'Moving-Zero' Solutions of Ward's Chiral Model

In [W1] Ward modifies the SU(2) chiral field equation in R^{2+1}:

$$\eta^{\mu\nu}\partial_\mu(J^{-1}\partial_\nu J) = 0, \tag{1}$$

which is Lorentz invariant but not integrable, to a field equation which is integrable, but has less symmetry. This equation is

$$\eta^{\mu\nu}\partial_\mu(J^{-1}\partial_\nu J) + V_\alpha\varepsilon^{\alpha\mu\nu}\partial_\mu(J^{-1}\partial_\nu J) = 0 \tag{2}$$

with $\varepsilon^{\alpha\mu\nu}$ being the alternating tensor with $\varepsilon^{012} = 1$, $\eta^{\mu\nu}$=diag$(-1,1,1)$, J a map from R^{2+1} to SU(2) and V_α a constant unit vector. We are using coordinates $x^\mu = (x^0, x^1, x^2) = (t, x, y)$, and $\partial_\mu = \partial/\partial x^\mu$. The conformal properties of V_α determine whether the symmetry group is SO(2) or SO(1,1). Ward chooses V to have the components $V_\alpha = (0,1,0)$, the space-like case. The reason for this is that the standard energy momentum-tensor for [1] -

$$T_{\mu\nu} = (-\delta^\alpha_\mu\delta^\beta_\nu + \frac{1}{2}\eta_{\mu\nu}\eta^{\alpha\beta})tr(J^{-1}\partial_\alpha J J^{-1}\partial_\beta J) \tag{3}$$

gives an energy-momentum vector $P_\mu = T_{\mu 0}$ which is divergence-free when both $V_0 = 0$ and J satisfies [2]. Hence we have a conserved energy $E = \int P_0 dx^1 dx^2$, where the integration is taken over planes of constant x^0, for solutions of [2]. This provides us with a natural boundary condition of finite energy, which means that $J = K + O(r^{-1})$ as $r \to \infty$, K being a (constant) element of SU(2), and $r^2 = (x^1)^2 + (x^2)^2$.

Ward then goes on to find soliton solutions of [2] using the method of 'Riemann problem with zeros', with the location of the zeros being fixed. I shall now outline the same technique, with the generalization that the zeros are not fixed.

We examine the Lax pair

$$(\lambda\partial_x - \partial_u)\Psi = A\Psi \tag{4}$$
$$(\lambda\partial_v - \partial_x)\Psi = B\Psi$$

where $\lambda \in C$, (u,v,x) are coordinates on R^{2+1}, A and B are su(2)-valued functions of (u,v,x) only, and $\Psi(u,v,x,\lambda)$ is a unimodular 2×2 matrix-valued function satisfying the reality condition

$$\Psi(u,v,x,\bar\lambda)^\dagger = \Psi(u,v,x,\lambda)^{-1}, \tag{5}$$

where $\dagger$ denotes the conjugate transpose matrix.

The Lax pair has the usual type of consistency conditions

$$\partial_x B = \partial_v A \quad \partial_x A - \partial_u B - [A, B] = 0, \tag{6}$$

and putting $J(u, v, x) = \Psi(u, v, x, \lambda = 0)^{-1}$ for Ψ a solution of [4], we get from [4], [6] that

$$\partial_x(J^{-1}\partial_x J) - \partial_v(J^{-1}\partial_u J) = 0.$$

This is precisely[2], as we can see by replacing $u = \frac{1}{2}(t + y), v = \frac{1}{2}(t - y)$, and the reality condition ensures that J is unitary. We can now use the Riemann method on [4] by using the Ansatz

$$\Psi_{ab}(u, v, x, \lambda) = \delta_{ab} + \frac{n_a m_b}{\lambda - \alpha}$$

where the vectors n,m, and the function α are independent of λ. (Ward has α constant - we are allowing the pole to move. His Ansatz also adds on other first order poles, but we shall just have the one pole. The generalization is straightforward).

