# Twistor Newsletter

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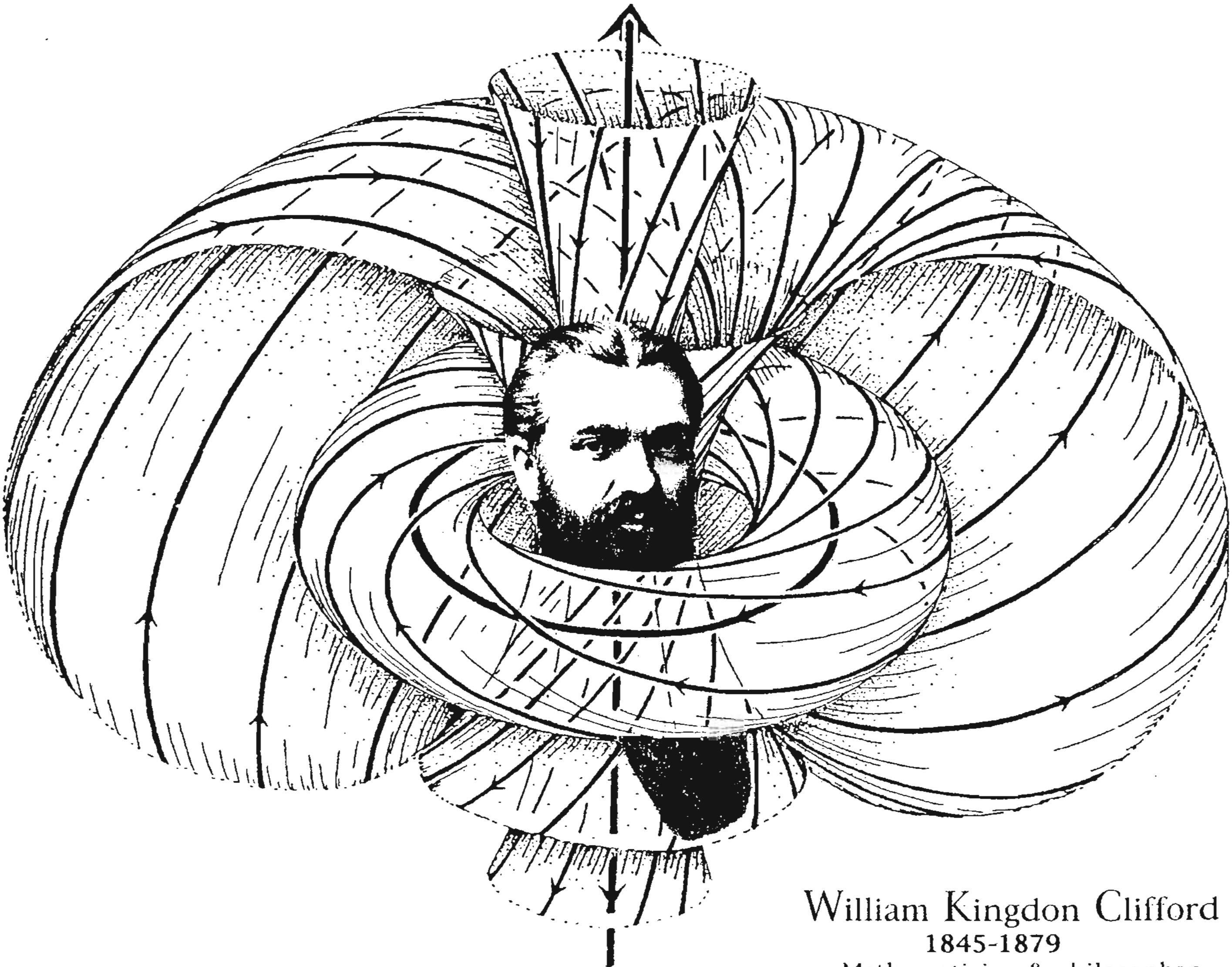

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# Concerning Space-Time Points for Spin 3/2 Twistor Spaces

In $\mathbb{T}$N 38 (pp. 1-9) (and see that article for earlier related references) I discussed various sheaves, exact sequences, etc. relevant to the spin 3/2 approach to the long-standing programme of defining the appropriate twistor space $\mathcal{T}$ for a general vacuum (Ricci-flat) space-time $\mathcal{M}$. Certain further ideas directed towards finding the twistors for $\mathcal{M}$ (i.e. points of $\mathcal{T}$) have arisen, but these remain inconclusive and will not be described here. Instead, I shall concern myself with a different question. Let us imagine that we have *found* the space $\mathcal{T}$. How would we go about re-constructing $\mathcal{M}$ from $\mathcal{T}$? In other words, what do we expect would be the definition of a *space-time point* (i.e. point of $\mathcal{M}$) in terms of $\mathcal{T}$?

Let us recall a way in which (complex) space-time points have been viewed in relation to the "googly" programme and non-linear graviton construction (R.P. in $\mathbb{T}$N 8, pp. 32-34). We first consider the standard (Poincaré-invariant) exact sequence for flat twistor space $\mathbb{T}$:

$$0 \rightarrow \mathbb{S}^A \xrightarrow{i} \mathbb{T}^\alpha \xrightarrow{p} \mathbb{S}_{A'} \rightarrow 0$$

where the second map ($i$) takes $\omega^A$ to $(\omega^A, 0)$, and the third ($p$) takes $(\omega^A, \pi_{A'})$ to $\pi_{A'}$. A point $q$ in $\mathbb{C}\mathbb{M}$ can be associated with either of the two maps $q, \hat{q}$ indicated by the dotted lines in

$$0 \rightarrow \mathbb{S}^A \underset{\hat{q}}{\overset{i}{\rightleftarrows}} \mathbb{T}^\alpha \underset{q}{\overset{p}{\rightleftarrows}} \mathbb{S}_{A'} \rightarrow 0$$

where $\hat{q}$ takes

$$(\omega^A, \pi_{A'}) \text{ to } \omega^A - iq^{AA'}\pi_{A'}$$

($q^a$ being the position vector of $q$), and the map $q$ takes

$$\pi_{A'} \quad \text{to} \quad (i q^{AA'} \pi_{A'}, \pi_{A'}).$$

The composition $p \circ q$ is the identity on $\mathbb{S}_{A'}$, and $\hat{q} \circ i$ is the identity on $\mathbb{S}^A$.

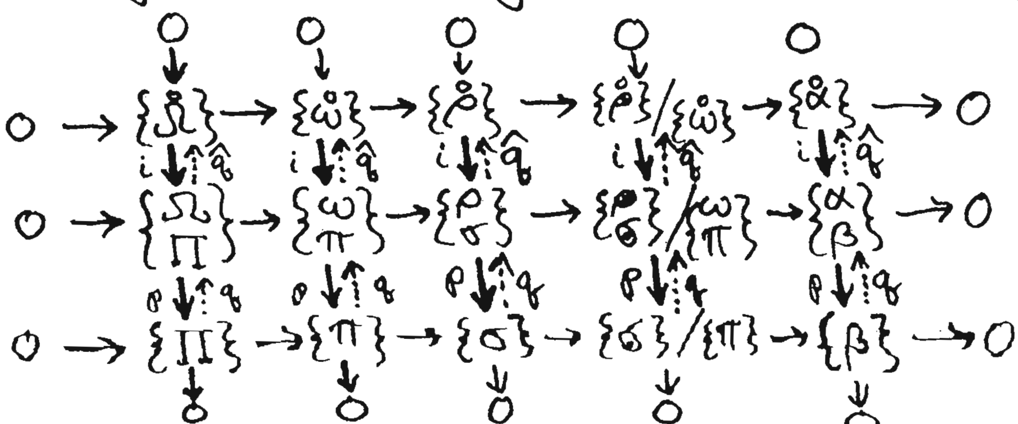
In the non-linear graviton construction for an anti-self-dual complex vacuum $\mathcal{M}$, the twistor space $\mathcal{T}$ still has a canonical projection $\mathcal{T} \xrightarrow{p} \mathbb{S}_{A'}$, and space-time points (cross-sections of $\mathcal{T}$ in its capacity as a fibration over $\mathbb{S}_{A'}$) can be identified with (holomorphic) maps $q : \mathbb{S}_{A'} \to \mathcal{T}$ for which $p \circ q$ is the identity on $\mathbb{S}_{A'}$

$$\mathcal{T} \underset{q}{\overset{p}{\rightleftarrows}} \mathbb{S}_{A'} \, .$$

(More correctly, $\mathbb{S}_{A'} - \{0\}$ should replace $\mathbb{S}_{A'}$ in the preceding assertion.) In the proposals for a "googly graviton" construction, the points of a self-dual complex vacuum $\mathcal{M}$ would be intended to arise correspondingly from the twistor space $\mathcal{T}$ in accordance with

$$\mathbb{S}^A \underset{q}{\overset{i}{\rightleftarrows}} \mathcal{T} \, .$$

It is of interest to explore the way in which space-time points might come about in relation to a "spin $3/2$ definition" of the twistor space $\mathcal{T}$ for a general complex vacuum $\mathcal{M}$. Let us recall the exact commutative diagram Ⓢ appearing on p. 7 of TN38, where I am also introducing additional maps $q, \hat{q}$ (to be explained shortly) in the columns (upwards)



In accordance with the notation of that article (TN 38, pp. 1-9), I have now used $(\Omega^A, \Pi_{A'})$, in place of $(\omega^A, \pi_{A'})$ as above, to denote <u>constant</u> twistors, therefore satisfying

$$\nabla_{AA'} \Omega_B = -i\varepsilon_{AB} \Pi_{A'}, \quad \nabla_{AA'} \Pi_{B'} = 0,$$

with, also,

$$\nabla_{AA'} \mathring{\Omega}_B = 0;$$

and the more general $(\omega^A, \pi_{A'})$ and $\mathring{\omega}^A$ now merely satisfy

$$\nabla_{A'}^{B} \omega_B = 2i\pi_{A'}, \quad \nabla_{B}^{A'} \pi_{A'} = 0$$

with

$$\nabla_{A'}^{B} \mathring{\omega}_B = 0.$$

The helicity 3/2 (Dirac-type) potential $\sigma_{A'B'C}$ (symmetric in $A'B'$), and the second potential $\rho_{A'BC}$ (symmetric in $BC$) satisfy

$$\nabla_{B'}^{B} \rho_{A'BC} = 2i\sigma_{A'B'C}, \quad \nabla_{B}^{B'} \sigma_{A'B'C} = 0$$

with

$$\nabla_{B'}^{B} \mathring{\rho}_{A'BC} = 0$$

(symmetric in $BC$) the gauge freedoms being

$$\rho_{A'BC} \mapsto \rho_{A'BC} - i\varepsilon_{BC}\pi_{A'} + \nabla_{CA'}\omega_B, \quad \sigma_{A'B'C} \mapsto \sigma_{A'B'C} + \nabla_{CB'}\pi_{A'}, \quad \mathring{\rho}_{A'BC} \mapsto \mathring{\rho}_{A'B'C} + \nabla_{CA'}\mathring{\omega}^B$$

The Frauendiener-Sparling-type quantities $\overset{\odot}{\rho}_{A'BC}$ and $\overset{\odot}{\sigma}_{A'B'C}$ (still symmetric in $BC$ and $A'B'$, respectively) generalize these equations to

$$\nabla_{(B'}^{B} \overset{\odot}{\rho}_{A')BC} = 2i\sigma_{A'B'C}, \quad \nabla_{(B}^{B'} \sigma_{C)A'B'} = 0, \quad \nabla_{(B'}^{B} \overset{\odot}{\rho}_{A')BC} = 0$$

(with the same gauge freedom as before) and the Frauendiener-type quantities $\alpha_A, \beta_{A'}, \mathring{\alpha}_A$ arise as

$$\alpha_C = \tfrac{1}{2}\nabla^{BB'} \overset{\odot}{\rho}_{B'BC}, \quad \beta_{A'} = \nabla^{BB'} \overset{\odot}{\sigma}_{A'B'B}, \quad \mathring{\alpha}_A = \tfrac{1}{2}\nabla^{BB} \mathring{\rho}_{B'BC}$$

and satisfy

$$\nabla_{A'}^{A} \alpha_A = 2i\beta_{A'}, \quad \nabla_{A}^{A'} \beta_{A'} = 0, \quad \nabla_{A'}^{A} \mathring{\alpha}_A = 0$$

All these relations hold consistently in Ricci-flat $\mathcal{M}$ — except for those involving $\Omega^A$, $\Pi_{A'}$, and $\mathring{\Omega}^A$, which require $\mathcal{M}$ to be flat.

How are we to define the maps $q$, $\hat{q}$ in each of the various columns? In the case of

$$0 \rightarrow \{\mathring{\Omega}^A\} \underset{\hat{q}}{\overset{i}{\rightleftarrows}} \left\{ \begin{matrix} \Omega^A \\ \Pi_{A'} \end{matrix} \right\} \underset{q}{\overset{p}{\rightleftarrows}} \{\Pi_{A'}\} \rightarrow 0,$$

these maps are just as defined earlier (for the flat case $\mathbb{T}^\alpha$), namely

$$\Pi_{A'} \overset{q}{\longmapsto} (iq^{AA'}\Pi_{A'}, \Pi_{A'}) \text{ and } (\Omega^A, \Pi_{A'}) \overset{\hat{q}}{\longmapsto} \Omega^A - iq^{AA'}\Pi_{A'}.$$

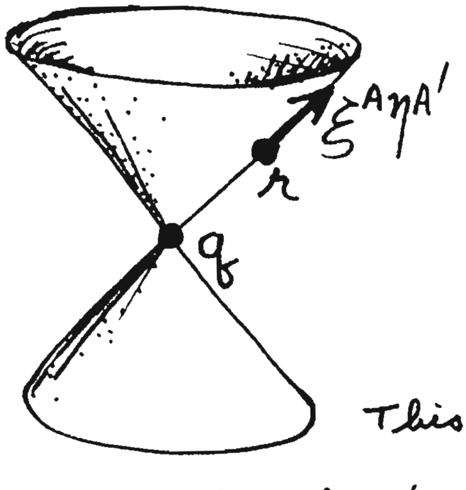
We need to generalize this to the remaining columns, and also in a way that makes sense in (Ricci-flat) $\mathcal{M}$. What this amounts to, in the case of the map $q$, is finding a way of <u>fixing</u> the "second potentials" $\omega_A$, $\rho_{A'BC}$, $\rho\!\!\!\circ_{A'BC}$ (mod $\omega_A$), and $\alpha_A$, in terms of the "first potentials" and a given point $q \in \mathcal{M}$. This is achieved by requiring that the appropriate <u>null-data</u> for these second-potential quantities are <u>zero</u> on the light cone of $q$. In the case of the map $\hat{q}$, we need a way of obtaining, as fixed by the point $q \in \mathcal{M}$, a definite "free-field" quantity $\mathring{\omega}_A$, $\mathring{\rho}_{A'BC}$, $\mathring{\rho\!\!\!\circ}_{A'BC}$ (mod $\mathring{\omega}_A$), and $\mathring{\alpha}_A$, given the respective "sourced" quantity $\omega_A$, $\rho_{A'BC}$, $\rho\!\!\!\circ_{A'BC}$ (mod $\omega_A$), and $\alpha_A$, with its **respective** "source" $\pi_{A'}$, $\sigma_{A'B'C}$, $\sigma\!\!\!\circ_{A'BC}$ (mod $\pi_{A'}$), and $\beta_{A'}$. This is achieved, in each case, by requiring that the null-data of the required "free fields" be equal to those of its corresponding "sourced fields" on the light cone of $q$.

