A Minimum Principle for the Cohomological Inner Product on Twistor Space

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In this note we describe how, in the Dolbeault framework, one can obtain the $SU(2,2)$-invariant inner product on the cohomology groups $H^1(\tilde{\mathcal{P}}_\pm\,,\,\mathcal{O}(-2-n))$ ($\tilde{\mathcal{P}}_\pm$ a small nbh. of $\overline{\mathcal{P}}_\pm \subset \mathcal{P}T$), $n \geq 0$, starting from a pos. def. inner product on representative $(0,1)$-forms and evaluating it for representatives with *minimal norm.* In the standard approach one puts a $SU(4)$-invariant Fubini-Study metric g on $\mathcal{P}T$ which gives a pos. def. $SU(4) \cap SU(2,2)$-invariant inner product on forms [R. et al.], [S]. But the fact that full $SU(2,2)$-invariance is recovered by going to representatives with minimal g-norm might have been overlooked.

It follows, at least for $n > 0$, from the general theory [FK] (as referred to in [S]) that the minima are attained by unique g-harmonic representatives satisfying $\bar{\partial}$-Neumann boundary conditions (3.2). Thus, one can just *calculate* to see that these minima agree with the $SU(2,2)$-invariant cohomological norm (for sufficiently many states (3.8)). We shall perform such a calculation (3.6), (3.12) for the case $n = 0$ where in fact one integrates over $\mathcal{P}_0$ generalising (1.1).

1. The case of $SU(1,1)$ as an Introduction: We consider the analogue of $H = \mathcal{L}^2(S^1)$ for the hypersurface $\mathcal{P}_0$ in $\mathcal{P}T$. The Hermitian inner product

$$\langle f \,|\, g\rangle = \frac{1}{2\pi i}\int_{|z|=1} \overline{f(z)}g(z)\frac{dz}{z} \quad , \quad z \in \mathbb{C}\,, \tag{1.1}$$

on H is positive definite and induces an orthogonal splitting $H = H_+ \oplus H_-$ into functions extending holomorphically over the upper (S_+) and lower (S_-) hemisphere of $\mathbb{C}P^1 \sim S^2$. It is furthermore invariant under the action

$$(g * f)(z) = \frac{1}{\bar{b}z + \bar{a}} f\Big(\frac{az+b}{\bar{b}z+\bar{a}}\Big) \quad , \quad g^{-1} = \begin{pmatrix} a & b \\ \bar{b} & \bar{a}\end{pmatrix} \in G = SU(1,1)\,. \tag{1.2}$$

This is best seen by thinking of f as a section of $\mathcal{O}(-1)$ over S_+ or S_- , represented by a function of homogeneity -1 in the homogeneous coordinates $[z_0\,,\,z_1]$, e.g.

$$f_a = 1/(a_0 z^0 + a_1 z^1) \quad , \quad a = [a_0\,,\,a_1] \in S_\pm \quad . \tag{1.3}$$

The action of G is then induced by the natural action on points $a \in S^2 \backslash S^1$ and the inner product is

$$\langle f_a \,|\, g_b\rangle = \pm(\bar{a}_0 b_0 - \bar{a}_1 b_1)^{-1} \text{ or zero} \tag{1.4}$$

depending on whether $a, b \in S_+$ or S_- . The sign is such that the expression is positive for $a = b$. In the same way one gets realisations on $\mathcal{O}(-1-n)$ for all the (holomorphic) discrete series representations of $SU(1,1)$ ($n \in \mathbb{N}$), an invariant inner product for $n > 0$ being [BE]

$$\langle f \,|\, g\rangle_n = \frac{1}{2\pi i}\int_{|z|<1} (1 - z\bar{z})^{n-1}\overline{f(z)}g(z)\, d\bar{z} \wedge dz \quad , \quad z \in \mathbb{C}\,. \tag{1.5}$$

