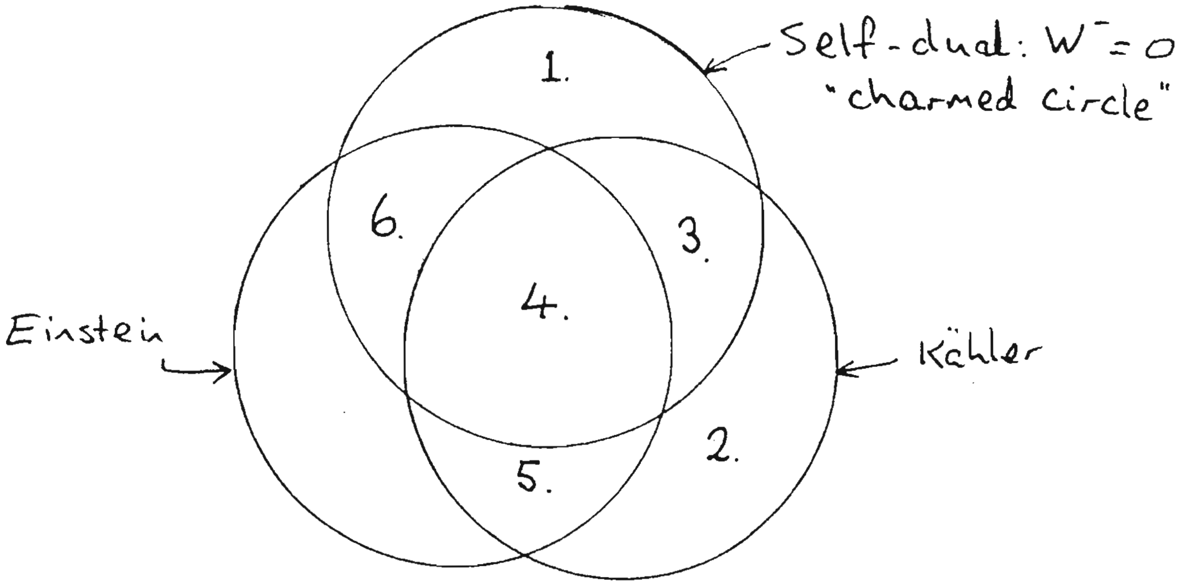
Self-dual Einstein metrics with symmetry

There are three different 'field equations' that one might impose on a four-dimensional Riemannian metric: one might require it to be Kähler or to be Einstein or to have self-dual Weyl tensor. These possibilities lead to an attractive Venn diagram in which the overlaps all have 'names' and have all been studied at one time or another. We may consider what happens to this diagram when the metric has a Killing vector. Again the regions usually have names as follows:



With symmetry, the regions are:

1: 3-dimensional Einstein-Weyl spaces (much studied);
2: Kähler-with-symmetry, studied by LeBrun (J.Diff.Geom. **34** 223 (1991);
3 and 4: scalar-flat Kähler and hyper-Kähler respectively with symmetry, solved by the SU(∞)-Toda field equation (LeBrun);
5: Einstein-Kähler with symmetry, studied by Pedersen and Poon (Comm.Math.Phys. **136** 309 (1991)) and solved by the 'Pedersen-Poon equation', equation (7) below;
6: self-dual Einstein or 'quaternionic-Kähler' with symmetry.

In this article, I will find a simple form for the field equations for region 6 corresponding to Einstein metrics with self-dual Weyl tensor and non-zero scalar curvature. After a sequence of transformations, the field equations end up being rather familiar.

As an introduction to the calculation, I will review some of the other regions in the diagram. For Kähler-plus-symmetry, we know from LeBrun's work (J.Diff.Geom. **34** 223 (1991)) that coordinates can be found in which the metric can be written in the form

$$ds^2 = W\left[e^u(dx^2+dy^2)+dz^2\right]+\frac{1}{W}(dt+\theta)^2. \tag{1}$$

The Kähler condition entails

$$d\theta = W_x\, dy\wedge dz + W_y\, dz\wedge dx + (We^u)_z\, dx\wedge dy \tag{2}$$

from which the function W must satisfy

$$W_{xx} + W_{yy} + (We^u)_{zz} = 0. \tag{3}$$

If we now require the scalar curvature to vanish then the function u must satisfy the 'SU(∞)-Toda field equation', namely:

$$u_{xx} + u_{yy} + (e^u)_{zz} = 0. \tag{4}$$

Thus a scalar-flat Kähler metric is determined by a solution of (4) together with a solution W of the linear equation (3). One particular solution of (3) is given by

$$W = c\,u_z\ ;\ c = \text{const.} \tag{5}$$

where u satisfies (4).