It may be recalled (cf. Penrose & Rindler, Vol 1, §§5.11, 5.12; R.P. (1980) Gen. Rel. Grav. 12, 225–64) that the null-datum at a point $r$ on the data light cone, for a field $\theta_{A\ldots DP'\ldots S'}$ $(=\theta_{(A\ldots D)(P'\ldots S')})$, is the quantity

$$\theta = \xi^A \ldots \xi^D \eta^{P'} \ldots \eta^{S'} \theta_{A\ldots DP'\ldots S'}$$

where $\xi^A\eta^{A'}$ is a (complex) tangent vector at $r$ to the light cone. In order that the null-data for fields $\theta_{\ldots}, \ldots, \chi_{\ldots}$ determine, freely, the fields themselves, we require that these fields constitute an exact set (i.e. that their totally symmetrized $n^{\text{th}}$ derivatives $(n=0,1,2,\ldots)$ be independent and sufficient to determine all the unsymmetrized $m^{\text{th}}$ derivatives $(m=0,1,2,\ldots)$. In the present context, we find that the appropriate null-data are, respectively

$$\xi^A\omega_A, \quad \begin{pmatrix} \eta^{A'}\xi^B\xi^C\rho_{A'BC} \\ \xi^A\xi^B\xi^C\nabla^{A'}_A\rho_{A'BC} \end{pmatrix}, \quad \begin{pmatrix} \eta^{A'}\xi^B\xi^C\mathring{\rho}_{A'BC} \\ \xi^A\xi^B\xi^C\nabla^{A'}_A\mathring{\rho}_{A'BC} \\ \xi^C\nabla^{BA'}\mathring{\rho}_{A'BC} \end{pmatrix}, \quad \xi^A\alpha_A$$




except that $\eta^{A'}\xi^B\xi^C\mathring{\rho}_{A'BC}$ is mere "gauge", and so does not contribute to $\{\mathring{\rho}\}/\{\omega\}$

This enables all the $q$ and $\hat{q}$ maps to be defined.

Thanks to Robin Graham for a valuable discussion in which he brought up the issue addressed by this article.

~ Roger Penrose

# Spin 3/2 zero rest-mass fields in the Schwarzschild space-time

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## 1 Introduction

Helicity 3/2 field equations seem to play a crucial role in general relativity as well as in twistor theory. Indeed, if we consider the Dirac equation for the first potential of a helicity 3/2 massless field,

$$\nabla^{AA'}\sigma^{C}_{A'B'} = 0\ , \quad \sigma^{C}_{A'B'} = \sigma^{C}_{(A'B')}, \tag{1}$$

the vanishing of the Ricci curvature can be taken as a consistency condition for such an equation in curved space-time (see for example [1]). Such a close connection with Einstein's vacuum equations is quite remarquable. Moreover, twistors in flat space-time are interpreted as charges for spin 3/2 massless fields (see [3]) and it is therefore hoped that these fields in vacuum space-times might be used to define twistors.

The approach adopted here is purely analytical. The idea is to set up a technical basis that will (maybe) lead to a better understanding of the analytic or geometric obstacles to the definition of a "II-charge" in Ricci-flat space-times. We chose to study the case of the Schwarzschild metric

$$g_{\mu\nu}dx^{\mu}dx^{\nu} = Fdt^2 - F^{-1}dr^2 - r^2d\omega^2 \tag{2}$$

where $F = 1 - 1/r$, the Schwarzschild radius being here equal to 1.

**Notations:** Let $(M, g)$ be a Riemannian manifold, $\mathcal{C}_0^\infty(M)$ denotes the set of $\mathcal{C}^\infty$functions with compact support in $M$, $H^k(M,g)$, $k \in I\!N$ is the Sobolev space, completion of $\mathcal{C}_0^\infty(M)$ for the norm

$$\|f\|^2_{H^k(M)} = \sum_{j=0}^{k} \int_M \left\langle \nabla^j f, \nabla^j f \right\rangle d\mu,$$

where $\nabla^j$, $d\mu$ and $<,>$ are respectively the covariant derivatives, the measure of volume and the hermitian product associated with the metric $g$. We write $L^2(M,g) = H^0(M,g)$.

The 2-dimensional euclidian sphere $S^2_\omega$ is endowed with its usual metric

$$d\omega^2 = d\theta^2 + sin^2\theta d\varphi^2\ ,\ 0 \le \theta \le \pi\ ,\ 0 \le \varphi < 2\pi.$$

## 2 The global Cauchy problem

In this space-time, we consider the null tetrad $l^a, m^a, \overline{m}^a, n^a$, where

$$l^a\nabla_a = \frac{1}{\sqrt{2}}\left(F^{-1/2}\frac{\partial}{\partial t} + F^{1/2}\frac{\partial}{\partial r}\right), \tag{3}$$

$$n^a\nabla_a = \frac{1}{\sqrt{2}}\left(F^{-1/2}\frac{\partial}{\partial t} - F^{1/2}\frac{\partial}{\partial r}\right), \tag{4}$$

$$m^a\nabla_a = \frac{1}{r\sqrt{2}}\left(\frac{\partial}{\partial\theta} + \frac{i}{sin\theta}\frac{\partial}{\partial\varphi}\right). \tag{5}$$

It is chosen so that the "extent" of each vector is 1, in order not to emphasize the importance of any of the null directions with respect to the others. If we consider this tetrad as being associated with a spin-frame $o^A, \iota^A$, i.e.

$$l^a = o^A o^{A'} \ , \ n^a = \iota^A \iota^{A'} \ , \ m^a = o^A \iota^{A'} \tag{6}$$

we can calculate the Infeld-Van der Waerden symbols

$$g_{AA'}{}^a = \begin{pmatrix} l^a & m^a \\ \overline{m}^a & n^a \end{pmatrix} \ , \ g_a{}^{AA'} = \begin{pmatrix} n_a & -\overline{m}_a \\ -m_a & l_a \end{pmatrix} \tag{7}$$

and the non-zero spin-coefficients are

$$\rho = \mu = -\frac{F^{1/2}}{r\sqrt{2}} \ , \ \varepsilon = \gamma = \frac{F'F^{-1/2}}{4\sqrt{2}} \ , \ \beta = -\alpha = \frac{cotg\theta}{2r\sqrt{2}}. \tag{8}$$

We now have all we need to translate equation (1) in terms of partial derivatives in a coordinate basis. We obtain 8 scalar equations, two of which can be transformed into constraints involving only space-like partial derivatives. The system in its hamiltonian form can be written:

$$U = {}^t\left(\sigma^0_{0'0'} \ , \ \sigma^0_{0'1'} \ , \ \sigma^0_{1'1'} \ , \ \sigma^1_{0'0'} \ , \ \sigma^1_{0'1'} \ , \ \sigma^1_{1'1'}\right), \tag{9}$$

$$\frac{\partial U}{\partial t} = \frac{\partial}{\partial t}\begin{pmatrix} \sigma^0_{0'0'} \\ \sigma^0_{0'1'} \\ \sigma^0_{1'1'} \\ \sigma^1_{0'0'} \\ \sigma^1_{0'1'} \\ \sigma^1_{1'1'} \end{pmatrix} = \begin{pmatrix} h\sigma^0_{0'0'} + \frac{F^{1/2}}{r}\overline{L_1}\,\sigma^0_{0'1'} \\ -h\sigma^0_{0'1'} - \frac{F}{r}\left(\sigma^0_{0'1'} + \sigma^1_{0'0'}\right) + \frac{F^{1/2}}{r}L_3\,\sigma^0_{0'0'} \\ -h\sigma^0_{1'1'} - \frac{F'}{2}\sigma^0_{1'1'} - \frac{F}{r}\sigma^1_{0'1'} + \frac{F^{1/2}}{r}L_2\,\sigma^0_{0'1'} \\ h\sigma^1_{0'0'} + \frac{F'}{2}\sigma^1_{0'0'} + \frac{F}{r}\sigma^0_{0'1'} + \frac{F^{1/2}}{r}\overline{L_2}\,\sigma^1_{0'1'} \\ h\sigma^1_{0'1'} + \frac{F}{r}\left(\sigma^1_{0'1'} + \sigma^0_{1'1'}\right) + \frac{F^{1/2}}{r}\overline{L_3}\,\sigma^1_{1'1'} \\ -h\sigma^1_{1'1'} + \frac{F^{1/2}}{r}L_1\,\sigma^1_{0'1'} \end{pmatrix} = HU \tag{10}$$

where

$$h = F\left(\frac{\partial}{\partial r} + \frac{F'}{4F} + \frac{1}{r}\right), \tag{11}$$

$$L_k = \frac{\partial}{\partial\theta} + \left(k - \frac{3}{2}\right) cotg\theta + \frac{i}{sin\theta}\frac{\partial}{\partial\varphi} \quad , \quad k = 1,2,3, \tag{12}$$

$$\overline{L_k} = \frac{\partial}{\partial\theta} + \left(k - \frac{3}{2}\right) cotg\theta - \frac{i}{sin\theta}\frac{\partial}{\partial\varphi} \quad , \quad k = 1,2,3, \tag{13}$$

together with the two constraints

$$2h\sigma^0_{0'1'} + \left(\frac{2F}{r} - \frac{F'}{2}\right)\sigma^0_{0'1'} + \frac{F}{r}\sigma^1_{0'0'} - \frac{F^{1/2}}{r}\left(L_3\ \sigma^0_{0'0'} - \overline{L_2}\ \sigma^0_{1'1'}\right) = 0, \tag{14}$$

$$2h\sigma^1_{0'1'} + \left(\frac{2F}{r} - \frac{F'}{2}\right)\sigma^1_{0'1'} + \frac{F}{r}\sigma^0_{1'1'} - \frac{F^{1/2}}{r}\left(L_2\ \sigma^1_{0'0'} - \overline{L_3}\ \sigma^1_{1'1'}\right) = 0. \tag{15}$$

We introduce the Hilbert space $\mathcal{H}$ defined by

$$\mathcal{H} = \left\{L^2\left(]1,+\infty[_r \times S^2_\omega; F^{-1}dr^2 + r^2d\omega^2\right)\right\}^6 \tag{16}$$

and the successive domains of $H$ in $\mathcal{H}$

$$D(H^k) = \left\{U \in \mathcal{H};\quad H^jU \in \mathcal{H},\ 1 \leq j \leq k\right\},\quad k \in IN^*. \tag{17}$$

We also consider the spaces $\mathcal{H}_c$ and $D(H^k)_c$, $k \in IN^*$, of the elements of $\mathcal{H}$ and $D(H^k)$, $k \in IN^*$, which satisfy the constraint equations (14), (15); i.e. if we write (14) in the following way

$$AU = 0,\quad A = \left(-\frac{F^{1/2}}{r}L_3\ ,\ 2h + \frac{2F}{r} - \frac{F'}{r}\ ,\ \frac{F^{1/2}}{r}\overline{L_2}\ ,\ \frac{F}{r}\ ,\ 0\ ,\ 0\right), \tag{18}$$

and in the same manner (15) could become

$$BU = 0,\quad B = \left(0\ ,\ 0\ ,\ \frac{F}{r}\ ,\ -\frac{F^{1/2}}{r}L_2\ ,\ 2h + \frac{2F}{r} - \frac{F'}{r}\ ,\ \frac{F^{1/2}}{r}\overline{L_3}\right), \tag{19}$$

then, we have simply

$$\mathcal{H}_c = KerA \bigcap KerB \tag{20}$$

where $KerA$ is the kernel of $A$ in $\mathcal{H}$, and for $k \in IN^*$,

$$D(H^k)_c = (KerA)_{D(H^k)} \bigcap (KerB)_{D(H^k)} = KerA \bigcap KerB \bigcap D(H^k), \tag{21}$$

$(KerA)_{D(H^k)}$ being the kernel of $A$ in $D(H^k)$. In these spaces with constraints, the following existence and uniqueness result holds:

**Theorem 1** *For any initial data $U_0 \in \mathcal{H}_c$ (resp. $U_0 \in D(H^k)_c$, $k \in IN^*$), equation (10) admits a unique solution $U$ such that*

$$U \in C\left(IR_t; \mathcal{H}_c\right) \quad \left(resp.\ U \in C\left(IR_t; D(H^k)_c\right)\right) \tag{22}$$

*and*

$$U|_{t=0} = U_0. \tag{23}$$

*Note that if $U_0 \in D(H^k)_c$, $k \in I\!N^*$, the solution $U$ has the following additional regularities which are straightforward consequences of (10)*

$$U \in \bigcap_{j=0}^{k} C^j \left( I\!R_t ; D(H^{k-j})_c \right) . \tag{24}$$

*Moreover, the Rarita-Schwinger 3-form*

$$\beta = i \sigma_{aC'} \, \overline{\sigma}_{bC} \; dx^a \wedge dx^b \wedge dx^c , \tag{25}$$

*is divergence-free, $\sigma^A_{B'C'}$ denoting the spinor, symmetric in $B', C'$, whose components satisfy (10), (22) and (23). In other words, if we consider the explicit translation of (25)*

$$< \xi, \eta >_\beta = \left( \xi^0_{0'0'}, \eta^0_{0'0'} \right)_{L^2} + \left( \xi^1_{1'1'}, \eta^1_{1'1'} \right)_{L^2} + \left( \xi^1_{0'1'}, \eta^1_{0'1'} \right)_{L^2} + \left( \xi^0_{0'1'}, \eta^0_{0'1'} \right)_{L^2}$$

$$+ \left( \xi^1_{0'1'}, \eta^0_{1'1'} \right)_{L^2} + \left( \xi^0_{1'1'}, \eta^1_{0'1'} \right)_{L^2} + \left( \xi^1_{0'0'}, \eta^0_{0'1'} \right)_{L^2} + \left( \xi^0_{0'1'}, \eta^1_{0'0'} \right)_{L^2} , \quad \xi, \eta \in \mathcal{H} \tag{26}$$

*where $( \, , \, )_{L^2}$ denotes the standard scalar product on $L^2 \left( ]1, +\infty[_r \times S^2_\omega, F^{-1} dr^2 + r^2 d\omega^2 \right)$, then for any $U, V \in D(H)_c$*

$$< HU, V >_\beta = - < U, HV >_\beta \tag{27}$$

*and if $U \in C \left( I\!R_t ; \mathcal{H}_c \right)$ is a solution of (10), the quantity $< U, U >_\beta$ is conserved throughout time.*

*Hints of the proof:* Firstly, we decompose equations (10), (14) and (15) into spin-weighted spherical harmonics. On each sub-space of given angular dependence, we prove a global existence and uniqueness result for solutions without constraints. This is done using a fixed point method: the evolution system (10) is expressed in the form of an integral equation, the fixed points of which are the solutions of (10). The next step is to prove that if the initial data have a given angular dependence and satisfy the constraints (14) and (15), then the solution associated with this initial data satisfies the constraints at each time $t$. To prove this, we show that the spaces with constraints are stable under $H$ from which we infer their stability under the one parameter continuous group generated by $H$. At this point, we have proved the theorem for initial data with a fixed angular dependence. By linearity, the same result holds for initial data involving only a finite number of harmonics, i.e. belonging to a dense sub-space of $\mathcal{H}_c$ (or $D(H^k)_c$, $k \geq 1$). The last step is to extend the propagator to the entire space using an energy estimate. For more details about these analytical methods, see for example [2].

