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Complex Structures via 3 and 4-Dimensional Cauchy Integrals

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Introduction

The existence of complex structures in quantum theory and their possible role in a theory of quantum gravity is a subject of much current interest. As Pauli points out these structures are essential in ordinary quantum mechanics in order that the state vector evolves unitarily subject to the appropriate field equation. Hawking and Gibbons have argued that the complex structures correspond in the 'history' of the universe to the emergence of our classical notion of time in thermodynamics. In the passage from a 4-dimensional real Euclidean universe to a real Lorentzian universe one acquires a real 'time' coordinate and thus a notion of positive and negative frequency. It is their view that the complex structure comes into play at the interface of these two regions where the real and imaginary time coordinates vanish.

In terms of the theory of Fock space, we have the two cases of a Hilbert space of *real* or of *complex* valued solutions to the field equation. In the former case, there is only *one* available complex structure since this must map a given real solution of the field equation to another *real* solution. In terms of the transform $\hat{\varphi}(k)$ of the field multiplication by a complex number $\alpha = x + iy$ is replaced by the action of the operator $x + Jy$ on φ where J acts linearly and multiplies the positive and negative frequency parts of the field by $+i$ and $-i$ respectively. We say that J is the *complex structure* acting on the Hilbert space. Thus the action of J amounts to multiplication by i but *within the Hilbert space*. In the case of a *complex* valued field there are in fact *two* possible complex structures – that functionally identical to the one given above, together with straightforward multiplication of $\hat{\varphi}$ by the complex number α. However it is only the former choice which leads to positive values for the expectation of the energy operator – since particles and their associated antiparticles both have *positive* rest energy we make *this* choice on physical grounds.

In this article we discuss the action of the complex structure J on a real or complex valued spacetime field φ with any index structure via multidimensional Cauchy integration. In the case of a 4-dimensional integral φ is a field required to be analytic on real compactified spacetime $M^{\#}$ and extending therefore to a 4-complex dimensional neighbourhood. In the 3-dimensional case we require further that the field satisfies $\nabla^2\varphi = 0$ throughout $CM^{\#}$.

The formulae we give are *essentially* different from the usual multidimensional Cauchy integral formulae one encounters in complex analysis. The latter are merely generalisations of the 1-dimensional residue calculus to integration around the boundary of a n-dimensional polydisk – a triviality by repeated integration.

The formulae we present have the property of Lorentz and conformal covariance.

A Reproducing Kernel (RK)

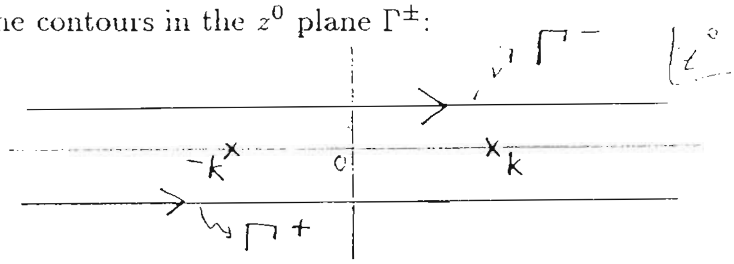
Holomorphic extensions and positive frequency.

We let $M^{\#}$ denote real compactified Minkowski spacetime. The invariant spacetime regions $CM^{\pm}$, CM^{0}, $M^{\#}$ are of considerable physical significance. A spacetime field φ which is holomorphic on $\overline{CM^{+}}$ (by which we mean holomorphic on some *open* neighbourhood $U \supset \overline{CM^{+}}$) is called *positive frequency*. Fields which are holomorphic on $\overline{CM^{-}}$ are called *negative frequency*. In contrast, and we will not be considering such, fields which are holomorphic on the *open* future or past tubes CM^{+}, CM^{-} are called *future* or *past analytic*. CM^{0} denotes the set of points of $M^{\#}$ with spacelike imaginary part.

