonians (12) and (19) are homogeneous. We have shown that for both the Coulomb and the linearized Schwarzschild backgrounds, shear is induced by considering the behaviour of neighbouring rays. This displacement does not vanish as one moves to large distances from the source and is described naturally in terms of the non-projective space $\mathbb{N}$. One can show that in the case of an electromagnetic dipole field, which decreases faster than the Coulomb field, the amount of shear decreases with increasing impact parameter. Further details, and the extension of the results to massive particles scattering off these sources (cf. Tod 1975, Tod & Perjés 1976), can be found in Roulstone (1994).

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