## 3 Where do we go from here?

Thus, the Cauchy problem is well-posed for a spin 3/2 potential $\sigma^C_{A'B'}$ in Schwarzschild's space-time and of course, it is also true for a pure gauge field which is simply given by a solution of Weyl's neutrino equation. In particular, a non global $\sigma$ can be propagated into its domain of dependence and the same holds for the gauge. Therefore, at least in the case of the Schwarzschild metric, we now have some information about the nature of the obstacle to the construction of a $\Pi$-charge. It was known that such an obstacle had to exists but it was not clear wether it resided in the propagation or in the patching of the potential modulo gauge (see [3]). The results obtained here seem to imply that there is nothing pathological about the propagation. Hence, the obstacle is more probably of a topological, rather than analytic, nature. More precisely, there might not exist at each time a proper covering of $S^2$.

Several directions of research can now be followed. Firstly, the same kind of study can be carried out in other Ricci-flat space-times. This would tell us wether the result obtained in Schwarzschild's space-time is an exception or if the propagator of the potential modulo gauge really is a "reliable" 1-parameter group in all vacuum space-times. Of course, one would also like to understand what features of the geometry cause the obstacle, be it topological or analytical, to arise. Studying the limit of a Schwarzschild black-hole when the mass goes to zero could shed some light on this conundrum. ¿From the viewpoint of analysis, it would be interesting to push further the study of spin 3/2 fields in black-hole backgrounds. An open problem is the construction of a time-dependent scattering theory in the Schwarzschild case. The technical difficulties are numerous, but it seems a fairly natural conjecture that the asymptotic behavior of linear spin 3/2 fields in the neighbourhood of horizons or in asymptotically flat regions can be described in terms of classical wave operators. Work is in progress.

## Acknowlegements

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# Two examples of classical scattering off fixed sources

By
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2nd March, 1995.

## 1 Introduction

Penrose, in Penrose & MacCallum (1972, §3.2), describes the scattering of zero rest-mass particles by impulsive electromagnetic and gravitational waves by constructing hamiltonian equations on twistor space. The fixed source problem was also addressed in Penrose & MacCallum (1972, §5.2) and formal expressions for the hamiltonians, obtained by use of the twistor transform, were given. The approach to scattering off fixed sources described here differs in two ways from that given in Penrose & MacCallum. Firstly, we consider the scattering of zero rest-mass particles off fields generated by zero rest-mass particles, whereas Penrose & MacCallum considered zero rest-mass particles scattering off a massive fixed source. Secondly, because the fields produced by the zero rest-mass particles are impulsive waves, we can utilize the explicit 'scissors and paste' methods used in Penrose (1968) and Penrose & MacCallum (1972, §3.2). This avoids the use of the twistor transform. The fields off which the zero rest-mass particles scatter are treated as fixed although the sources are moving with the speed of light. In contrast to scattering off a fixed Coulomb field (with timelike source), the whole description is manifestly Lorentz covariant.

We construct the hamiltonian equations by applying the 'scissors and paste' method, in which the zero rest-mass particle undergoing scattering is described by a null twistor, to find a solution to the Lorentz force equation and to construct a space-time consisting of two regions of Minkowski space, $\mathcal{M}$, separated by an impulsive gravitational wave. The scattering is thus described in terms of the unfolding of a canonical transformation on the symplectic manifold of null twistors.

In the following sections we utilize the results of Bonnor (1969a,b) to construct fixed backgrounds generated by null fluids. These solutions to the Maxwell and Einstein equations have the form of plane-fronted waves which, upon dropping differentiability requirements, may be impulsive (i.e. with $\delta$-function amplitude).

## 2 A twistor hamiltonian approach to scattering off fixed sources

### 2.1 *The Lorentz force problem*

Bonnor (1969a) showed that one can obtain plane wave solutions of Maxwell's equations where the source was taken to be a charge moving with the speed of light.

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Introduce a spin-frame $\alpha^A$, $\beta^A$, with normalization $\alpha_A\beta^A = 1$, and the associated null tetrad $l^a = \alpha^A\overline{\alpha}^{A'}, n^a = \beta^A\overline{\beta}^{A'}, m^a = \alpha^A\overline{\beta}^{A'}, \overline{m}^a = \beta^A\overline{\alpha}^{A'}$. A position vector takes the form $x^a = vl^a + un^a + \zeta m^a + \overline{\zeta}\overline{m}^a$.

The field is generated by the vector potential

$$\Phi^a = (2\varphi, 0, 0, 0) , \tag{1}$$

where Bonnor takes the function $\varphi$ as $C^1$ and piecewise $C^2$. This condition ensures the continuity of the Maxwell field $F^{ab}$ and precludes surface charges and surface currents. (Later we will drop this requirement for the sake of considering fields with $\delta$-function amplitude.) The function $\varphi$ takes the form $\varphi = \varphi(u, \zeta, \overline{\zeta})$. The components of the Maxwell 2-form, $F_{ab}$, are given by $F_{ab} = \nabla_{[a}\Phi_{b]}$. The components of the 4-current $J^a = (4\pi)^{-1}\nabla_b F^{ab}$ are

$$J^a = (2\rho, 0, 0, 0) , \tag{2}$$

which implies $J^a \propto l^a$ and $4\pi\rho = -\nabla^2\varphi$. Thus in a charge-free region where $J^a = 0$, $\varphi$ satisfies

$$\frac{\partial^2\varphi}{\partial\zeta\partial\overline{\zeta}} = 0 . \tag{3}$$

The electromagnetic field is *null*, that is, it satisfies $F^{ab}F_{ab} = 0$ and $\varepsilon^{abcd}F_{ab}F_{cd} = 0$.

We now consider a specific choice for the function $\varphi$. For $r = (\zeta\overline{\zeta})^{\frac{1}{2}} \geq a$, $a \in \mathbb{R}$, a positive constant, take

$$\varphi = \delta(u)[2\log(r/a) + 1] .$$

(Bonnor chooses $\varphi = \psi(u)[2\log(r/a) + 1]$ for $r \geq a$, and $\varphi = \psi(u)r^2/a^2$, for $r \leq a$, with $\psi \in C^2$. So we are dropping the differentiability requirements, and we have no need of the solution for $r \leq a$. We note that Bonnor shows that such a solution can be obtained from an advanced potential, but the total energy of the field due to a single charge moving with the speed of light, diverges. When charges of both signs are present (so that the total charge is zero), he shows that the field has finite energy.)

We find that the electromagnetic spinor takes the form

$$\phi_{11} = \phi_{AB}\beta^A\beta^B = \delta(u)\frac{2a}{\overline{\zeta}} , \tag{4}$$

and remark

$$\phi_{AB} \propto \alpha_A\alpha_B .$$

All other components of the anti-self-dual part of the field vanish.

We wish to solve the Lorentz force equation in this background. The Lorentz force equation may be written in the form $P_b\nabla^b P^a = eF^{ab}P_b$, which gives well-defined equations of motion for null momenta $P_a = \overline{\pi}_A\pi_{A'}$. This equation reduces to

$$\overline{\pi}_B\pi_{B'}\nabla^{BB'}\overline{\pi}^A = e\phi^{AB}\overline{\pi}_B . \tag{5}$$

In order to solve (5) in the background (4) we make the ansatz

$$\overline{\pi}^A = \overline{\pi}^A_{\circ} + \Theta(u)[\overline{\pi}^A] . \tag{6}$$

Here, $\overline{\pi}_o^A$ is the initial momentum, $[\overline{\pi}_A]$ is the change in momentum and $\Theta(u)$ is the Heaviside step function. So we find upon substituting (6) into (5), and using (4),

$$[\overline{\pi}^A] = -\frac{2a\, e\, \alpha^A}{\zeta\, \pi_{oB'}\, \overline{\alpha}^{B'}}\,, \tag{7}$$

where we have inserted the initial momentum into $\overline{\pi}_A \pi_{A'} \nabla^{AA'}$. From here on we shall drop the constant $2a$.

We follow Penrose & MacCallum (1972, §3.2) in finding a twistor hamiltonian formulation of (7). The hamiltonian takes the form $\mathcal{H}(Z^\alpha, \overline{Z}_\alpha) = \mathcal{H}^+(Z^\alpha) + \mathcal{H}^-(\overline{Z}_\alpha)$, where

$$\mathcal{H}^+(Z^\alpha) = \overline{\mathcal{H}^-(\overline{Z}_\alpha)} = e\int^{\overline{\zeta}} f(x)\,\mathrm{d}x \tag{8}$$

and $f(x) = x^{-1}$. On the surface $u = 0$, the coordinate $\overline{\zeta}$ is given by

$$i\overline{\zeta} = \frac{\alpha_A \omega^A}{\overline{\alpha}^{A'} \pi_{A'}}\,. \tag{9}$$

Then

$$\delta\overline{\pi}_A = i\frac{\partial \mathcal{H}^+}{\partial \omega^A} = \frac{e\,\alpha_A}{\overline{\zeta}\overline{\alpha}^{A'}\pi_{A'}}\,, \tag{10}$$

and

$$\delta\overline{\omega}^{A'} = i\frac{\partial \mathcal{H}^+}{\partial \pi_{A'}} = -i\frac{e\overline{\alpha}^{A'}}{\overline{\alpha}^{B'}\pi_{B'}}\,, \tag{11}$$

where the hamiltonian (8) is

$$\mathcal{H}^+(Z^\alpha) = e\,\log\left(-i\frac{A_\alpha Z^\alpha}{\overline{B}_\alpha Z^\alpha}\right)\,, \tag{12}$$

with $A_\alpha \leftrightarrow (\alpha_A, 0)$ and $\overline{B}_\alpha \leftrightarrow (0, \overline{\alpha}^{A'})$. We observe that the hamiltonian is homogeneous of degree zero in $Z^\alpha$. Equations (10) and (11) may be expressed in the manifestly twistorial form

$$\delta\overline{Z}_\alpha = i\frac{\partial \mathcal{H}^+}{\partial Z^\alpha}\,.$$

## 2.2 *Scattering off an impulsive gravitational wave*

We now turn to the formulation of a twistor hamiltonian approach to scattering off a fixed gravitational source. Here we make use of the result of Bonnor (1969b, §7), to obtain an exact solution of Einstein's equations that represents the gravitational field of a zero rest-mass particle. The field has the structure of a plane-fronted impulsive gravitational wave and hence our twistor hamiltonian approach to scattering null geodesics off this field will follow the 'scissors and paste' technique of Penrose (1968).

The metric for plane-fronted gravitational waves can be written in the form

$$\mathrm{d}s^2 = 2\mathrm{d}u\,\mathrm{d}v - 2\mathrm{d}\zeta\,\mathrm{d}\overline{\zeta} + 2A(u,\zeta,\overline{\zeta})\mathrm{d}u^2\,. \tag{13}$$

The function $A(u, \zeta, \overline{\zeta})$ is taken by Bonnor to be piecewise $C^1$, however we will suspend this differentiability requirement to allow for a $\delta$-function behaviour in $u$. For this metric $l^a$ is a Killing vector. The Ricci tensor for (13) is given by

$$R^{ab} = -\frac{1}{2}\frac{\partial^2 A}{\partial\zeta\partial\overline{\zeta}}\, l^a l^b\,.$$

From the field equations

$$R^{ab} - \frac{1}{2}g^{ab}R = -8\pi T^{ab} ,$$

we have

$$T^{ab} = \frac{1}{16\pi}\frac{\partial^2 A}{\partial\zeta\partial\overline{\zeta}}\, l^a l^b$$

and thus we may interpret this as the energy tensor of a fluid of zero rest-mass particles. In (13) we choose

$$A = \delta(u)\,\log\left(\frac{r}{a}\right) ,$$

where $a$ is a constant and $r^2 = \zeta\overline{\zeta} \geq a$.

We now outline the 'scissors and paste' method as applied to the impulsive gravitational wave. Consider two regions of Minkowski space, $\mathcal{M}$ and $\mathcal{M}^*$, separated by a hypersurface $\Sigma$ given by $u = 0 = u^*$.

$$\begin{aligned} \mathcal{M} &: \ ds^2 = 2du\, dv - 2d\zeta\, d\overline{\zeta} , \\ \mathcal{M}^* &: \ ds^2 = 2du^*\, dv^* - 2d\zeta^*\, d\overline{\zeta}^* , \end{aligned}$$

and identify the two coordinate patches according to $\zeta = \zeta^*$ and $v = v^* + s(\zeta, \overline{\zeta})$, where $s(\zeta, \overline{\zeta}) = \log\left(\frac{r}{a}\right)$. This identification creates the curvature.