Positive frequency is equivalent to holomorphic extension to the forward tube defined by

$$CM^{+} \equiv \{(x^{a} - iy^{a}) : y^{0} > 0,\ (y^{0})^{2} > |\mathbf{y}|^{2}\}.$$

We now define the contours in the z^{0} plane $\Gamma^{\pm}$:



Both contours are future pointing in real time. Consider

$$I = \int_{\Gamma^{-}} \frac{\varphi^{+}(z)}{z^{4}} d^{4}z$$

$$T = \int_{\Gamma^{-}} \frac{\varphi^{+}(z^{0}, \mathbf{z})}{(z^{0} - k)^{2}(z^{0} + k)^{2}} dz^{0}$$

where $k = |\mathbf{z}|$; $\mathbf{z}$ is assumed to be real. Setting $f(z^{0}) = \frac{\varphi^{+}(z^{0}, \mathbf{z})}{(z^{0}-k)^{2}(z^{0}+k)^{2}}$ we calculate the residues of f at $z^{0} = \pm k$, $k \neq 0$.

$$\text{residue at } k = \frac{-\varphi^{+}(k, \mathbf{z})}{4k^{3}} + \frac{1}{(2k)^{2}} \frac{\partial \varphi^{+}}{\partial t}|_{t=k}$$

Similarly, but noting sign differences

$$\text{residue at } -k = \frac{\varphi^{+}(-k, \mathbf{z})}{4k^{3}} + \frac{1}{(-2k)^{2}} \frac{\partial \varphi^{+}}{\partial t}|_{t=-k}.$$

Since the field is positive frequency we may close off the contour in the lower half z^{0} plane, and the residue theorem gives

$$T = \frac{\pi i}{2}[\frac{\varphi^{+}(k) - \varphi^{+}(-k)}{k^{3}}] + -\frac{\pi i}{2} \frac{[\partial \varphi^{+}/\partial t(k) + \partial \varphi^{+}/\partial t(-k)]}{k^{2}}.$$

For $k = 0$

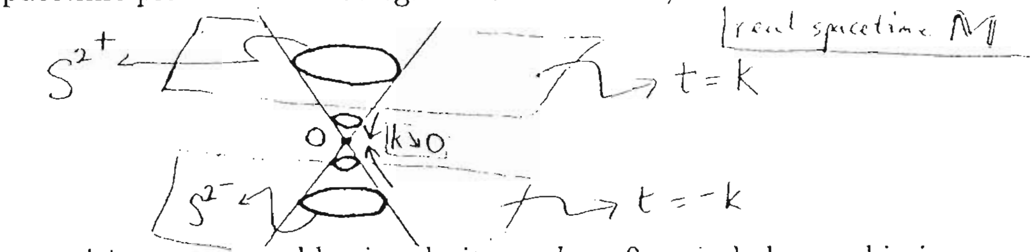
$$T = -2\pi i \cdot \operatorname{res}\varphi^+/(z_0)^4|_0 = -\frac{1}{3}\pi i \frac{\partial^3 \varphi^+}{\partial t^3}|_0$$

which is just a finite number.
Now

$$I = \int_{\Gamma^-} \frac{\varphi^+(z)}{z^4} d^4 z = \frac{\pi i}{2} \int_{S^3} [\frac{\varphi^+(k) - \varphi^+(-k)}{k^3}] - \frac{[\partial\varphi^+/\partial t(k) + \partial\varphi^+/\partial t(-k)]}{k^2}$$

$$= \frac{\pi i}{2} \int_0^\infty \int_{S^2} [\frac{\varphi^+(k) - \varphi^+(-k)}{k}] - [\partial\varphi^+/\partial t(k) + \partial\varphi^+/\partial t(-k)] dk\, d^2 S.$$

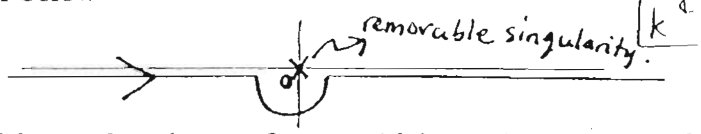
The term from $k = 0$ in T contributes zero since it is a simple discontinuity in the k integration. The spacetime picture of this integral is shown below,