In this case, the metric (1) is actually hyper-Kähler or equivalently is Ricci-flat with self-dual Weyl tensor (and so corresponds to region 4 in the figure).

If, instead of vanishing scalar-curvature, we seek Kähler metrics with vanishing trace-free Ricci tensor then in place of (5) we must take

$$W = \frac{u_z}{\Lambda z + M} \tag{6}$$

where M is a constant (of integration) and Λ is proportional to the Ricci scalar. Now in place of (4) we find that u must satisfy the equation:

$$u_{xx} + u_{yy} + (e^u)_{zz} = \frac{2\Lambda e^u u_z}{\Lambda z + M} \tag{7}$$

an equation found by Pedersen and Poon (Comm.Math.Phys. **136** 309 (1991)). This corresponds to region 5 in the figure.

If Λ is non-zero, then we may absorb M into z whereupon Λ disappears from (7). If is zero then this case reduces to the previous one.

With Λ non-zero, the field-equations lie outside the 'charmed circle' at the top in the figure. Everything inside the circle has self-dual Weyl tensor, so can be solved by a twistor construction and so all symmetry reductions of them should be integrable; outside the circle there is no expectation of integrability.

Region 6 in the figure, corresponding to self-dual Einstein metrics with symmetry and with Λ non-zero, lies inside the charmed circle and so should lead to integrable equations. My purpose in this note is to find these equations. What turns up is eventually very similar to the scalar-flat Kähler case.

We begin by finding a canonical form for the metric in this case. Suppose then that we have a Killing vector K in a 4-dimensional Riemannian space. In terms of spinors, the derivative of K decomposes as:

$$\nabla_a K_b = \varphi_{AB}\, \epsilon_{A'B'} + \psi_{A'B'}\, \epsilon_{AB} \tag{8}$$

and the following identity, true for any Killing vector,

$$\nabla_a \nabla_b K_c = R_{bcad} K^d \tag{9}$$

entails

$$\nabla_{AA'}\varphi_{BC} = -\psi_{BCAD}K^{D}_{A'} + \Lambda\,\epsilon_{A(B}K_{C)A'}$$
$$\nabla_{AA'}\psi_{B'C'} = \Lambda\,\epsilon_{A'(B'}K_{C')A}. \tag{10}$$

Define the scalar ψ by $2\psi^2 = \psi^{A'B'}\psi_{A'B'}$, and define the tensor $J_a{}^b$ by

$$J_a{}^b = \frac{1}{\psi}\,\delta_A{}^B\,\psi_{A'}{}^{B'} \tag{11}$$

then J is an almost-complex structure. It is a straightforward calculation based on (8) and (10) to see that this complex structure is actually integrable. This fact, which is crucial in what follows, was pointed out to me by Lionel Mason. Thus the metric is Hermitian, but will not be Kähler unless Λ is zero.

Contracting the second of equations (10) with $\psi^{A'B'}$ enables us to see that

$$2\psi\nabla_a\psi = \psi^{B'C'}\nabla_a\psi_{B'C'} = \Lambda\psi_{A'B'}K^{B'}{}_A \tag{12}$$

so that $J_{ab}K^b = \dfrac{2}{\Lambda}\nabla_a\psi$.

We may now follow what is essentially LeBrun's argument to arrive at the form of the metric. If the Killing vector is $K^{\cdot} = \partial/\partial t$ in contravariant form then in covariant form it can be written

$$K_{\cdot} = \frac{1}{W}(dt+\theta) \tag{13}$$

in terms of a 1-form θ and a scalar $W = (K_a K^a)^{-1}$.