The deflection of a worldline passing through $\Sigma$ is written in terms of the transformation of the spin frame $\alpha^A$, $\beta^A$,

$$\beta^A \mapsto \beta^A + \frac{\partial s}{\partial\overline{\zeta}}\alpha^A , \tag{14}$$

$$\alpha^A \mapsto \alpha^A . \tag{15}$$

Whence

$$\delta\pi_{A'} = (\pi_{B'}\overline{\alpha}^{B'})\,\frac{\partial s}{\partial\zeta}\,\overline{\alpha}_{A'} , \tag{16}$$

and by construction

$$\delta X^{AA'} = -s(\zeta, \overline{\zeta})\,\alpha^A\overline{\alpha}^{A'} ,$$

so that for $\delta\omega^A$ we have

$$\begin{aligned} \delta\omega^A &= iX^{AA'}\delta\pi_{A'} + i\delta X^{AA'}\pi_{A'}, \\ &= iX^{AA'}(\pi_{B'}\overline{\alpha}^{B'})\,\frac{\partial s}{\partial\zeta}\overline{\alpha}_{A'} - is(\zeta,\overline{\zeta})\,\alpha^A\overline{\alpha}^{A'}\pi_{A'} . \end{aligned} \tag{17}$$

Introducing the hamiltonian

$$\mathcal{H}(Z^\alpha, \overline{Z}_\alpha) = (\overline{\alpha}^{B'}\pi_{B'})(\alpha^B\overline{\pi}_B)\, s(\zeta, \overline{\zeta}) ,$$

we may write the equations (16) and (17) for the scattering of the null geodesics in twistor hamiltonian form

$$\delta Z^\alpha = -i\frac{\partial\mathcal{H}}{\partial\overline{Z}_\alpha} .$$

The spacelike coordinates on $\Sigma$ may be written in terms of $\overline{Z}_\alpha$:

$$\zeta = i\frac{\overline{A}^\alpha\overline{Z}_\alpha}{B^\alpha\overline{Z}_\alpha}, \quad \text{where } A_\alpha \leftrightarrow (\alpha_A, 0),\ \overline{B}_\alpha \leftrightarrow (0, \overline{\alpha}^{A'}), \tag{18}$$

and therefore the twistor hamiltonian is given by

$$\mathcal{H}(Z^\alpha, \overline{Z}_\alpha) = \frac{1}{2}(\overline{B}_\alpha Z^\alpha)(B^\beta \overline{Z}_\beta) \log \left[ \left( i\frac{\overline{A}^\alpha \overline{Z}_\alpha}{B^\beta \overline{Z}_\beta} \right) \left( -i\frac{A_\gamma Z^\gamma}{\overline{B}_\sigma Z^\sigma} \right) \right] . \tag{19}$$

This is homogeneous of degree 1 in $Z^\alpha$ and $\overline{Z}_\alpha$. We may write the function (19) as the sum of two expressions which involve functions that are homogeneous separately in $Z^\alpha$ and $\overline{Z}_\alpha$. To see this (cf. Penrose & MacCallum 1972, §3.2; Tod 1975), introduce two such functions $g^+$ and $g^-$, where

$$g^+ = \frac{i}{2}(\overline{B}_\alpha Z^\alpha)^2 \int^{\overline{\zeta}} \frac{dx}{x} ,$$

and $g^- = \overline{g^+}$. Clearly, $g^+$ and $g^-$ are homogeneous of degree 2 in $Z^\alpha$ and $\overline{Z}_\alpha$ respectively. Then we see that the hamiltonian (19) can be written as $\mathcal{H}(Z^\alpha, \overline{Z}_\alpha) = \mathcal{H}^+ + \mathcal{H}^-$, with

$$\mathcal{H}^+ = \overline{Z}_\alpha I^{\alpha\beta} \frac{\partial g^+(Z^\alpha)}{\partial Z^\beta}$$

where $I^{\alpha\beta}$ is the infinity twistor, together with a similar expression for $\mathcal{H}^-$.

## 3 The scattering transformation

The results (10), (11), (16) and (17) for spin-1 and spin-2 monopole scattering may be analyzed both in terms of space-time and twistor space kinematics and dynamics. We begin with a brief analysis of the scattering dynamics for the case of the fixed electromagnetic source.

Let $X_o^a$ be the position vector with respect to the origin $O$, of the point where the worldline of the zero rest-mass particle meets the wave. Having such a vector, we construct the position vectors for the particle before (with subscript 'b') and after (with subscript 'a') scattering in the following manner

$$x_b^{AA'} = X_o^{AA'} + \lambda \overline{\pi}_o^A \pi_o^{A'} ,$$

$$x_a^{AA'} = X_o^{AA'} + \lambda \overline{\pi}_*^A \pi_*^{A'} ,$$

for varying $\lambda \in \mathbb{R}$. The initial momenta are $\overline{\pi}_o^A \pi_o^{A'}$, and the final momenta $\overline{\pi}_*^A \pi_*^{A'}$ are obtained from Hamilton's equations. In the case of scattering off an electromagnetic wave $\delta X_o^{AA'} = 0$, so the change in $x^{AA'}$ given by

$$\delta x^{AA'} = x_a^{AA'} - x_b^{AA'} ,$$

may be obtained by the substitution $\pi_*^{A'} = \pi_o^{A'} + [\pi^{A'}]$, as follows

$$\delta x^{AA'} = \lambda(\overline{\pi}_o^A [\pi^{A'}] + \pi_o^{A'} [\overline{\pi}^A] + [\pi^{A'}][\overline{\pi}^A]) . \tag{20}$$

We may describe some of the salient features of the dynamics, in terms of the magnitudes of 'time delay' and 'deflection' vectors, obtained by projecting (20) into timelike and spacelike 2-surfaces respectively.

If we denote the spacelike 2-surface by $S$ then the projection operator taking a vector at some point of $S$ into the 2-surface is (see e.g. Penrose & Rindler 1984, p. 271)

$$S_{BB'}^{AA'} = -\alpha^A \overline{\beta}^{A'} \beta_B \overline{\alpha}_{B'} - \beta^A \overline{\alpha}^{A'} \alpha_B \overline{\beta}_{B'} ,$$

and we can project the connecting vector between the scattered and unscattered geodesics (in a spacelike direction) into the 2-surface

$$\delta\tilde{x}^{AA'} = S^{AA'}_{BB'}\delta x^{BB'} .$$

We find that the magnitude of $\delta\tilde{x}^{AA'}$ is

$$\left|\delta\tilde{x}^{AA'}\delta\tilde{x}_{AA'}\right| \propto \sqrt{\frac{e^2}{\zeta\overline{\zeta}}} .$$

Thus the deflection decreases as $r^{-1}$. We may examine the magnitude of the time delay in a similar manner. In this case we use a timelike 2-surface $T$, with the projection operator $T^a_b$ that takes a 4-vector at some point of $T$ into the 2-surface

$$T^{AA'}_{BB'} = \alpha^A\overline{\alpha}^{A'}\beta_B\overline{\beta}_{B'} + \beta^A\overline{\beta}^{A'}\alpha_B\overline{\alpha}_{B'} ,$$

which, together with the expression for the change in $x^{AA'}$, gives

$$T^{AA'}_{BB'}\delta x^{BB'} \equiv \delta\hat{x}^{AA'} = -\frac{e\beta_B\overline{\pi}^B\alpha^A\overline{\alpha}^{A'}}{\zeta\alpha^B\overline{\pi}_B} - \frac{e\overline{\beta}_{B'}\pi^{B'}\alpha^A\overline{\alpha}^{A'}}{\overline{\zeta}\overline{\alpha}^{B'}\pi_{B'}} .$$

The vector $\delta\hat{x}^{AA'}$ is *null* and therefore the magnitude of the time delay is a constant (zero) and independent of the impact parameter.

We may consider the time delay for the scattering of a null geodesic off the impulsive gravitational wave. For the space-time given by (13) $t^a$ is not a Killing vector. The transformation for $t^a$, $t^a \mapsto t^{a*}$, is given by, from (14) and (15),

$$\frac{1}{2}\left(\alpha^A\overline{\alpha}^{A'} + \beta^A\overline{\beta}^{A'}\right) \mapsto \frac{1}{2}\left(t^{AA'} + \beta^A\overline{\alpha}^{A'}\frac{\partial s}{\partial\overline{\zeta}} + \alpha^A\overline{\beta}^{A'}\frac{\partial s}{\partial\zeta} + \alpha^A\overline{\alpha}^{A'}\frac{\partial s}{\partial\zeta}\frac{\partial s}{\partial\overline{\zeta}}\right) .$$

Thus, for instance, a 2-surface with projection operator $T^a_b$, will have no invariant meaning. However, as Penrose & MacCallum (1972) remarked, there is no 'absolute' concept of time delay in general relativity and thus the time delay that we shall calculate here is not inconsistent with that obtained from examination of orbits in the Schwarzschild solution.

The deflection may be calculated in a straightforward manner. Write down an infinitesimal change in the coordinates of some position vector as

$$\begin{aligned}\delta x^{AA'} &= \delta X^{AA'} + \lambda\delta\overline{\pi}^A\pi^{A'} + \lambda\overline{\pi}^A\delta\pi^{A'} ,\\ &= -s(\zeta,\overline{\zeta})\alpha^A\overline{\alpha}^{A'} + \lambda(\overline{\pi}^B\alpha_B)\frac{\partial s}{\partial\zeta}\alpha^A\pi^{A'} + \lambda(\overline{\alpha}_{B'}\pi^{B'})\frac{\partial s}{\partial\overline{\zeta}}\overline{\pi}^A\overline{\alpha}^{A'}. \qquad (21)\end{aligned}$$

Then using (14) and (15), for the spacelike surface we calculate $S^{*BB'}_{AA'}$ and for the timelike surface we calculate $T^{*BB'}_{AA'}$. Thus the 'time delay' and 'deflection' may be computed from $\delta x^{AA'}T^{*BB'}_{AA'}$ and $\delta x^{AA'}S^{*BB'}_{AA'}$ and from the norms we obtain the $r^{-1}$ dependence for the deflection and zero for the time delay (the vector being null). (Remark: if we had chosen to transvect (21) with $\beta_B\overline{\beta}_{B'}$ then the function $s(\zeta,\overline{\zeta})$ would contribute to the 'null delay'.)

We have shown that the behaviour of null geodesics in flat space-time with an impulsive wave background may be understood in terms of the unfolding of a canonical transformation of null twistors. Penrose (1968) discusses the transformation properties of the canonical equations on twistor space and in particular refers to the invariant structures of the system. Following Penrose *op. cit.*, we shall examine the invariance of the twistor norm $Z^\alpha \overline{Z}_\alpha$ and the inner product $Y^\alpha \overline{X}_\alpha$, under the action of the transformation $\delta$, generated by the hamiltonians (12) and (19). To examine the symplectic invariance, we consider certain differential forms: $\vartheta = iZ^\alpha \mathbf{d}\overline{Z}_\alpha$ and the symplectic two-form $\Omega = \mathbf{d}\vartheta = i\mathbf{d}Z^\alpha \wedge \mathbf{d}\overline{Z}_\alpha$.

If $\mathcal{H}$ is homogeneous of degree $m$ in $Z^\alpha$ and $\overline{Z}_\alpha$, then $\delta(Z^\alpha \overline{Z}_\alpha) = 0$ and the norm is part of the invariant structure. This holds for hamiltonians (12) and (19). It follows that this may be interpreted as the invariance of the helicity under $\delta$. The symplectic two-form is the natural integral invariant associated with the hamiltonian system. It is invariant under the action of $\delta$. Examination of $\delta\vartheta$ shows that it will vanish iff $\mathcal{H}$ is homogeneous of degree one in $Z^\alpha$. Thus $\vartheta$ is an invariant for the hamiltonian (19) but not for the electromagnetic case (12).

Of particular interest here is the effect of the transformations on the twistor inner product $Y^\alpha \overline{X}_\alpha$, which, in general, will not be preserved under the action of the transformation. We construct a measure of the shift induced by the transformation generated by (12) on a point of $\mathbb{P}\mathbb{N}$. Let two rays $\mu$ and $\mu'$ be described by the two null twistors $X^\alpha$ and $Y^\alpha$

$$X^\alpha \longleftrightarrow (ix^{AA'}\eta_{A'}, \eta_{A'}) \text{ and } Y^\alpha \longleftrightarrow (iy^{AA'}\xi_{A'}, \xi_{A'}) \ .$$

Choose the two rays to be abreast and focused at infinity. That is

$$y^{AA'} - x^{AA'} \equiv \Delta^{AA'} = m\,\alpha^A\overline{\beta}^{A'} - \overline{m}\,\beta^A\overline{\alpha}^{A'} ,$$

where $m, \overline{m} \in \mathbb{C}$, $\Delta^a\nabla_a\overline{\beta}^{A'} = 0$, and $Y^\alpha \overline{X}_\alpha = 0$. We then scatter the two rays using our hamiltonian prescription and then consider the product $X^{*\alpha}\overline{Y}^*_\alpha$. We find

$$\delta(Y^\alpha \overline{X}_\alpha) = ie\left(\frac{(\zeta\overline{\zeta})_x - (\zeta\overline{\zeta})_y}{\zeta_y\overline{\zeta}_x}\right) \tag{22}$$

(where $\zeta_{x/y}$ denotes the $\zeta$-coordinate of the vector $x^{AA'}$ or $y^{AA'}$). This will vanish only if $X$ and $Y$ are coincident. In other words, even at large values of $\zeta$, the expression will only become small if the values of the $x$ and $y$ coordinates are close together. Hence the scattering transformation induces a shift in viewpoint from $\mathbb{P}\mathbb{N}$ to $\mathbb{N}$. This is a realization of the fact that, in general, the canonical transformation will induce shear on a bundle of shear-free rays. The asymptotic behaviour is typical of the Coulomb problem, where at large distances from the source the field has an effect on the dynamics unless the interaction is switched off (for further discussion see Hodges 1983, 1985; Roulstone 1994).

A similar calculation for the scattering of two parallel rays off an impulsive gravitational wave gives, from (14) *et seq.*, the following for the change in the inner product

$$\begin{aligned}
\delta(Y^\alpha \overline{X}_\alpha) &= -i\overline{X}_\alpha\frac{\partial\mathcal{H}}{\partial\overline{Y}_\alpha} + iY^\alpha\frac{\partial\mathcal{H}}{\partial X^\alpha} , \\
&= -\frac{i}{2}\overline{B}_\alpha Y^\alpha B^\beta \overline{X}_\beta \log(\zeta\overline{\zeta})_y - \frac{i}{2}\overline{B}_\alpha Y^\alpha B^\beta \overline{Y}_\beta \left(\frac{\overline{A}^\gamma \overline{X}_\gamma}{\overline{A}^\delta \overline{Y}_\delta} - \frac{B^\gamma \overline{X}_\gamma}{B^\delta \overline{Y}_\delta}\right) \\
&\quad + \frac{i}{2}\overline{B}_\alpha Y^\alpha B^\beta \overline{X}_\beta \log(\zeta\overline{\zeta})_x + \frac{i}{2}\overline{B}_\alpha X^\alpha B^\beta \overline{X}_\beta \left(\frac{A_\gamma Y^\gamma}{A_\delta X^\delta} - \frac{\overline{B}_\gamma Y^\gamma}{\overline{B}_\delta X^\delta}\right) , \\
&\neq 0 ,
\end{aligned} \tag{23}$$

as $\zeta \to \infty$. Thus we see that the $s_x$ and $s_y$ terms now contribute directly to the displacement.