Note that the integrand has a removable singularity at $k = 0$ so is holomorphic in an open neighbourhood of the real k axis, and that, analytically continued as a function of both positive and negative k, is an even function. The second term in square brackets contributes zero since for each term separately the contour can be closed in the upper/lower half plane enclosing no singularities. We may therefore replace the k integration by the contour integral in the complex k plane

$$\frac{1}{2} \int_C [\frac{\varphi^+(k) - \varphi^+(-k)}{k}] dk$$

where C is shown below



(Clearly we could have also chosen C to avoid $k = 0$ in the upper-half plane – this choice will not affect the final result.) Now we can evaluate the k integral by residues; writing this as $\frac{1}{2}\int_C [\frac{\varphi^+(k)}{k}] - \frac{1}{2}\int_C [\frac{\varphi^+(-k)}{k}] dk$, we see that the first term contributes nothing as C can be closed in the lower-half k-plane. The second term on the other hand contributes $-\pi i \varphi^+(0)$, since in this case C can be closed off in the upper-half k-plane. The S^2 integration contributes a factor of 4π. Hence

$$I = 2\pi^3 \varphi^+(0).$$

We have thus eliminated any problems in dealing with the spatial 2−sphere dependence of the field. In a similar way

$$\int_{\Gamma^+} \frac{\varphi^-(z)}{z^4} d^4 z = 2\pi^3 \varphi^-(0).$$

Note that Hartog's theorem does *not* apply. A simple corollary is the following

REMOVABILITY LEMMA (RL) : Given f holomorphic on $\overline{CM^{\pm}}$, then

$$\int_{\Gamma^{\mp}} \frac{f}{z^{2n}} d^4 z = 0, \quad n = 1, 0, -1, \ldots$$

Proof: Write the integrand as $\frac{f z^{4-2n}}{z^4}$ and use the reproducing kernel.

The Complex Structure for Spacetime Fields.

We introduce the complex structure J on spacetime fields as follows. Let J be a linear map from V to itself

$$J : V \longrightarrow V.$$

Then J is defined to act on a field ϕ by multiplying its positive frequency part by i and its negative frequency part by $-i$. So writing $\phi = \phi^+ + \phi^-$

$$J[\phi] = J[\phi^+ + \phi^-] = J[\phi^+] + J[\phi^-] = i\phi^+ - i\phi^-.$$

Clearly then

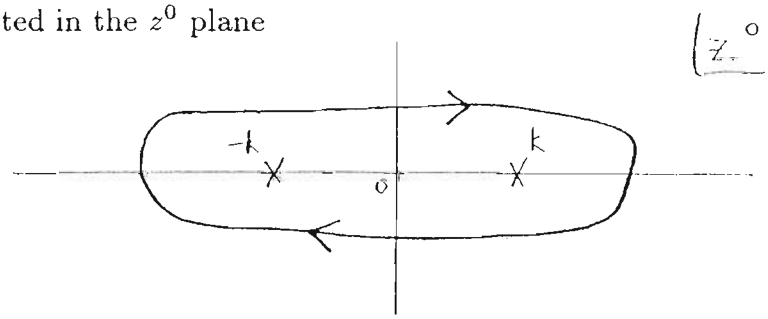
$$J^2 \equiv -1 \text{ acting on all } V$$

as required for J to be a complex structure on V. Note that J maps real fields to real fields. The action of J constitutes a *repolarisation* of the field $\varphi \mapsto i(\varphi^+ - \varphi^-)$.