All of this has analogues for projective (flat) twistor space $\mathcal{PT} \sim \mathbb{C}P^3$ although one has to work with cohomology groups. S^1 is replaced by the five-dimensional real hypersurface

$$\mathcal{P}_0 = \{ [z] \,|\, h(z,z) = 0 \} = (SU(2) \times SU(2))/U(1) \quad , \tag{1.6}$$

$$\text{where } h(z,z) = z^0\overline{z}^0 + z^1\overline{z}^1 - z^2\overline{z}^2 - z^3\overline{z}^3 \quad , \tag{1.7}$$

which divides $\mathcal{PT}$ into $\mathcal{P}_+$ and $\mathcal{P}_-$. We have "elementary states" (sections of $\mathcal{O}(-2)$)

$$f_{ab} = \frac{1}{(a_i z^i)(b_i z^i)} \quad , \quad a \wedge b \in \mathcal{P}_\pm \tag{1.8}$$

which generate 1-cocycles in Čech cohomology via the connecting homomorphism in the Mayer-Vietoris long exact sequence on a cover with two Stein sets [EH]. The positive definite $SU(2,2)$-invariant inner product on these is

$$\langle [f_{ab}] \,|\, [f_{cd}] \rangle_{h,-2} = (h(a,c)h(b,d) - h(a,d)h(b,c))^{-1} \text{ or zero,} \tag{1.9}$$

depending on the respective position of the lines $a \wedge b$, $c \wedge d$.

2. **(0,1)-forms on $\mathcal{P}_0$:** In a fixed basis where h has the form (1.7) we fix

$$g(z,z) = z^0\overline{z}^0 + z^1\overline{z}^1 + z^2\overline{z}^2 + z^3\overline{z}^3 \ . \tag{2.1}$$

Simultaneously, g denotes the Fubini-Study metric induced on $\mathcal{PT}$. We use affine coordinates

$$(\zeta, \eta, \xi) = (z^1/z^0 \, , \, z^2/z^0 \, , \, z^3/z^0) \tag{2.2}$$

on a dense open cell $\mathbb{C}P^3 \backslash \mathbb{C}P^2$. A g-normal to $\mathcal{P}_0$ of constant g-length is given by

$$n = \eta\partial_\eta + \xi\partial_\xi + \overline{\eta}\partial_{\overline{\eta}} + \overline{\xi}\partial_{\overline{\xi}} \ . \tag{2.3}$$

We then naturally define (0,1)-forms ω on $\mathcal{P}_0$ to be those (C^∞) 1-forms which vanish on holomorphic tangent vectors. If such a (0,1)-form ω is the restriction of a $\overline{\partial}$-*closed* form on a nbh. $\mathcal{U}$ of $\mathcal{P}_0$ then it satisfies the boundary CR-equations

$$\overline{\partial}_b\omega = 0 \iff v \lrcorner d\omega = 0 \ \forall v \in T^{0,1}\mathcal{PT} \cap T_{\mathbb{C}}\mathcal{P}_0 \tag{2.4}$$

where d is the exterior derivative on $\mathcal{P}_0$. Now we can complete to a Hilbert space of (0,1)-forms on $\mathcal{P}_0$ with the inner product

$$\langle \omega \,|\, \eta \rangle_g = \int_{\mathcal{P}_0} \eta \wedge *_{\tilde{g}}(\omega) \tag{2.5}$$

where $*_{\tilde{g}}$ is the Hodge star with respect to the metric $\tilde{g}$ induced from g on $\mathcal{P}_0$. This is obviously a positive definite inner product on arbitrary (square-integrable) 1-forms on $\mathcal{P}_0$, invariant under the transformations in $SU(2,2)_h \cap SU(4)_g$. It is however *not* invariant under general transformations in $SU(2,2)$. ¿From (1.1) it is clear that one should try to construct a fully invariant — on the level of cohomology — inner product from the "boundary values" of elements in the cohomology groups $H^1(\mathcal{P}_\pm \, , \, \mathcal{O}(-2))$ of $\mathcal{P}_\pm$. It is surprising that this can be achieved with a formula (2.5), which depends upon the choice of g, by picking unique g-harmonic representatives satisfying the $\bar{\partial}$-Neumann conditions on the boundary $\mathcal{P}_0$.