## 4 Summary

We conclude that the unfolding of a canonical transformation on the symplectic manifold of null twistors can describe the scattering off a fixed background and preserves the usual integral invariants. In contrast to the formal inhomogeneous expressions for the hamiltonians given in Penrose & MacCallum (1972, §5.2), and by virtue of their construction, the hamiltonians (12) and (19) are homogeneous. We have shown that for both the Coulomb and the linearized Schwarzschild backgrounds, shear is induced by considering the behaviour of neighbouring rays. This displacement does not vanish as one moves to large distances from the source and is described naturally in terms of the non-projective space $\mathbb{N}$. One can show that in the case of an electromagnetic dipole field, which decreases faster than the Coulomb field, the amount of shear decreases with increasing impact parameter. Further details, and the extension of the results to massive particles scattering off these sources (cf. Tod 1975, Tod & Perjés 1976), can be found in Roulstone (1994).

Thanks especially to K.P.T.

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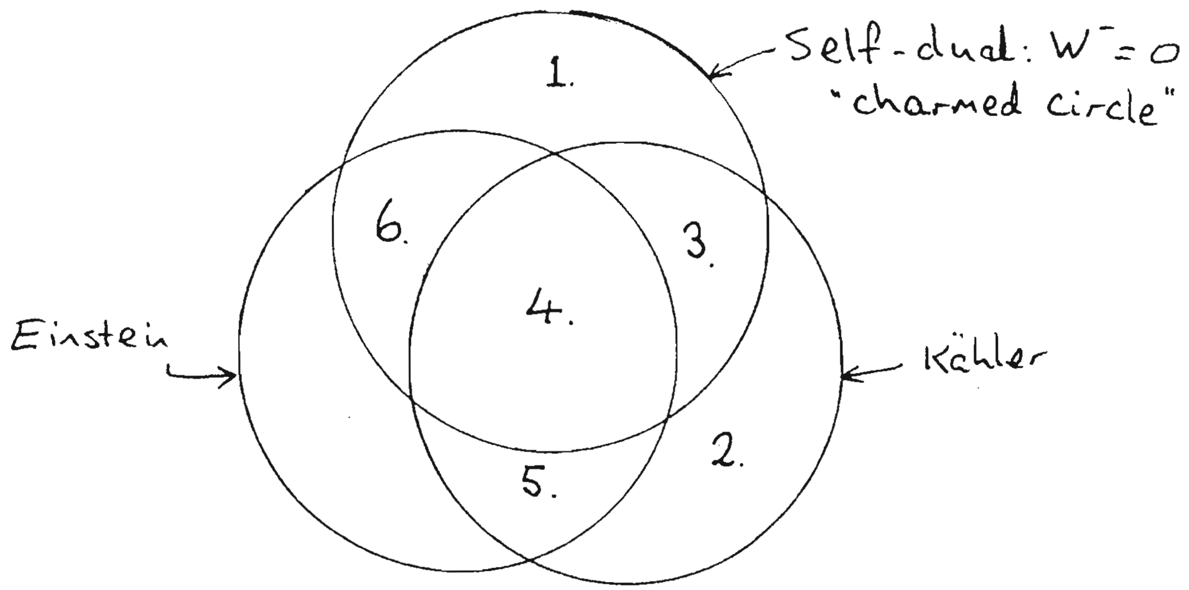
## Self-dual Einstein metrics with symmetry

There are three different 'field equations' that one might impose on a four-dimensional Riemannian metric: one might require it to be Kähler or to be Einstein or to have self-dual Weyl tensor. These possibilities lead to an attractive Venn diagram in which the overlaps all have 'names' and have all been studied at one time or another. We may consider what happens to this diagram when the metric has a Killing vector. Again the regions usually have names as follows:




With symmetry, the regions are:

1: 3-dimensional Einstein-Weyl spaces (much studied);
2: Kähler-with-symmetry, studied by LeBrun (J.Diff.Geom. **34** 223 (1991);
3 and 4: scalar-flat Kähler and hyper-Kähler respectively with symmetry, solved by the SU($\infty$)-Toda field equation (LeBrun);
5: Einstein-Kähler with symmetry, studied by Pedersen and Poon (Comm.Math.Phys. **136** 309 (1991)) and solved by the 'Pedersen-Poon equation', equation (7) below;
6: self-dual Einstein or 'quaternionic-Kähler' with symmetry.

In this article, I will find a simple form for the field equations for region 6 corresponding to Einstein metrics with self-dual Weyl tensor and non-zero scalar curvature. After a sequence of transformations, the field equations end up being rather familiar.

As an introduction to the calculation, I will review some of the other regions in the diagram. For Kähler-plus-symmetry, we know from LeBrun's work (J.Diff.Geom. **34** 223 (1991)) that coordinates can be found in which the metric can be written in the form

$$ds^2 = W[e^u(dx^2+dy^2)+dz^2] + \frac{1}{W}(dt+\theta)^2. \quad (1)$$

The Kähler condition entails

$$d\theta = W_x\, dy\wedge dz + W_y\, dz\wedge dx + (We^u)_z\, dx\wedge dy \quad (2)$$

from which the function W must satisfy

$$W_{xx} + W_{yy} + (We^u)_{zz} = 0. \quad (3)$$

If we now require the scalar curvature to vanish then the function u must satisfy the 'SU($\infty$)-Toda field equation', namely:

$$u_{xx} + u_{yy} + (e^u)_{zz} = 0. \quad (4)$$

Thus a scalar-flat Kähler metric is determined by a solution of (4) together with a solution W of the linear equation (3). One particular solution of (3) is given by

$$W = c\,u_z \; ; \; c = \text{const.} \quad (5)$$

where u satisfies (4).

In this case, the metric (1) is actually hyper-Kähler or equivalently is Ricci-flat with self-dual Weyl tensor (and so corresponds to region 4 in the figure).

If, instead of vanishing scalar-curvature, we seek Kähler metrics with vanishing trace-free Ricci tensor then in place of (5) we must take

$$W = \frac{u_z}{\Lambda z + M} \quad (6)$$

where M is a constant (of integration) and $\Lambda$ is proportional to the Ricci scalar. Now in place of (4) we find that u must satisfy the equation:

$$u_{xx} + u_{yy} + (e^u)_{zz} = \frac{2\Lambda e^u u_z}{\Lambda z + M} \tag{7}$$

an equation found by Pedersen and Poon (Comm.Math.Phys. **136** 309 (1991)). This corresponds to region 5 in the figure.

If $\Lambda$ is non-zero, then we may absorb M into z whereupon $\Lambda$ disappears from (7). If is zero then this case reduces to the previous one.

With $\Lambda$ non-zero, the field-equations lie outside the 'charmed circle' at the top in the figure. Everything inside the circle has self-dual Weyl tensor, so can be solved by a twistor construction and so all symmetry reductions of them should be integrable; outside the circle there is no expectation of integrability.

Region 6 in the figure, corresponding to self-dual Einstein metrics with symmetry and with $\Lambda$ non-zero, lies inside the charmed circle and so should lead to integrable equations. My purpose in this note is to find these equations. What turns up is eventually very similar to the scalar-flat Kähler case.

We begin by finding a canonical form for the metric in this case. Suppose then that we have a Killing vector K in a 4-dimensional Riemannian space. In terms of spinors, the derivative of K decomposes as:

$$\nabla_a K_b = \varphi_{AB}\,\epsilon_{A'B'} + \psi_{A'B'}\,\epsilon_{AB} \tag{8}$$

and the following identity, true for any Killing vector,

$$\nabla_a \nabla_b K_c = R_{bcad} K^d \tag{9}$$

entails

$$\nabla_{AA'}\varphi_{BC} = -\psi_{BCAD}K^{D}_{A'} + \Lambda\epsilon_{A(B}K_{C)A'}$$
$$\nabla_{AA'}\psi_{B'C'} = \Lambda\epsilon_{A'(B'}K_{C')A}. \tag{10}$$

Define the scalar $\psi$ by $2\psi^2 = \psi^{A'B'}\psi_{A'B'}$, and define the tensor $J_a{}^b$ by

$$J_a{}^b = \frac{1}{\psi}\delta_A{}^B\psi_{A'}{}^{B'} \tag{11}$$

then $J$ is an almost-complex structure. It is a straightforward calculation based on (8) and (10) to see that this complex structure is actually integrable. This fact, which is crucial in what follows, was pointed out to me by Lionel Mason. Thus the metric is Hermitian, but will not be Kähler unless $\Lambda$ is zero.

Contracting the second of equations (10) with $\psi^{A'B'}$ enables us to see that

$$2\psi\nabla_a\psi = \psi^{B'C'}\nabla_a\psi_{B'C'} = \Lambda\psi_{A'B'}K^{B'}{}_A \tag{12}$$

so that $J_{ab}K^b = \frac{2}{\Lambda}\nabla_a\psi$.

We may now follow what is essentially LeBrun's argument to arrive at the form of the metric. If the Killing vector is $K^{\cdot} = \partial/\partial t$ in contravariant form then in covariant form it can be written

$$K_{\cdot} = \frac{1}{W}(dt + \theta) \tag{13}$$

in terms of a 1-form $\theta$ and a scalar $W = (K_a K^a)^{-1}$.

We define a coordinate $z = \frac{2\psi}{\Lambda}$ (which incidentally makes the limit $\Lambda \to 0$ one to be taken with care). The 2-blades containing K and dz are eigenspacres of the complex structure and so are integrable, as are their orthogonal complements. Introduce a complex coordinate $\zeta$ = x+iy on the orthogonal complement, then the metric can be written exactly as in (1) again.

This time the metric is not Kähler, so we don't have equation (2). To see what we have instead, note that from (8) and (12)

$$K^b \nabla_a K_b = -\varphi_{AB} K^B{}_{A'} - \psi_{A'B'} K^{B'}{}_A = -\frac{\nabla_a W}{2W^2}$$

$$K^b * (\nabla_a K_b) = \varphi_{AB} K^B{}_{A'} - \psi_{A'B'} K^{B'}{}_A \tag{14}$$
$$= \frac{\nabla_a W}{2W^2} - \frac{4\psi}{\Lambda} \nabla_a \psi .$$

Knowing these we can calculate that

$$d\theta = W_x \, dy \wedge dz + W_y \, dz \wedge dx + e^u (W_z - 2\Lambda z W^2) \, dx \wedge dy \tag{15}$$

This has an integrability condition which naturally differs from that in (3).

The next step is to impose the conditions on the metric (1) that the trace-free Ricci tensor vanish and that the Weyl tensor be self-dual. I carried out this calculation following the formalism described in my article in *Twistor Theory* ed. Stephen Huggett (Dekker; 1995), the proceedings of the Seale Hayne twistor conference. The details are unilluminating, but the result is that W is determined as

$$-2\Lambda W = \frac{1}{z} u_z - \frac{2}{z^2} \tag{16}$$

where u satisfies the equation

$$u_{xx} + u_{yy} + e^u \left( u_{zz} + u_z^2 - \frac{6}{z} u_z + \frac{12}{z^2} \right) = 0. \tag{17}$$

Equations (15), (16) and (17) with the metric (1) form our principal conclusion. It can be checked that the integrability condition for (15) is automatically satisfied.

One simple class of solutions to (17) is the separable solutions: write u in a separated form $u = f(x,y) + g(z)$ then the general such solution of (17) is

$$e^u = \frac{z^2 (4k + bz + az^2)}{(1 + k(x^2 + y^2))^2} \tag{18}$$

where a,b and k are arbitrary constants. The resulting metric is the metric with U(2) symmetry found by Pedersen (Math.Ann. 274 35 (1986)).

What is unexpected, and is a disappointment, is that (17) can be transformed back into (4); in other words, *self-dual Einstein metrics with non-zero scalar curvature are determined by the SU(∞)-Toda field equation.*

To see this, introduce new variables w = 1/z and v = u - 4logz, then

$$\begin{aligned} & v_{xx} + v_{yy} + (e^{v})_{ww} \\ &= u_{xx} + u_{yy} + e^{u}\left(u_{zz} + u_z^2 - \frac{6}{z}u_z + \frac{12}{z^2}\right) \qquad (19) \\ \therefore\ &= 0 \end{aligned}$$

The metric (1) transforms to the following form:

$$ds^2 = \frac{P}{w^2}\left[e^{v}(dx^2 + dy^2) + dw^2\right] + \frac{1}{Pw^2}(dt + \theta)^2 \qquad (20)$$

where v satisfies the SU(∞)-Toda field equation (in x,y and w), P is given by

$$-2\Lambda P = 2 - wv_w \qquad (21)$$

and $\theta$ is determined by

$$\begin{aligned} d\theta &= -P_x\, dy\wedge dw - P_y\, dw\wedge dx - e^{v}\left(P_w + \frac{2}{w}P + \frac{2\Lambda}{w}P^2\right)dx\wedge dy \qquad (22) \\ &= \frac{-1}{2\Lambda}\left\{wv_{wx}\, dy\wedge dw + wv_{wy}\, dw\wedge dx + e^{v}(wv_{ww} - v_w + wv_w^2)\, dx\wedge dy\right\} \end{aligned}$$

**Acknowledgement**
I am grateful to Lionel Mason for the observation that the complex structure $J$ of equation (11) is integrable for a self-dual Einstein space with symmetry.