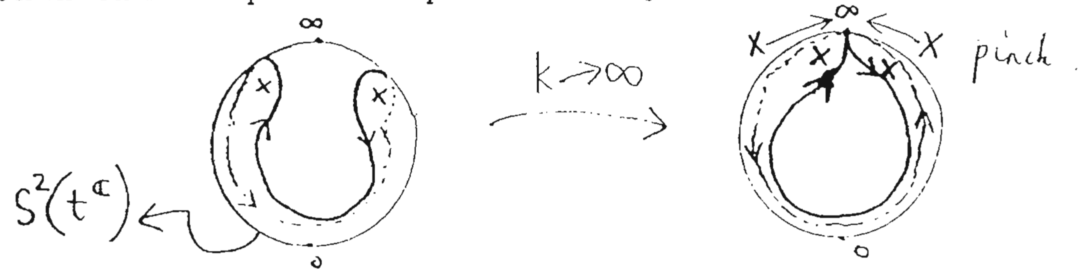
Now from our reproducing kernel above we see that

$$J[\varphi(r)] = \frac{i}{2\pi^3} \int_{\Gamma_J} \frac{\varphi(z)}{(z-r)^4} d^4 z$$

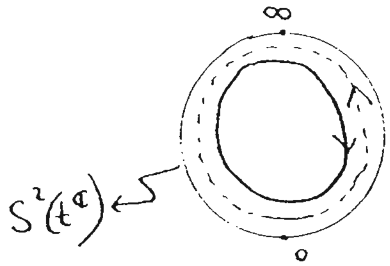
where Γ_J is depicted in the z^0 plane



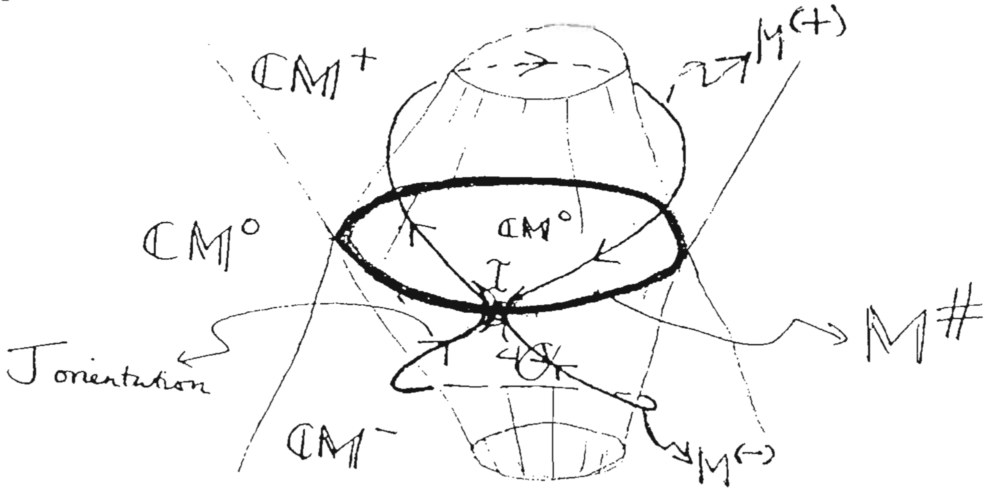
Note that under time reflection T: $z^0 \mapsto -z^0$, the contour reverses orientation so that $J[\varphi]$ changes sign, as must be the case since T interchanges positive and negative frequency parts. On the Riemann sphere of complexified time Γ_J is



We see that as $k \to \infty$, Γ_J *pinches* at ∞ in z^0 and splits into two components in the future and past tubes



with opposite orientation with respect to the great circle of real time. It is instructive to see the geometry of the contour in the full four dimensions:

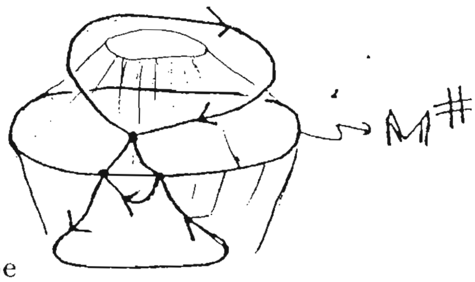


$M^\#$, $M^{(\pm)}$ have topology $S^3 \times S^1$, I denotes the point at infinity. There is only *one* region CM^0.