3. The $SU(2,2)$-invariance on normalised representatives : An orthogonal basis of the Hilbert space completion of $H^1(\widetilde{\mathcal{P}}_+, \mathcal{O}(-2))$ with respect to the cohomological inner product is given by the classes generated by the following g-harmonic $(0,1)$-forms [EP],[S], written in homogeneous coordinates

$$\omega_{n,ij} = a_{n,ij}(\bar{z}_0 d\bar{z}_1 - \bar{z}_1 d\bar{z}_0) \quad , \quad a_{n,ij} = \frac{\bar{z}_0^{n-i}\bar{z}_1^{i}\, z_2^{n-j}\, z_3^{j}}{(z_0\bar{z}_0 + z_1\bar{z}_1)^{2+n}} \quad , \quad n \in \mathbb{N}\, ;\, i,j = 0,\ldots,n\ . \tag{3.1}$$

Furthermore, they satisfy the $\bar{\partial}$-Neumann boundary conditions [FK]

$$\sigma(\bar{\partial}_g^*, dr)\omega = -\pi_{0,1}(n)\lrcorner\omega = -n\lrcorner\omega = 0 \quad \text{at } \mathcal{P}_0 \tag{3.2}$$

where σ is the symbol of an operator and $\pi_{0,1}$ is the antiholomorphic projection of a vector field. Notice that the $\omega_{n,ij}$ are also h-harmonic in the sense

$$\bar{\partial}\omega_{n,ij} = \bar{\partial}_h^*\omega_{n,ij} = 0 \quad \forall n,i,j \tag{3.3}$$

but the associated Laplacian is not elliptic. If we are presented with $(0,1)$-forms ω, η on some nbh. $\mathcal{U}$ of $\mathcal{P}_0$ which extend $\tilde{\omega}$, $\tilde{\eta}$ and one of which satisfies (3.2) then (2.5) simplifies to

$$\langle\tilde{\omega}\,|\,\tilde{\eta}\rangle_g = \int_{\mathcal{P}_0} g^*(\omega\,,\,\eta)\, n\lrcorner\Omega_g \tag{3.4}$$

where g^* is the dual of g and Ω_g is its determinant. (This is probably the "better" definition anyway although seemingly less intrinsic.) We calculate

$$\langle\omega_{n,ij}\,|\,\omega_{m,kl}\rangle_{g,-2} = \int_{\mathcal{P}_0} \frac{\bar{a}_{n,ij}a_{m,kl}(z^0\bar{z}^0 + z^1\bar{z}^1)\, g(z,z)}{g^{-2}(z,z)}\, n\lrcorner\Omega_g \quad \text{with}$$

$$n\lrcorner\Omega_g = n\lrcorner(\frac{1}{2i})^3 \frac{d\zeta\wedge d\bar{\zeta}\wedge d\eta\wedge d\bar{\eta}\wedge d\xi\wedge d\bar{\xi}}{(1+\zeta\bar{\zeta}+\eta\bar{\eta}+\xi\bar{\xi})^4} = \frac{2}{(2i)^3}\frac{(\eta\bar{\eta}+\xi\bar{\xi})d\zeta\wedge d\bar{\zeta}\wedge d\eta\wedge d\bar{\eta}\wedge d\xi/\xi}{(1+\zeta\bar{\zeta}+\eta\bar{\eta}+\xi\bar{\xi})^4}\ , \tag{3.5}$$

and thus

$$\begin{aligned}\langle\omega_{n,ij}\,|\,\omega_{m,kl}\rangle_{g,-2} &= \frac{1}{(2i)^3}\int_{\mathcal{P}_0} \frac{\zeta^i\bar{\zeta}^k\bar{\eta}^{n-j}\eta^{m-l}\bar{\xi}^j\xi^l d\zeta\wedge d\bar{\zeta}\wedge d\eta\wedge d\bar{\eta}\wedge d\xi/\xi}{(1+\zeta\bar{\zeta})^{3+m+n}} \\ &= \pi^3\delta_{nm}\delta_{ik}\delta_{jl}\Big((1+n)^2\binom{n}{i}\binom{n}{j}\Big)^{-1}\ .\end{aligned} \tag{3.6}$$