Paul Tod

# 'Moving-Zero' Solutions of Ward's Chiral Model

In [W1] Ward modifies the SU(2) chiral field equation in $R^{2+1}$:

$$\eta^{\mu\nu}\partial_\mu(J^{-1}\partial_\nu J) = 0, \tag{1}$$

which is Lorentz invariant but not integrable, to a field equation which is integrable, but has less symmetry. This equation is

$$\eta^{\mu\nu}\partial_\mu(J^{-1}\partial_\nu J) + V_\alpha\varepsilon^{\alpha\mu\nu}\partial_\mu(J^{-1}\partial_\nu J) = 0 \tag{2}$$

with $\varepsilon^{\alpha\mu\nu}$ being the alternating tensor with $\varepsilon^{012} = 1$, $\eta^{\mu\nu}$=diag$(-1,1,1)$, $J$ a map from $R^{2+1}$ to SU(2) and $V_\alpha$ a constant unit vector. We are using coordinates $x^\mu = (x^0, x^1, x^2) = (t, x, y)$, and $\partial_\mu = \partial/\partial x^\mu$. The conformal properties of $V_\alpha$ determine whether the symmetry group is SO(2) or SO(1,1). Ward chooses V to have the components $V_\alpha = (0,1,0)$, the space-like case. The reason for this is that the standard energy momentum-tensor for [1] -

$$T_{\mu\nu} = (-\delta^\alpha_\mu\delta^\beta_\nu + \frac{1}{2}\eta_{\mu\nu}\eta^{\alpha\beta})tr(J^{-1}\partial_\alpha J J^{-1}\partial_\beta J) \tag{3}$$

gives an energy-momentum vector $P_\mu = T_{\mu 0}$ which is divergence-free when both $V_0 = 0$ and J satisfies [2]. Hence we have a conserved energy $E = \int P_0 dx^1 dx^2$, where the integration is taken over planes of constant $x^0$, for solutions of [2]. This provides us with a natural boundary condition of finite energy, which means that $J = K + O(r^{-1})$ as $r \to \infty$, K being a (constant) element of SU(2), and $r^2 = (x^1)^2 + (x^2)^2$.

Ward then goes on to find soliton solutions of [2] using the method of 'Riemann problem with zeros', with the location of the zeros being fixed. I shall now outline the same technique, with the generalization that the zeros are not fixed.

We examine the Lax pair

$$(\lambda\partial_x - \partial_u)\Psi = A\Psi \tag{4}$$
$$(\lambda\partial_v - \partial_x)\Psi = B\Psi$$

where $\lambda \in C$, $(u,v,x)$ are coordinates on $R^{2+1}$, A and B are su(2)-valued functions of $(u,v,x)$ only, and $\Psi(u,v,x,\lambda)$ is a unimodular $2\times 2$ matrix-valued function satisfying the reality condition

$$\Psi(u,v,x,\bar\lambda)^\dagger = \Psi(u,v,x,\lambda)^{-1}, \tag{5}$$

where $\dagger$ denotes the conjugate transpose matrix.

The Lax pair has the usual type of consistency conditions

$$\partial_x B = \partial_v A \quad \partial_x A - \partial_u B - [A, B] = 0, \tag{6}$$

and putting $J(u, v, x) = \Psi(u, v, x, \lambda = 0)^{-1}$ for $\Psi$ a solution of [4], we get from [4], [6] that

$$\partial_x(J^{-1}\partial_x J) - \partial_v(J^{-1}\partial_u J) = 0.$$

This is precisely[2], as we can see by replacing $u = \frac{1}{2}(t + y), v = \frac{1}{2}(t - y)$, and the reality condition ensures that $J$ is unitary. We can now use the Riemann method on [4] by using the Ansatz

$$\Psi_{ab}(u, v, x, \lambda) = \delta_{ab} + \frac{n_a m_b}{\lambda - \alpha}$$

where the vectors n,m, and the function $\alpha$ are independent of $\lambda$. (Ward has $\alpha$ constant - we are allowing the pole to move. His Ansatz also adds on other first order poles, but we shall just have the one pole. The generalization is straightforward).

The reality condition[5] gives us n in terms of m:

$$n_a = -\frac{\bar{m}_a(\bar{\alpha} - \alpha)}{m_0\bar{m}_0 + m_1\bar{m}_1}.$$

There is a 'gauge' homogeneity in that we can multiply $m_0$ and $m_1$ by the same holomorphic function without altering the resulting $\Psi$. Using this, we set $m_a = (1, f)$.

Substituting the Ansatz into [4] gives us differential equations for $f$ and $\alpha$. Firstly we find that

$$\alpha\partial_x\alpha - \partial_u\alpha = \alpha\partial_u\alpha - \partial_v\alpha = 0,$$

which is satisfied when there is an F such that

$$F(\alpha^2 u + \alpha x + v, \alpha) = 0,$$

giving us $\alpha$. Then the condition on $f$ is that

$$\alpha\partial_x f - \partial_u f = \alpha\partial_u f - \partial_v f = 0,$$

which holds when $f = f(\alpha^2 u + \alpha x + v, \alpha)$ .

Thus we can generate a J satisfying [2] by prescribing: a function F of two variables; one of the solutions $\alpha$ to the equation $F(\alpha^2 u+\alpha x+v, \alpha) = 0$; and a function $f(\alpha^2 u+\alpha x+v, \alpha)$. J is then given by

$$(J^{-1})_{ab} = \Delta^{-\frac{1}{2}}(\delta_{ab} + \bar{m}_a m_b \frac{(\bar{\alpha} - \alpha)}{(1 + f\bar{f})}),$$

where $\Delta = \bar{\alpha}/\alpha$ is the determinant of the term in brackets on the RHS, and $(m_0, m_1) = (1, f)$. It should be noted that normalizing $J$ so that it is unimodular does not affect $J$'s being a solution of [2]. Some choices of F will give $\alpha$ such that J has singularities, and so care must be taken with this choice.

We can now go on to find the energy $P_0 = T_{00}$ of such a J, using expression [3]. This turns out to be

$$\frac{1}{2}(Im(\frac{\alpha\bar{A}(1+\bar{a}^2)}{\alpha\bar{\alpha}}))^2 + \frac{A\bar{A}}{\alpha\bar{\alpha}}(Im(\alpha))^2 + 2\frac{(1+\alpha\bar{\alpha})^2|f_1 B + f_2 A|^2}{\alpha\bar{\alpha}(1+f\bar{f})^2}(Im(\alpha))^2,$$

where $A$ and $B$ are given by

$$A = -\frac{F_1}{(2\alpha u + x)F_1 + F_2} \quad B = \frac{F_2}{(2\alpha u + x)F_1 + F_2}$$

with $F_1, F_2$ being the derivatives of $F$ with respect to its first and second parameters respectively, and similarly $f_1$, $f_2$ the derivatives of $f$.

The first two terms are entirely due to the determinant of $J$, and provide a 'vacuum' over which the third term of the energy density moves. In the case of constant $\alpha$, this vacuum vanishes, and we are left with equation (A9) of [W1].

The simplest interesting case is obtained by choosing

$$F(\alpha^2 u + \alpha x + v, \alpha) = (\alpha^2 u + \alpha x + v) + \frac{i}{2}(\alpha^2 + 1) = 0, \tag{7}$$

the imaginary term being necessary to avoid singularities in $J$. This gives us two choices for $\alpha$, but they both give the similar $J$'s. We still have left a choice of $f$. The simplest choice is a projection onto either of the parameters. We shall choose projection onto the first, so that $f = (\alpha^2 u + \alpha x + v)$. Figure 1 shows the vacuum state of this solution at t=5. The two crests move away from each other with velocity 1, maintaining their height (which tends to 2 along the crests). There is also a circle of radiation, spreading outwards from the origin, with velocity 1. At t=0, the circle is not apparent, and the crests form a single wavefront.

On top of this vacuum, we get a soliton-like object travelling in a straight line along the y-axis, in tandem with one of the crests. Unfortunately, its amplitude decays rapidly away from $t = 0$, and so gets swamped by the vacuum. Figure 2 shows this 'soliton' at $t = 1$. It should be emphasized that due to the vacuum term, no solutions will have finite-energy for this choice of $\alpha$.

**Figure 1**

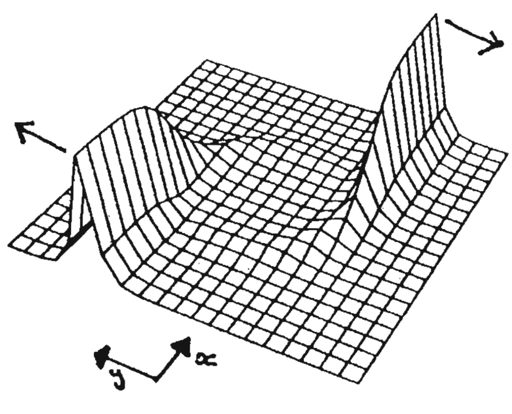
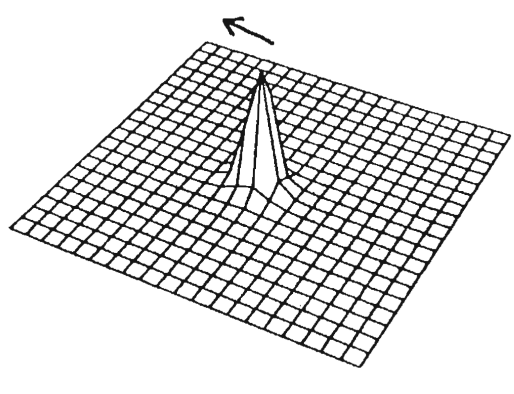
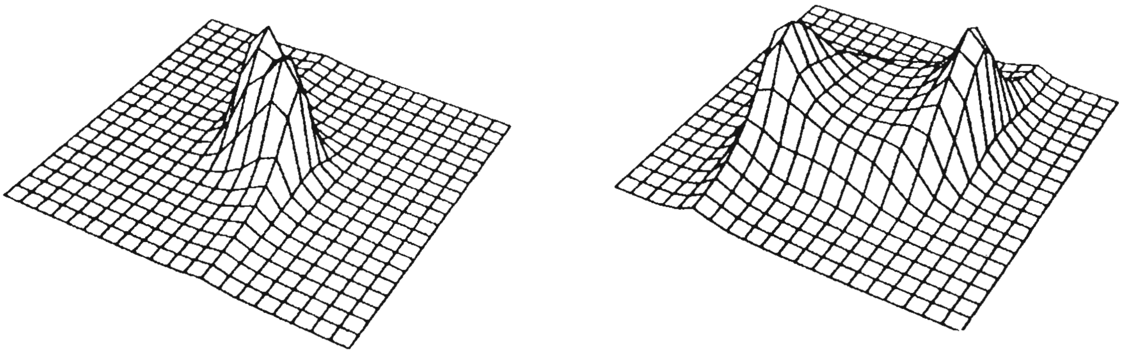


**Figure 2**



If $f = \alpha$, we get the appearance of two solitons coming towards each other, colliding at $t = 0$, and scattering of at an angle (see below for rescaled views of this at $t$=-2,5). They decay in amplititude away from $t = 0$. This is an interesting result, since there has long been numerical evidence for scattering solitons for [2], but no analytic explanation. It is unfortunate that the vacuum is once again dominant.



The above example has a motivation from twistor theory, in that we would like to interpret $\Psi$ as the patching matrix for a bundle over minitwistor space, or a compactification. (Minitwistor space is $O(2)$, the line bundle over $CP^1$ with Chern class 2. For more details see [W2],[MS]). If we write, suppressing matrix indices,

$$\Psi(\pi_A, \xi) = I + \frac{L(\pi_A, \xi)}{\frac{i}{2}(\pi_0^2 + \pi_1^2) - \xi},$$

where $\pi_A, \xi = \pi_0^2 u + \pi_0 \pi_1 x + \pi_1^2 v$ are coordinates on minitwistor space, then we can split this into partial fractions, writing $\lambda = \pi_0/\pi_1$, to obtain

$$\Psi = I + \frac{M}{\lambda - \alpha} + \frac{N}{\lambda - \beta}$$

with $\alpha, \beta$ the roots of [7], which is very similar to our Ansatz.

The consequences of adding the term corresponding to second root are currently being investigated, but it is already known that the vacuum term will mean that the boundary conditions cannot be satisfied.This is not unexpected, as there are geometric reasons for believing it necessary to use an $F$ that is quadratic in its first parameter before the boundary conditions can be satisfied.

Many thanks to Lionel Mason for suggesting this topic, and for all his help.

G Kelly

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# Complex Structures via 3 and 4-Dimensional Cauchy Integrals

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## Introduction

The existence of complex structures in quantum theory and their possible role in a theory of quantum gravity is a subject of much current interest. As Pauli points out these structures are essential in ordinary quantum mechanics in order that the state vector evolves unitarily subject to the appropriate field equation. Hawking and Gibbons have argued that the complex structures correspond in the 'history' of the universe to the emergence of our classical notion of time in thermodynamics. In the passage from a 4-dimensional real Euclidean universe to a real Lorentzian universe one acquires a real 'time' coordinate and thus a notion of positive and negative frequency. It is their view that the complex structure comes into play at the interface of these two regions where the real and imaginary time coordinates vanish.

In terms of the theory of Fock space, we have the two cases of a Hilbert space of *real* or of *complex* valued solutions to the field equation. In the former case, there is only *one* available complex structure since this must map a given real solution of the field equation to another *real* solution. In terms of the transform $\hat{\varphi}(k)$ of the field multiplication by a complex number $\alpha = x + iy$ is replaced by the action of the operator $x + Jy$ on $\varphi$ where $J$ acts linearly and multiplies the positive and negative frequency parts of the field by $+i$ and $-i$ respectively. We say that $J$ is the *complex structure* acting on the Hilbert space. Thus the action of $J$ amounts to multiplication by $i$ but *within the Hilbert space*. In the case of a *complex* valued field there are in fact *two* possible complex structures – that functionally identical to the one given above, together with straightforward multiplication of $\hat{\varphi}$ by the complex number $\alpha$. However it is only the former choice which leads to positive values for the expectation of the energy operator – since particles and their associated antiparticles both have *positive* rest energy we make *this* choice on physical grounds.

In this article we discuss the action of the complex structure $J$ on a real or complex valued spacetime field $\varphi$ with any index structure via multidimensional Cauchy integration. In the case of a 4-dimensional integral $\varphi$ is a field required to be analytic on real compactified spacetime $M^{\#}$ and extending therefore to a 4-complex dimensional neighbourhood. In the 3-dimensional case we require further that the field satisfies $\nabla^2\varphi = 0$ throughout $CM^{\#}$.

The formulae we give are *essentially* different from the usual multidimensional Cauchy integral formulae one encounters in complex analysis. The latter are merely generalisations of the 1-dimensional residue calculus to integration around the boundary of a n-dimensional polydisk – a triviality by repeated integration.

The formulae we present have the property of Lorentz and conformal covariance.