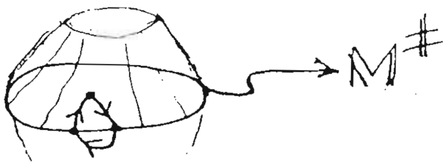
Up to an overall sign, there is another orientation of Γ_J – obtained by just flipping the orientation of one of its constituents – this would simply reproduce any field of mixed frequency.

Note that both $M^{(\pm)}$ are contractible – they sit inside $CM^\pm$ which are contractible manifolds – however the positive and negative frequency parts of φ extend only to an open *neighbourhood* of $M^\#$ thus preventing contraction of the contours.

Note that the contour must pinch at a point of $M^\#$ – for example for the reproducing contour if we instead chose



the extra contribution would be



which is divergent since this encloses a dense set of singularities of the integrand.

The Complex Structure on Arbitrary Spacelike 3-Surfaces

It is essential in the consideration of scalar products for zero rest mass fields to understand how the results obtained above enable one to perform J via an integral over an *arbitrary* spacelike 3-surface. In essence, we wish to understand how to obtain a 3-surface integral for J from the 4-dimensional contour integral above, by insertion, holomorphically, of a delta function in imaginary time. We have the following result.

THEOREM:

The action of the complex structure J on any spacetime field or potential (with any index structure) φ satisfying $\nabla^2\varphi = 0$ with the decay $\varphi \sim r^{-(1+\gamma)}$, $\gamma > 0$ at spatial infinity, is given by the 3-surface integral

$$J[\varphi] = \frac{1}{2\pi^2}\int_{\Sigma^0} \frac{1}{K^2(x,x')}(\overleftarrow{\nabla}'^a - \overrightarrow{\nabla}'^a)\varphi(x')d^3\Sigma'_a \qquad (*)$$

where $K^2 = (x^a - x'^a)^2$, Σ_a is future pointing, and Σ^0 is once differentiable and constrained to intersect the light cone of x at its vertex.

Remark on time symmetry. K^2 determines no time orientation. However both sides of the above repolarisation formula reverse sign under time reflection – the positive and negative frequency parts are interchanged and the normal to the surface of integration changes sign, i.e. the repolarisation formula is *consistent* under T, time reflection symmetry.
Remarks on K. If $\delta x = x - x'$ then $K^2 = 0$ if and only if δx has real and imaginary parts which are spacelike, null, or zero, and are orthogonal. Therefore $1/K^2$ is non-singular if $\delta x \in CM^\pm$. K is symmetric in x, x'.
Note that our formula is manifestly Lorentz covariant.

Proof: We rely on the results above for 4-dimensional integrals in an essential way. Consider the general expression

$$\omega(f,g) = \int_\Sigma f(\overleftarrow{\nabla^a} - \overrightarrow{\nabla^a}) g \, d^3\Sigma_a.$$

This is a symplectic functional of f, g and the integrand is divergence free provided $\nabla^2 f = \nabla^2 g = 0$. We consider the case $f = 1/K^2$, $g = \varphi$. Note that $\nabla^2 f \equiv 0$ in M even where f itself is singular (see for example Schwinger's papers on quantum electrodynamics c. 1949). However as we shall see the 2-point kernel and the field do *not* play entirely symmetric roles in an important way.

Now we complexify the r.h.s. of $(*)$ above, i.e. allow $x \mapsto z = x - iy$ for real x, y. Then the *complex* exterior derivative $d = \partial + \bar\partial$ of the form in $(*)$ vanishes since the integrand is independent of $\bar z$, and is now divergence free in z. Thus $(*)$ is independent of Σ^0 provided $K \neq 0$ and no singularities of φ are encountered. In real terms we have the freedom of deformation of a 3-dimensional contour in 8-dimensional space. Now define the linear functionals $I^\pm$:

$$I^\pm[\varphi] := \int_{\Sigma^\mp} \frac{1}{K^2} (\overleftarrow{\nabla'^a} - \overrightarrow{\nabla'^a}) \varphi(x') \, d^3\Sigma'_a$$