The reality condition[5] gives us n in terms of m:

$$n_a = -\frac{\bar{m}_a(\bar{\alpha} - \alpha)}{m_0\bar{m}_0 + m_1\bar{m}_1}.$$

There is a 'gauge' homogeneity in that we can multiply m_0 and m_1 by the same holomorphic function without altering the resulting Ψ. Using this, we set $m_a = (1, f)$.

Substituting the Ansatz into [4] gives us differential equations for f and α. Firstly we find that

$$\alpha\partial_x\alpha - \partial_u\alpha = \alpha\partial_u\alpha - \partial_v\alpha = 0,$$

which is satisfied when there is an F such that

$$F(\alpha^2 u + \alpha x + v, \alpha) = 0,$$

giving us α. Then the condition on f is that

$$\alpha\partial_x f - \partial_u f = \alpha\partial_u f - \partial_v f = 0,$$

which holds when $f = f(\alpha^2 u + \alpha x + v, \alpha)$.

Thus we can generate a J satisfying [2] by prescribing: a function F of two variables; one of the solutions α to the equation $F(\alpha^2 u + \alpha x + v, \alpha) = 0$; and a function $f(\alpha^2 u + \alpha x + v, \alpha)$. J is then given by

$$(J^{-1})_{ab} = \Delta^{-\frac{1}{2}}(\delta_{ab} + \bar{m}_a m_b \frac{(\bar{\alpha} - \alpha)}{(1 + f\bar{f})}),$$

where $\Delta = \bar{\alpha}/\alpha$ is the determinant of the term in brackets on the RHS, and $(m_0, m_1) = (1, f)$. It should be noted that normalizing J so that it is unimodular does not affect J's being a solution of [2]. Some choices of F will give α such that J has singularities, and so care must be taken with this choice.

We can now go on to find the energy $P_0 = T_{00}$ of such a J, using expression [3]. This turns out to be

$$\frac{1}{2}(Im(\frac{\alpha\bar{A}(1+\bar{a}^2)}{\alpha\bar{\alpha}}))^2 + \frac{A\bar{A}}{\alpha\bar{\alpha}}(Im(\alpha))^2 + 2\frac{(1+\alpha\bar{\alpha})^2|f_1B + f_2A|^2}{\alpha\bar{\alpha}(1+f\bar{f})^2}(Im(\alpha))^2,$$

where A and B are given by

$$A = -\frac{F_1}{(2\alpha u + x)F_1 + F_2} \quad B = \frac{F_2}{(2\alpha u + x)F_1 + F_2}$$

with F_1, F_2 being the derivatives of F with respect to its first and second parameters respectively, and similarly f_1, f_2 the derivatives of f.

The first two terms are entirely due to the determinant of J, and provide a 'vacuum' over which the third term of the energy density moves. In the case of constant α, this vacuum vanishes, and we are left with equation (A9) of [W1].

The simplest interesting case is obtained by choosing

$$F(\alpha^2 u + \alpha x + v, \alpha) = (\alpha^2 u + \alpha x + v) + \frac{i}{2}(\alpha^2 + 1) = 0, \tag{7}$$

the imaginary term being necessary to avoid singularities in J. This gives us two choices for α, but they both give the similar J's. We still have left a choice of f. The simplest choice is a projection onto either of the parameters. We shall choose projection onto the first, so that $f = (\alpha^2 u + \alpha x + v)$. Figure 1 shows the vacuum state of this solution at t=5. The two crests move away from each other with velocity 1, maintaining their height (which tends to 2 along the crests). There is also a circle of radiation, spreading outwards from the origin, with velocity 1. At t=0, the circle is not apparent, and the crests form a single wavefront.

On top of this vacuum, we get a soliton-like object travelling in a straight line along the y-axis, in tandem with one of the crests. Unfortunately, its amplitude decays rapidly away from $t = 0$, and so gets swamped by the vacuum. Figure 2 shows this 'soliton' at $t = 1$. It should be emphasized that due to the vacuum term, no solutions will have finite-energy for this choice of α.