**A Reproducing Kernel (RK)**

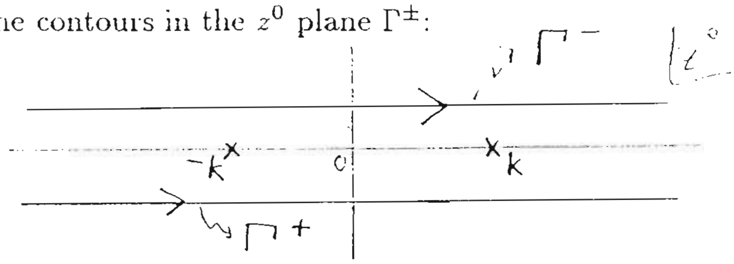
*Holomorphic extensions and positive frequency.*

We let $M^{\#}$ denote real compactified Minkowski spacetime. The invariant spacetime regions $CM^{\pm}$, $CM^{0}$, $M^{\#}$ are of considerable physical significance. A spacetime field $\varphi$ which is holomorphic on $\overline{CM^{+}}$ (by which we mean holomorphic on some *open* neighbourhood $U \supset \overline{CM^{+}}$) is called *positive frequency*. Fields which are holomorphic on $\overline{CM^{-}}$ are called *negative frequency*. In contrast, and we will not be considering such, fields which are holomorphic on the *open* future or past tubes $CM^{+}$, $CM^{-}$ are called *future* or *past analytic*. $CM^{0}$ denotes the set of points of $M^{\#}$ with spacelike imaginary part.

Positive frequency is equivalent to holomorphic extension to the forward tube defined by

$$CM^{+} \equiv \{(x^{a} - iy^{a}) : y^{0} > 0, (y^{0})^{2} > |\mathbf{y}|^{2}\}.$$

We now define the contours in the $z^{0}$ plane $\Gamma^{\pm}$:




Both contours are future pointing in real time. Consider

$$I = \int_{\Gamma^{-}} \frac{\varphi^{+}(z)}{z^{4}} d^{4}z$$

$$T = \int_{\Gamma^{-}} \frac{\varphi^{+}(z^{0}, \mathbf{z})}{(z^{0} - k)^{2}(z^{0} + k)^{2}} dz^{0}$$

where $k = |\mathbf{z}|$; $\mathbf{z}$ is assumed to be real. Setting $f(z^{0}) = \frac{\varphi^{+}(z^{0}, \mathbf{z})}{(z^{0} - k)^{2}(z^{0} + k)^{2}}$ we calculate the residues of $f$ at $z^{0} = \pm k$, $k \neq 0$.

$$\text{residue at } k = \frac{-\varphi^{+}(k, \mathbf{z})}{4k^{3}} + \frac{1}{(2k)^{2}} \frac{\partial \varphi^{+}}{\partial t}|_{t=k}$$

Similarly, but noting sign differences

$$\text{residue at } -k = \frac{\varphi^{+}(-k, \mathbf{z})}{4k^{3}} + \frac{1}{(-2k)^{2}} \frac{\partial \varphi^{+}}{\partial t}|_{t=-k}.$$

Since the field is positive frequency we may close off the contour in the lower half $z^{0}$ plane, and the residue theorem gives

$$T = \frac{\pi i}{2}\left[\frac{\varphi^{+}(k) - \varphi^{+}(-k)}{k^{3}}\right] + -\frac{\pi i}{2} \frac{[\partial \varphi^{+}/\partial t(k) + \partial \varphi^{+}/\partial t(-k)]}{k^{2}}.$$

For $k = 0$

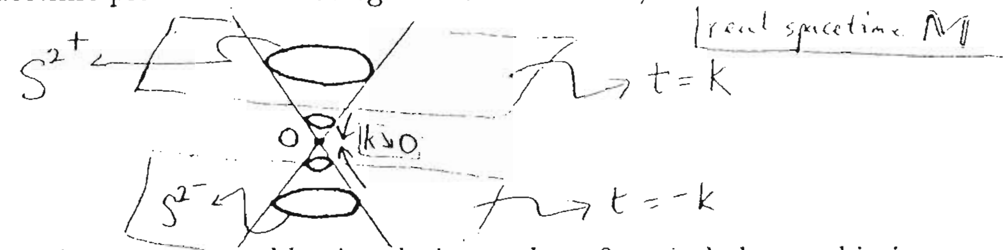
$$T = -2\pi i \cdot \text{res}\varphi^+/(z_0)^4|_0 = -\frac{1}{3}\pi i \frac{\partial^3 \varphi^+}{\partial t^3}|_0$$

which is just a finite number.
Now

$$I = \int_{\Gamma^-} \frac{\varphi^+(z)}{z^4} d^4 z = \frac{\pi i}{2} \int_{S^3} [\frac{\varphi^+(k) - \varphi^+(-k)}{k^3}] - \frac{[\partial\varphi^+/\partial t(k) + \partial\varphi^+/\partial t(-k)]}{k^2}$$

$$= \frac{\pi i}{2} \int_0^\infty \int_{S^2} [\frac{\varphi^+(k) - \varphi^+(-k)}{k}] - [\partial\varphi^+/\partial t(k) + \partial\varphi^+/\partial t(-k)] dk\, d^2 S.$$

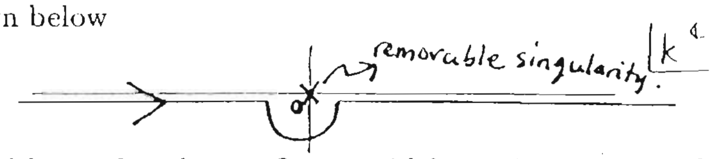
The term from $k = 0$ in $T$ contributes zero since it is a simple discontinuity in the $k$ integration. The spacetime picture of this integral is shown below,




Note that the integrand has a removable singularity at $k = 0$ so is holomorphic in an open neighbourhood of the real $k$ axis, and that, analytically continued as a function of both positive and negative $k$, is an even function. The second term in square brackets contributes zero since for each term separately the contour can be closed in the upper/ lower half plane enclosing no singularities. We may therefore replace the $k$ integration by the contour integral in the complex $k$ plane

$$\frac{1}{2} \int_C [\frac{\varphi^+(k) - \varphi^+(-k)}{k}] dk$$

where $C$ is shown below




(Clearly we could have also chosen $C$ to avoid $k = 0$ in the upper-half plane – this choice will not affect the final result.) Now we can evaluate the $k$ integral by residues; writing this as $\frac{1}{2}\int_C[\frac{\varphi^+(k)}{k}] - \frac{1}{2}\int_C[\frac{\varphi^+(-k)}{k}]dk$, we see that the first term contributes nothing as $C$ can be closed in the lower-half $k$-plane. The second term on the other hand contributes $-\pi i \varphi^+(0)$, since in this case $C$ can be closed off in the upper-half $k$-plane. The $S^2$ integration contributes a factor of $4\pi$. Hence

$$I = 2\pi^3 \varphi^+(0).$$

We have thus eliminated any problems in dealing with the spatial 2−sphere dependence of the field. In a similar way

$$\int_{\Gamma^+} \frac{\varphi^-(z)}{z^4} d^4 z = 2\pi^3 \varphi^-(0).$$

Note that Hartog's theorem does *not* apply. A simple corollary is the following

REMOVABILITY LEMMA (RL) : Given $f$ holomorphic on $\overline{CM^{\pm}}$, then

$$\int_{\Gamma^{\mp}} \frac{f}{z^{2n}} d^4 z = 0, \quad n = 1, 0, -1, \ldots$$

*Proof*: Write the integrand as $\frac{f z^{4-2n}}{z^4}$ and use the reproducing kernel.

**The Complex Structure for Spacetime Fields.**

We introduce the complex structure $J$ on spacetime fields as follows. Let $J$ be a linear map from $V$ to itself

$$J : V \longrightarrow V.$$

Then $J$ is defined to act on a field $\phi$ by multiplying its positive frequency part by $i$ and its negative frequency part by $-i$. So writing $\phi = \phi^+ + \phi^-$

$$J[\phi] = J[\phi^+ + \phi^-] = J[\phi^+] + J[\phi^-] = i\phi^+ - i\phi^-.$$

Clearly then

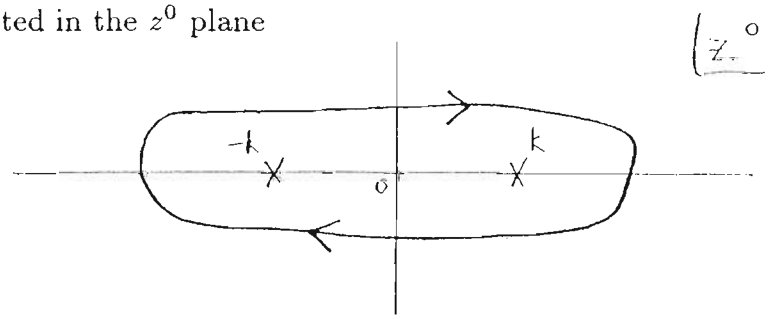
$$J^2 \equiv -1 \text{ acting on all } V$$

as required for $J$ to be a complex structure on $V$. Note that $J$ maps real fields to real fields. The action of $J$ constitutes a *repolarisation* of the field $\varphi \mapsto i(\varphi^+ - \varphi^-)$.

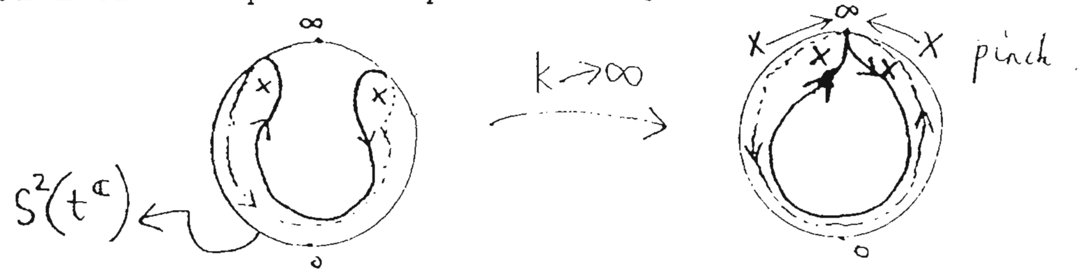
Now from our reproducing kernel above we see that

$$J[\varphi(r)] = \frac{i}{2\pi^3} \int_{\Gamma_J} \frac{\varphi(z)}{(z-r)^4} d^4 z$$

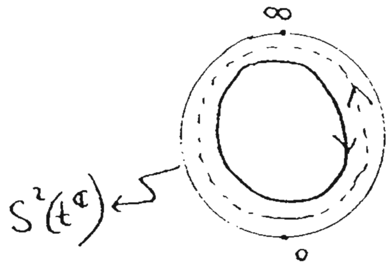
where $\Gamma_J$ is depicted in the $z^0$ plane



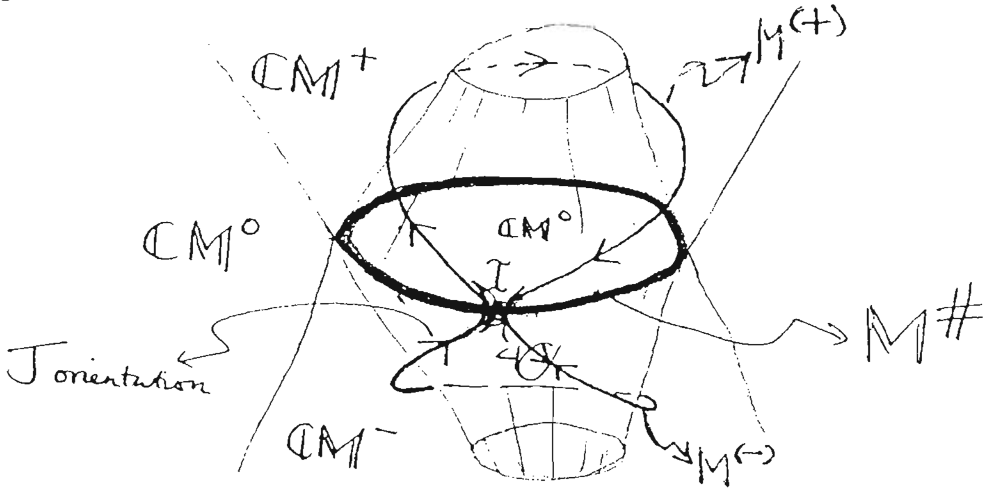
Note that under time reflection T: $z^0 \mapsto -z^0$, the contour reverses orientation so that $J[\varphi]$ changes sign, as must be the case since T interchanges positive and negative frequency parts. On the Riemann sphere of complexified time $\Gamma_J$ is




We see that as $k \to \infty$, $\Gamma_J$ *pinches* at $\infty$ in $z^0$ and splits into two components in the future and past tubes




with opposite orientation with respect to the great circle of real time. It is instructive to see the geometry of the contour in the full four dimensions:

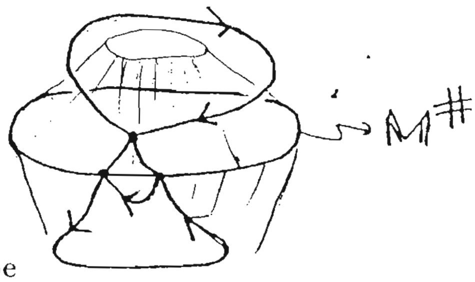



$M^\#$, $M^{(\pm)}$ have topology $S^3 \times S^1$, I denotes the point at infinity. There is only *one* region $CM^0$.

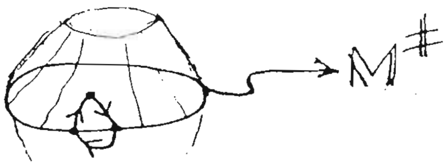
Up to an overall sign, there is another orientation of $\Gamma_J$ – obtained by just flipping the orientation of one of its constituents – this would simply reproduce any field of mixed frequency.

Note that both $M^{(\pm)}$ are contractible – they sit inside $CM^\pm$ which are contractible manifolds – however the positive and negative frequency parts of $\varphi$ extend only to an open *neighbourhood* of $M^\#$ thus preventing contraction of the contours.

Note that the contour must pinch at a point of $M^{\#}$ – for example for the reproducing contour if we instead chose




the extra contribution would be




which is divergent since this encloses a dense set of singularities of the integrand.

**The Complex Structure on Arbitrary Spacelike 3-Surfaces**

It is essential in the consideration of scalar products for zero rest mass fields to understand how the results obtained above enable one to perform $J$ via an integral over an *arbitrary* spacelike 3-surface. In essence, we wish to understand how to obtain a 3-surface integral for $J$ from the 4-dimensional contour integral above, by insertion, holomorphically, of a delta function in imaginary time. We have the following result.