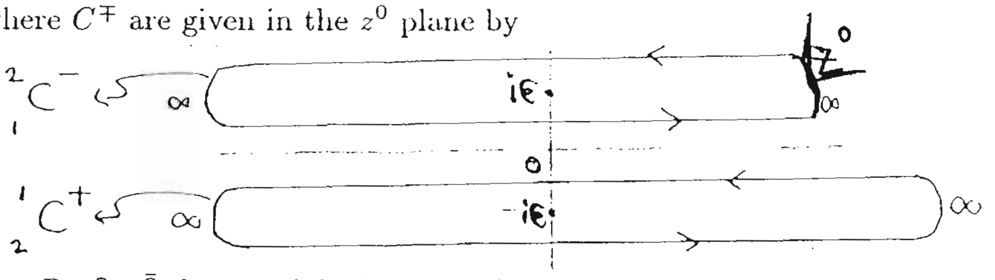
($d^3\Sigma_a = \epsilon_{abcd} dz^b \wedge dz^c \wedge dz^d$), where

$$\Sigma^\mp = \{z^a |\ z^0 = \pm i\epsilon,\ \epsilon > 0 \text{ is real and small; } \mathbf{z} = \mathbf{x} \text{ is real}\}.$$

Then we insert a delta function of imaginary time holomorphically:

$$I^\pm[\varphi] = \frac{1}{2\pi i} \int_{C^\mp} \frac{d^4 z}{z^0 \mp i\epsilon} \left[\frac{-2\varphi(z^a) z^0}{z^4} - \frac{1}{z^2} \cdot \frac{\partial \varphi(z^a)}{\partial z^0}\right]$$

where $C^\mp$ are given in the z^0 plane by



By $\partial + \bar\partial$ closure of the form in $(*)$ these expressions are all independent of $\epsilon > 0$. Note that no singularities are encountered since φ extends holomorphically in a neighbourhood of real spacetime, and $K^2 \neq 0$ since the contours remain in the future or past tubes. Now clearly $I^\pm[\varphi^\mp] = 0$ since by surface independence the contours can be taken to infinity in the past/future tubes where φ^- / φ^+ decay to zero.

To calculate $I^+[\varphi^+]$ note that in $\int_{C_1^-}$ the second term contributes zero by RL, and by RK the first term is proportional to $\frac{\varphi^+(z) z^0}{z^0 - i\epsilon}|_0 = 0$. Thus $\int_{C_1^-} = 0$. Then for $\int_{C_2^-}$ the first term is clearly continuous *at* $\epsilon = 0$, and then by RK is equal to $4\pi^3 \varphi^+(0)$. The second term is also continuous at $\epsilon = 0$ where it is equal to

$$-\oint \frac{1}{z_0} \cdot \frac{1}{z^2} \cdot \frac{\partial \varphi^+}{\partial z^0} d^4 z.$$

Now RL does not apply because of the pole at $z^0 = 0$, and this integral is

$$2\pi i \int \frac{1}{|\mathbf{x}|^2} \cdot \frac{\partial \varphi^+}{\partial z^0}(0, \mathbf{x})\, d^3\mathbf{x}.$$

Hence

$$I^+[\varphi^+] = -2i\pi^2 \varphi^+(0) + \int \frac{1}{|\mathbf{x}|^2} \cdot \frac{\partial \varphi^+}{\partial t}(0, \mathbf{x})\, d^3\mathbf{x}.$$

Similarly but taking note of the change in sign in the first term

$$I^-[\varphi^-] = 2i\pi^2 \varphi^-(0) + \int \frac{1}{|\mathbf{x}|^2} \cdot \frac{\partial \varphi^-}{\partial t}(0, \mathbf{x})\, d^3\mathbf{x}.$$