Figure 1

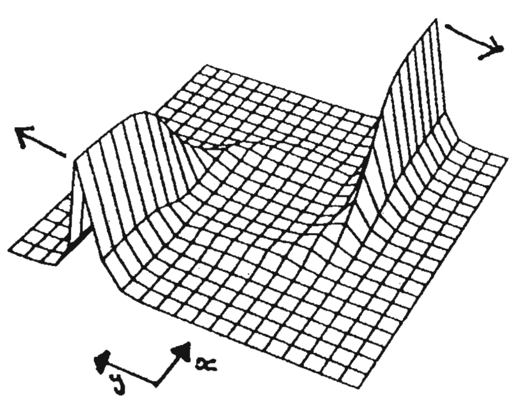
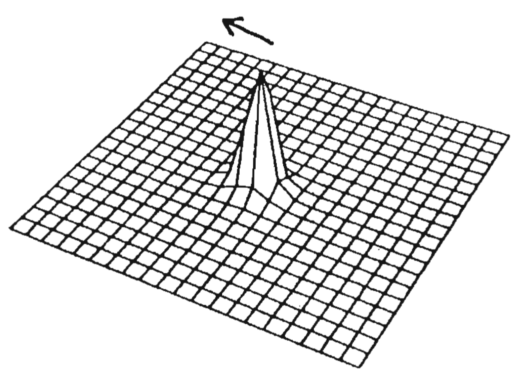
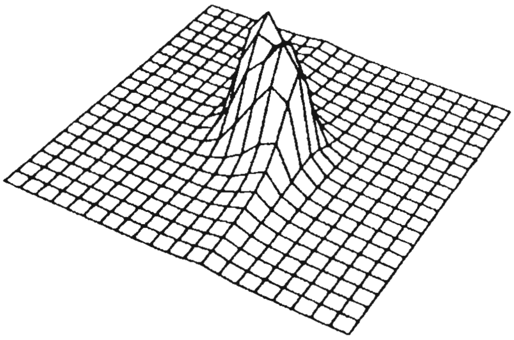
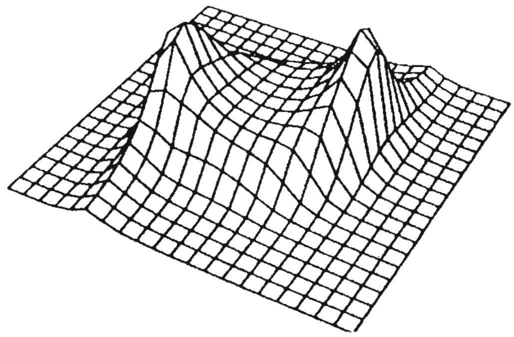


Figure 2



If $f = \alpha$, we get the appearance of two solitons coming towards each other, colliding at $t = 0$, and scattering of at an angle (see below for rescaled views of this at t=-2,5). They decay in amplititude away from $t = 0$. This is an interesting result, since there has long been numerical evidence for scattering solitons for [2], but no analytic explanation. It is unfortunate that the vacuum is once again dominant.





The above example has a motivation from twistor theory, in that we would like to interpret Ψ as the patching matrix for a bundle over minitwistor space, or a compactification. (Minitwistor space is $O(2)$, the line bundle over CP^1 with Chern class 2. For more details see [W2],[MS]). If we write, suppressing matrix indices,

$$\Psi(\pi_A, \xi) = I + \frac{L(\pi_A, \xi)}{\frac{i}{2}(\pi_0^2 + \pi_1^2) - \xi},$$

where $\pi_A, \xi = \pi_0^2 u + \pi_0 \pi_1 x + \pi_1^2 v$ are coordinates on minitwistor space, then we can split this into partial fractions, writing $\lambda = \pi_0/\pi_1$, to obtain

$$\Psi = I + \frac{M}{\lambda - \alpha} + \frac{N}{\lambda - \beta}$$

with α, β the roots of [7], which is very similar to our Ansatz.

The consequences of adding the term corresponding to second root are currently being investigated, but it is already known that the vacuum term will mean that the boundary conditions cannot be satisfied.This is not unexpected, as there are geometric reasons for believing it necessary to use an F that is quadratic in its first parameter before the boundary conditions can be satisfied.

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References

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