The space generated by the g-harmonics $\{\omega_{n,ij}\}$ is not invariant under $SU(2,2)$. Rather, we obtain an basis of an $sl_{\mathbb{C}}(4)$-module, orthogonal in cohomology, from the lowest weight vector $\omega_0 := \omega_{0,00}$ by the application of step operators

$$X_+^i\, Y_+^j\, E_+^n\,\omega_0 \quad , \quad n \in \mathbb{N}\, ;\, i,j = 0,\ldots,n \tag{3.7}$$

where X_+, Y_+ are the step operators in $su_{\mathbb{C}}(2)\otimes\mathbb{I}$ and $\mathbb{I}\otimes su_{\mathbb{C}}(2)$ complemented by E_+ (specified below) to generate the whole action of $sl_{\mathbb{C}}(4)$. Since $\langle\ |\ \rangle_g$ is invariant under $SU(2)\times SU(2)$ we only have to compare

$$\langle[E_+^n\,\omega_0]\,|\,[E_+^n\,\omega_0]\rangle_h \leftrightarrow \langle\omega_{n,00}\,|\,\omega_{n,00}\rangle_g \tag{3.8}$$

in order to ascertain the full invariance of $\langle\ |\ \rangle_g$ on normalised representatives, because we claim

Lemma: $E_+^n\,\omega_0$ and $c_n\,\omega_{n,00}$ are *cohomologous* for the right choice of $c_n\in\mathbb{Z}$ and $E_+\in sl_{\mathbb{C}}(4)$. For the $\bar\partial$-closed (0,1)-forms

$$\omega_{ab} := \frac{(a_i\hat z^i)(b_i d\hat z^i)-(b_i\hat z^i)(a_i d\hat z^i)}{[(a_i z^i)(b_i\hat z^i)-(b_i z^i)(a_i\hat z^i)]^2} \tag{3.9}$$

which correspond to the Čech representatives f_{ab} of (1.8) one obtains the cohomological inner product as in (1.9). Here $z\to\hat z$ is the map

$$(z^0,\, z^1,\, z^2,\, z^3)\to(\hat z^0,\,\hat z^1,\,\hat z^2,\,\hat z^3)=(-\bar z^1,\,\bar z^0,\,-\bar z^3,\,\bar z^2) \tag{3.10}$$

which is invariant under $SU(2)\times SU(2)$ and can be viewed as the antipodal map on the fibres of a fibration [A] $S^2\longrightarrow\mathcal{PT}\longrightarrow S^4$ which restricts to a fibration $S^2\to\mathcal{P}_0\to S^3$ of $\mathcal{P}_0$. We choose $E_+=\partial_{a_2}(.)|(a_i=\delta_i^0)$ such that

$$E_+^n\,\omega_0=(c_i\partial_{a_i})^n\,\omega_{ab}\big|_{a_i\,=\,\delta_i^0\,,\ b_i\,=\,\delta_i^1\,,\ c_i\,=\,\delta_i^2\,,\ d_i\,=\,\delta_i^3}\ , \tag{3.11}$$

(similarly, we could choose $X_+=b_i\partial_{a_i}(.)|_{\dots}$ and $Y_+=d_i\partial_{c_i}(.)|_{\dots}$). See [BE] §11.4 for a similar notation. From (1.9) one quickly computes

$$\langle[E_+^n\,\omega_0]\,|\,[E_+^n\,\omega_0]\rangle_h=\partial_{\bar a_2}^n\,\partial_{a_2}^n\,\langle[\omega_{ab}]\,|\,[\omega_{ab}]\rangle_h|_{\dots}=(n!)^2\ . \tag{3.12}$$