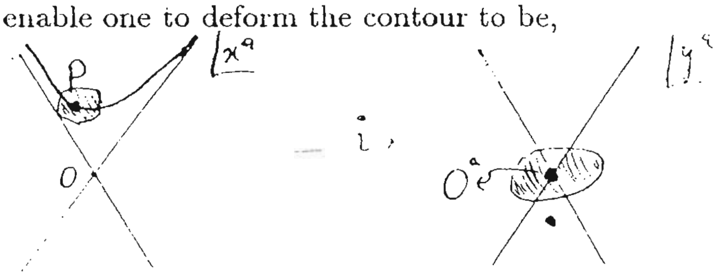
Adding these equations gives

$$I^+[\varphi] + I^-[\varphi] = -2i\pi^2[\varphi^+(0) - \varphi^-(0)] + \int \frac{1}{|\mathbf{x}|^2} \cdot \frac{\partial \varphi}{\partial t}(0, \mathbf{x})\, d^3\mathbf{x}.$$

PROPOSITION:

$I^{\pm}[\varphi] \equiv 0$ for all φ decomposable into $\varphi = \varphi^+ + \varphi^-$.

Proof: It suffices to show that $I^+[\varphi^+] = 0$. Surface independence and the decay of φ at spatial infinity enable one to deform the contour to be,



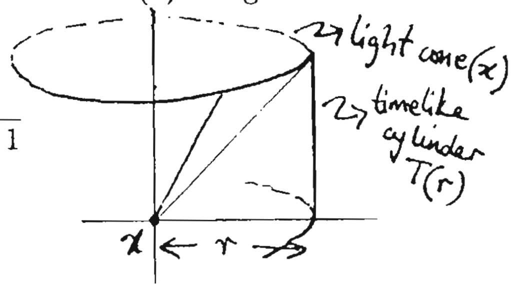
Then for the fixed real part P shown since P lies strictly in the *interior* of the future real light cone of x, the integrand is $\partial + \bar{\partial}$ closed for all z in an open neighbourhood of the union of $\overline{CM^+}$, $\overline{CM^-}$. In particular we have $\partial + \bar{\partial}$ closure at $y^a = 0$ since the integrand is holomorphic on any complex line through $P^a + i0^a$ in a sufficiently small neighbourhood. This enables us to deform Σ^- passing *through* real spacetime to eventually lie at infinity in the future tube where the integral is zero.

Hence,

$$J[\varphi(x)] = \frac{1}{2\pi^2} \int \frac{1}{|\mathbf{x} - \mathbf{y}|^2} \cdot \frac{\partial \varphi}{\partial t}(0, \mathbf{y})\, d^3\mathbf{y}.$$

Now in the case of a *flat* Σ^0 in $(*)$ the term from the derivative acting to the left contributes zero, since this creates a factor $(x - x')^a$ which is orthogonal to the measure of integration. Now by surface independence we claim our result is established. It suffices to verify that (a) the integration over the timelike cylinder at infinity vanishes and (b) that we may apply Stokes' theorem to obtain surface independence for a general spacelike 3-surface. As we shall see the latter in fact requires the surface to satisfy a differentiability condition.

To verify (a) if we set $\varphi \sim 1/r^{1+\gamma}$ then the second term of $(*)$ integrated over the timelike cylinder shown gives



$$r^{-(1+\gamma)} \int_0^1 \frac{du}{(u - i\epsilon/r)^2 - 1}$$

$$\sim r^{-(1+\gamma)}[\log r + A]$$

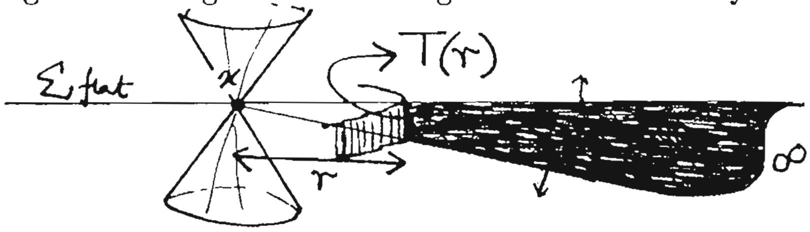
for some constant A. The first term contributes

$$r^{-(1+\gamma)} \int_0^1 \frac{du}{[(u - i\epsilon/r)^2 - 1]^2}$$

$$\sim -\frac{B}{r^{1+\gamma}} + \frac{1}{i\epsilon r^\gamma}$$

for some constant B. Thus a necessary and sufficient condition for the integral to vanish at infinity is $\gamma > 0$ as stated above.