We define a coordinate $z = \dfrac{2\psi}{\Lambda}$ (which incidentally makes the limit $\Lambda\to 0$ one to be taken with care). The 2-blades containing K and dz are eigenspacres of the complex structure and so are integrable, as are their orthogonal complements. Introduce a complex coordinate ζ = x+iy on the orthogonal complement, then the metric can be written exactly as in (1) again.

This time the metric is not Kähler, so we don't have equation (2). To see what we have instead, note that from (8) and (12)

$$K^b \nabla_a K_b = -\varphi_{AB} K^B{}_{A'} - \psi_{A'B'} K^{B'}{}_A = -\frac{\nabla_a W}{2W^2}$$

$$\begin{aligned} K^b * (\nabla_a K_b) &= \varphi_{AB} K^B{}_{A'} - \psi_{A'B'} K^{B'}{}_A \\ &= \frac{\nabla_a W}{2W^2} - \frac{4\psi}{\Lambda} \nabla_a \psi. \end{aligned} \tag{14}$$

Knowing these we can calculate that

$$d\theta = W_x \, dy \wedge dz + W_y \, dz \wedge dx + e^u (W_z - 2\Lambda z W^2) \, dx \wedge dy \tag{15}$$

This has an integrability condition which naturally differs from that in (3).

The next step is to impose the conditions on the metric (1) that the trace-free Ricci tensor vanish and that the Weyl tensor be self-dual. I carried out this calculation following the formalism described in my article in *Twistor Theory* ed. Stephen Huggett (Dekker; 1995), the proceedings of the Seale Hayne twistor conference. The details are unilluminating, but the result is that W is determined as

$$-2\Lambda W = \frac{1}{z} u_z - \frac{2}{z^2} \tag{16}$$

where u satisfies the equation

$$u_{xx} + u_{yy} + e^u \left(u_{zz} + u_z^2 - \frac{6}{z} u_z + \frac{12}{z^2} \right) = 0. \tag{17}$$

Equations (15), (16) and (17) with the metric (1) form our principal conclusion. It can be checked that the integrability condition for (15) is automatically satisfied.

One simple class of solutions to (17) is the separable solutions: write u in a separated form $u = f(x, y) + g(z)$ then the general such solution of (17) is

$$e^u = \frac{z^2 (4k + bz + az^2)}{(1 + k(x^2 + y^2))^2} \tag{18}$$

where a,b and k are arbitrary constants. The resulting metric is the metric with U(2) symmetry found by Pedersen (Math.Ann. 274 35 (1986)).

What is unexpected, and is a disappointment, is that (17) can be transformed back into (4); in other words, *self-dual Einstein metrics with non-zero scalar curvature are determined by the SU(∞)-Toda field equation.*

To see this, introduce new variables w = 1/z and v = u - 4logz, then

$$\begin{aligned} & v_{xx} + v_{yy} + (e^{v})_{ww} \\ &= u_{xx} + u_{yy} + e^{u}\left(u_{zz} + u_z^2 - \frac{6}{z}u_z + \frac{12}{z^2}\right) \\ \therefore\ &= 0 \end{aligned} \tag{19}$$

The metric (1) transforms to the following form:

$$ds^2 = \frac{P}{\omega^2}\left[e^{v}(dx^2 + dy^2) + d\omega^2\right] + \frac{1}{P\omega^2}(dt + \theta)^2 \tag{20}$$

where v satisfies the SU(∞)-Toda field equation (in x,y and w), P is given by

$$-2\Lambda P = 2 - \omega v_{\omega} \tag{21}$$

and θ is determined by

$$\begin{aligned} d\theta &= -P_x\, dy\wedge d\omega - P_y\, d\omega\wedge dx - e^{v}\left(P_\omega + \frac{2}{\omega}P + \frac{2\Lambda}{\omega}P^2\right) dx\wedge dy \\ &= \frac{-1}{2\Lambda}\left\{\omega v_{\omega x}\, dy\wedge d\omega + \omega v_{\omega y}\, d\omega\wedge dx + e^{v}\left(\omega v_{\omega\omega} - v_\omega + \omega v_\omega^2\right) dx\wedge dy\right\} \end{aligned} \tag{22}$$

Acknowledgement
I am grateful to Lionel Mason for the observation that the complex structure J of equation (11) is integrable for a self-dual Einstein space with symmetry.

Paul Tod