Proof of the Lemma: In affine coordinates one computes that on $\mathcal{O}(-2)$ one has

$$E_+=\mathcal{L}_v-2\eta \text{ where } v=\bar\xi\partial_{\bar\zeta}-\eta(\zeta\partial_\zeta+\eta\partial_\eta+\xi\partial_\xi) \tag{3.13}$$

and

$$(\mathcal{L}_v-2\eta)\frac{\eta^n\,d\bar\zeta}{(1+\zeta\bar\zeta)^{2+n}}=-(n+2)\frac{\eta^{n+1}\,d\bar\zeta}{(1+\zeta\bar\zeta)^{3+n}}+\bar\partial\{\frac{\eta^n\bar\xi}{(1+\zeta\bar\zeta)^{2+n}}\}\ . \tag{3.14}$$

Thus, by induction

$$[E_+^n\,\omega_0]=(-1)^n(n+1)![\omega_{n,00}]\ . \tag{3.15}$$

A comparison of (3.6) and (3.12) now yields

Theorem: Let $\langle\ |\ \rangle_h$ denote the positive definite inner product on the cohomology groups $H^1(\widetilde{\mathcal{P}}_\pm,\,\mathcal{O}(-2))$ invariant under $SU(2,2)=SU_h$ (for example given by linear extension of (1.9) but properly defined by the twistor transform [DE]), where h is an indefinite Hermitian form of signature $(+,+,-,-)$ on $\mathbb{C}^4$ which defines $\mathcal{P}_\pm$ and $\mathcal{P}_0$. Let g be the positive definite

Hermitian form obtained from h by a flip of signs (in some h-diagonal basis) with associated unitary group SU_g and Fubini-Study metric g_{-2} on $\mathcal{O}(-2)$ and define

$$\langle \omega \,|\, \eta \rangle_{g,-2} = \int_{\mathcal{P}_0} \eta \wedge \bar{*}_{\tilde{g}_{-2}}(\omega) \tag{3.16}$$

where $\tilde{g}_{-2}$ is the restriction of g_{-2} to $\mathcal{P}_0$ invariant under $SU_h \cap SU_g$ and ω, η are square integrable (0,1)-forms on $\mathcal{P}_0$. Let c_i $(i = 1, 2)$ be in the above cohomology groups. One has, for some g-independent constant $k \in \mathbb{R}_+$

$$\langle c_1 \,|\, c_2 \rangle_h = k \, \langle \omega_1^g \,|\, \omega_2^g \rangle_{g,-2} \tag{3.17}$$

where $\omega_i^g \in c_i$ are the (restrictions of the) unique g-harmonic representatives which satisfy the $\bar{\partial}$-Neumann boundary conditions. They are characterised by having minimal g-norm within their class. □

The last statement follows from the fact that the $\bar{\partial}$-Neumann conditions (3.2) on ω are equivalent to

$$\langle \omega \,|\, \bar{\partial}\phi \rangle_{g,-2} = 0 \quad \forall \phi \in \Gamma(\mathcal{U}(\mathcal{P}_0), \mathcal{O}(-2)) \ . \tag{3.18}$$

Remarks: We can consider the two spaces $H^1(\tilde{\mathcal{P}}_\pm, \mathcal{O}(-2))$ simultaneously and obtain an orthogonal splitting of "boundary values" as in the case of $SU(1,1)$. A weak form of (2.4) should be enough to characterise representatives in this total space.
We also remark that (3.6) makes it very easy to sum the g-harmonics to a (Szegö-type) kernel which is in fact different from the usual one [W] appearing in the twistor transform, the latter being a *g-unbounded* operator ! One would expect this kernel K to fit naturally into an integral representation (a so-called homotopy formula [HP]) of an arbitrary $(0,1)$-form on $\mathcal{P}_+$, say, viz.

$$\omega(z) = c\Big(\int_{\mathcal{P}_0} \omega(\zeta) \wedge K(\zeta, z) - \int_{\mathcal{P}_+} \bar{\partial}\omega(\zeta) \wedge K_1(\zeta, z)\Big) + \bar{\partial}_z \int_{\mathcal{P}_+} \omega(\zeta) \wedge K_2(\zeta, z) \Big) \ . \tag{3.19}$$

One can ponder the themes in this article from a physical point of view: Broken symmetry, normalisation, minimum principle, ...

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