To verify (b) it suffices to show that the integral over the small timelike surface $T(r)$ shown below tends to zero as the vertex of the light cone x is approached – then surface independence follows from Stokes' theorem applied to the shaded region since the form is closed throughout this region and non-singular on the boundary.



We take the general timelike displacement of the surface to vary with the spacelike distance r from the vertex according to kr^α (†) for real constants k, α such that the surface intersects the light cone at its vertex only, or equivalently $\alpha \geq 1$. Then since the field φ and its gradient are bounded and continuous at x the contribution from the second term of $(*)$ is

$$\sim \int_0^{kr^\alpha(<r)} \frac{r^2\,dt}{(t^2 - r^2)}$$

$$\propto r \log[\frac{1 - kr^{\alpha-1}}{1 + kr^{\alpha-1}}].$$

Now for all $\alpha \geq 1$ and k such that the surface is spacelike at x this contribution vanishes in the limit $r \to 0$. The contribution from the first term of $(*)$ is proportional to

$$r^3 \int_0^{kr^\alpha} \frac{dt}{(t^2 - r^2)^2}$$

$kr^\alpha < r$

$$\propto \tanh^{-1}(kr^{\alpha-1}) + \frac{kr^{\alpha-1}}{1 - k^2r^{2\alpha-2}}.$$

Now for all $\alpha > 1$ this contribution *vanishes* in the limit $r \to 0$. Only in the case $\alpha = 1$ are we left with the contribution

$$\propto \tanh^{-1} k + \frac{k}{1-k^2}$$

wherein $k < 1$ so that the surface is spacelike. This is the case of a conical 3-surface with vertex at the field point x. Thus if the surface is only once differentiable the contribution from $T(0^+)$ is zero since we can always bound this contribution above by that from our generic form (†) in the case $\alpha > 1$. This completes the proof of the theorem. I am grateful to N. Woodhouse for suggesting the existence of the extra contribution due to a surface with singular extrinsic curvature.

We have the trivial corollary in terms of the 3-dimensional Laplacian:

$$J[\varphi(\mathbf{x},t)] = -i\Delta^{-1/2}[\frac{\partial\varphi(\mathbf{x},t)}{\partial t}]$$

where $\Delta^{-1/2} f(\mathbf{x}) \equiv \pm \frac{i}{2\pi^2} \int \frac{f(\mathbf{y})}{|\mathbf{x}-\mathbf{y}|^2} d^3\mathbf{y}$. Thus we see that the action of J is *non-local* with respect to the Cauchy data $(\varphi, \partial\varphi/\partial n)$ and that in the case that Σ is flat is determined *alone* by the free data of the normal derivative. This gives a relation between elliptic operators on 3-spaces and restrictions to Euclidean spaces of hyperbolic operators on pseudo-Riemannian spaces.

Concluding remarks

These results may be used to derive expressions for positive definite norms of massless bosonic fields of arbitrary integer spin as two-point configuration space integrals, requiring *no extraction* of frequency parts or potentials for the principal fields. It is especially interesting in the case of linear gravity that one obtains a positive definite norm as an integral over the linearised phase space variables. This is well defined provided *only* that one can make a choice of foliation of real spacetime by a family of spacelike hypersurfaces – thus the vacuum Einstein equations may *fail* to hold. This gives in principal a way of measuring the difference of two neighbouring spacetime geometries even in the presence of matter. Details of this will appear in a subsequent publication.

I am grateful to L.P. Hughston for stimulating discussions.