THEOREM:

The action of the complex structure $J$ on any spacetime field or potential (with any index structure) $\varphi$ satisfying $\nabla^2\varphi = 0$ with the decay $\varphi \sim r^{-(1+\gamma)}$, $\gamma > 0$ at spatial infinity, is given by the 3-surface integral

$$J[\varphi] = \frac{1}{2\pi^2}\int_{\Sigma^0} \frac{1}{K^2(x,x')}(\overleftarrow{\nabla}'^a - \overrightarrow{\nabla}'^a)\varphi(x')d^3\Sigma'_a \qquad (*)$$

where $K^2 = (x^a - x'^a)^2$, $\Sigma_a$ is future pointing, and $\Sigma^0$ is once differentiable and constrained to intersect the light cone of $x$ at its vertex.

Remark on time symmetry. $K^2$ determines no time orientation. However both sides of the above repolarisation formula reverse sign under time reflection – the positive and negative frequency parts are interchanged and the normal to the surface of integration changes sign, i.e. the repolarisation formula is *consistent* under T, time reflection symmetry.
Remarks on $K$. If $\delta x = x - x'$ then $K^2 = 0$ if and only if $\delta x$ has real and imaginary parts which are spacelike, null, or zero, and are orthogonal. Therefore $1/K^2$ is non-singular if $\delta x \in CM^{\pm}$. $K$ is symmetric in $x, x'$.
Note that our formula is manifestly Lorentz covariant.

*Proof*: We rely on the results above for 4-dimensional integrals in an essential way. Consider the general expression

$$\omega(f,g) = \int_\Sigma f(\overleftarrow{\nabla^a} - \overrightarrow{\nabla^a}) g \, d^3\Sigma_a.$$

This is a symplectic functional of $f, g$ and the integrand is divergence free provided $\nabla^2 f = \nabla^2 g = 0$. We consider the case $f = 1/K^2$, $g = \varphi$. Note that $\nabla^2 f \equiv 0$ in $M$ even where $f$ itself is singular (see for example Schwinger's papers on quantum electrodynamics c. 1949). However as we shall see the 2-point kernel and the field do *not* play entirely symmetric roles in an important way.

Now we complexify the r.h.s. of $(*)$ above, i.e. allow $x \mapsto z = x - iy$ for real $x, y$. Then the *complex* exterior derivative $d = \partial + \bar\partial$ of the form in $(*)$ vanishes since the integrand is independent of $\bar z$, and is now divergence free in $z$. Thus $(*)$ is independent of $\Sigma^0$ provided $K \neq 0$ and no singularities of $\varphi$ are encountered. In real terms we have the freedom of deformation of a 3-dimensional contour in 8-dimensional space. Now define the linear functionals $I^\pm$:

$$I^\pm[\varphi] := \int_{\Sigma^\mp} \frac{1}{K^2}(\overleftarrow{\nabla'^a} - \overrightarrow{\nabla'^a})\varphi(x')\, d^3\Sigma'_a$$

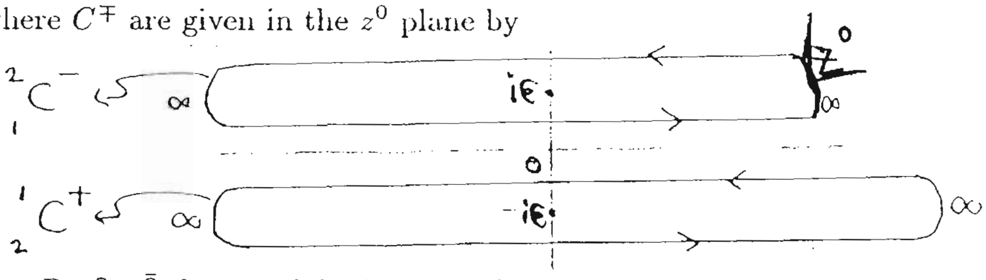
($d^3\Sigma_a = \epsilon_{abcd} dz^b \wedge dz^c \wedge dz^d$), where

$$\Sigma^\mp = \{z^a |\ z^0 = \pm i\epsilon,\ \epsilon > 0 \text{ is real and small; } \mathbf{z} = \mathbf{x} \text{ is real}\}.$$

Then we insert a delta function of imaginary time holomorphically:

$$I^\pm[\varphi] = \frac{1}{2\pi i}\int_{C^\mp} \frac{d^4 z}{z^0 \mp i\epsilon}\left[\frac{-2\varphi(z^a)z^0}{z^4} - \frac{1}{z^2}\cdot\frac{\partial\varphi(z^a)}{\partial z^0}\right]$$

where $C^\mp$ are given in the $z^0$ plane by




By $\partial + \bar\partial$ closure of the form in $(*)$ these expressions are all independent of $\epsilon > 0$. Note that no singularities are encountered since $\varphi$ extends holomorphically in a neighbourhood of real spacetime, and $K^2 \neq 0$ since the contours remain in the future or past tubes. Now clearly $I^\pm[\varphi^\mp] = 0$ since by surface independence the contours can be taken to infinity in the past/future tubes where $\varphi^-$ / $\varphi^+$ decay to zero.

To calculate $I^+[\varphi^+]$ note that in $\int_{C_1^-}$ the second term contributes zero by RL, and by RK the first term is proportional to $\frac{\varphi^+(z)\, z^0}{z^0 - i\epsilon}|_0 = 0$. Thus $\int_{C_1^-} = 0$. Then for $\int_{C_2^-}$ the first term is clearly continuous *at* $\epsilon = 0$, and then by RK is equal to $4\pi^3\varphi^+(0)$. The second term is also continuous at $\epsilon = 0$ where it is equal to

$$-\oint \frac{1}{z_0}\cdot\frac{1}{z^2}\cdot\frac{\partial\varphi^+}{\partial z^0}\, d^4 z.$$

Now RL does not apply because of the pole at $z^0 = 0$, and this integral is

$$2\pi i \int \frac{1}{|\mathbf{x}|^2} \cdot \frac{\partial \varphi^+}{\partial z^0}(0, \mathbf{x})\, d^3\mathbf{x}.$$

Hence

$$I^+[\varphi^+] = -2i\pi^2 \varphi^+(0) + \int \frac{1}{|\mathbf{x}|^2} \cdot \frac{\partial \varphi^+}{\partial t}(0, \mathbf{x})\, d^3\mathbf{x}.$$

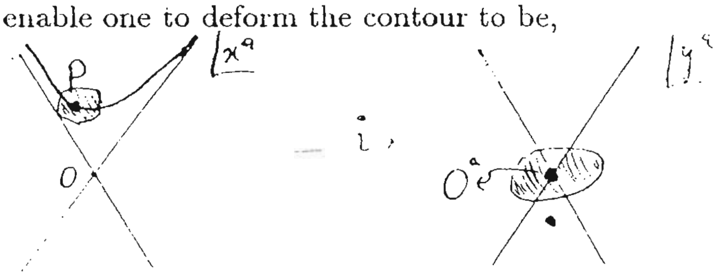
Similarly but taking note of the change in sign in the first term

$$I^-[\varphi^-] = 2i\pi^2 \varphi^-(0) + \int \frac{1}{|\mathbf{x}|^2} \cdot \frac{\partial \varphi^-}{\partial t}(0, \mathbf{x})\, d^3\mathbf{x}.$$

Adding these equations gives

$$I^+[\varphi] + I^-[\varphi] = -2i\pi^2[\varphi^+(0) - \varphi^-(0)] + \int \frac{1}{|\mathbf{x}|^2} \cdot \frac{\partial \varphi}{\partial t}(0, \mathbf{x})\, d^3\mathbf{x}.$$

PROPOSITION:
$I^\pm[\varphi] \equiv 0$ for all $\varphi$ decomposable into $\varphi = \varphi^+ + \varphi^-$.
*Proof*: It suffices to show that $I^+[\varphi^+] = 0$. Surface independence and the decay of $\varphi$ at spatial infinity enable one to deform the contour to be,



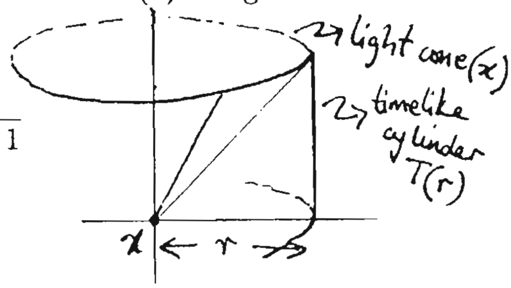

Then for the fixed real part $P$ shown since $P$ lies strictly in the *interior* of the future real light cone of $x$, the integrand is $\partial + \bar{\partial}$ closed for all $z$ in an open neighbourhood of the union of $\overline{CM^+}$, $\overline{CM^-}$. In particular we have $\partial + \bar{\partial}$ closure at $y^a = 0$ since the integrand is holomorphic on any complex line through $P^a + i0^a$ in a sufficiently small neighbourhood. This enables us to deform $\Sigma^-$ passing *through* real spacetime to eventually lie at infinity in the future tube where the integral is zero.

Hence,

$$J[\varphi(x)] = \frac{1}{2\pi^2} \int \frac{1}{|\mathbf{x} - \mathbf{y}|^2} \cdot \frac{\partial \varphi}{\partial t}(0, \mathbf{y})\, d^3\mathbf{y}.$$

Now in the case of a *flat* $\Sigma^0$ in $(*)$ the term from the derivative acting to the left contributes zero, since this creates a factor $(x - x')^a$ which is orthogonal to the measure of integration. Now by surface independence we claim our result is established. It suffices to verify that (a) the integration over the timelike cylinder at infinity vanishes and (b) that we may apply Stokes' theorem to obtain surface independence for a general spacelike 3-surface. As we shall see the latter in fact requires the surface to satisfy a differentiability condition.

To verify (a) if we set $\varphi \sim 1/r^{1+\gamma}$ then the second term of $(*)$ integrated over the timelike cylinder shown gives




$$r^{-(1+\gamma)} \int_0^1 \frac{du}{(u - i\epsilon/r)^2 - 1}$$

$$\sim r^{-(1+\gamma)}[\log r + A]$$

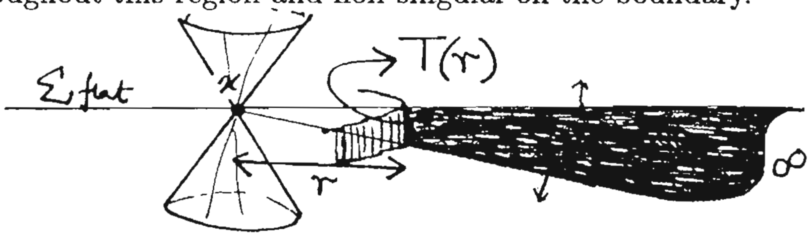
for some constant $A$. The first term contributes

$$r^{-(1+\gamma)} \int_0^1 \frac{du}{[(u - i\epsilon/r)^2 - 1]^2}$$

$$\sim -\frac{B}{r^{1+\gamma}} + \frac{1}{i\epsilon r^\gamma}$$

for some constant $B$. Thus a necessary and sufficient condition for the integral to vanish at infinity is $\gamma > 0$ as stated above.

To verify (b) it suffices to show that the integral over the small timelike surface $T(r)$ shown below tends to zero as the vertex of the light cone $x$ is approached – then surface independence follows from Stokes' theorem applied to the shaded region since the form is closed throughout this region and non-singular on the boundary.




We take the general timelike displacement of the surface to vary with the spacelike distance $r$ from the vertex according to $kr^\alpha$ (†) for real constants $k, \alpha$ such that the surface intersects the light cone at its vertex only, or equivalently $\alpha \geq 1$. Then since the field $\varphi$ and its gradient are bounded and continuous at $x$ the contribution from the second term of $(*)$ is

$$\sim \int_0^{kr^\alpha(<r)} \frac{r^2\,dt}{(t^2 - r^2)}$$

$$\propto r \log[\frac{1 - kr^{\alpha-1}}{1 + kr^{\alpha-1}}].$$

Now for all $\alpha \geq 1$ and $k$ such that the surface is spacelike at $x$ this contribution vanishes in the limit $r \to 0$. The contribution from the first term of $(*)$ is proportional to

$$r^3 \int_0^{kr^\alpha} \frac{dt}{(t^2 - r^2)^2}$$

$kr^\alpha < r$

$$\propto \tanh^{-1}(kr^{\alpha-1}) + \frac{kr^{\alpha-1}}{1 - k^2 r^{2\alpha-2}}.$$

Now for all $\alpha > 1$ this contribution *vanishes* in the limit $r \to 0$. Only in the case $\alpha = 1$ are we left with the contribution

$$\propto \tanh^{-1} k + \frac{k}{1-k^2}$$

wherein $k < 1$ so that the surface is spacelike. This is the case of a conical 3-surface with vertex at the field point $x$. Thus if the surface is only once differentiable the contribution from $T(0^+)$ is zero since we can always bound this contribution above by that from our generic form (†) in the case $\alpha > 1$. This completes the proof of the theorem. I am grateful to N. Woodhouse for suggesting the existence of the extra contribution due to a surface with singular extrinsic curvature.

We have the trivial corollary in terms of the 3-dimensional Laplacian:

$$J[\varphi(\mathbf{x},t)] = -i\Delta^{-1/2}[\frac{\partial\varphi(\mathbf{x},t)}{\partial t}]$$

where $\Delta^{-1/2} f(\mathbf{x}) \equiv \pm\frac{i}{2\pi^2}\int \frac{f(\mathbf{y})}{|\mathbf{x}-\mathbf{y}|^2} d^3\mathbf{y}$. Thus we see that the action of $J$ is *non-local* with respect to the Cauchy data $(\varphi, \partial\varphi/\partial n)$ and that in the case that $\Sigma$ is flat is determined *alone* by the free data of the normal derivative. This gives a relation between elliptic operators on 3-spaces and restrictions to Euclidean spaces of hyperbolic operators on pseudo-Riemannian spaces.

**Concluding remarks**

These results may be used to derive expressions for positive definite norms of massless bosonic fields of arbitrary integer spin as two-point configuration space integrals, requiring *no extraction* of frequency parts or potentials for the principal fields. It is especially interesting in the case of linear gravity that one obtains a positive definite norm as an integral over the linearised phase space variables. This is well defined provided *only* that one can make a choice of foliation of real spacetime by a family of spacelike hypersurfaces – thus the vacuum Einstein equations may *fail* to hold. This gives in principal a way of measuring the difference of two neighbouring spacetime geometries even in the presence of matter. Details of this will appear in a subsequent publication.

I am grateful to L.P. Hughston for stimulating discussions.

# A Minimum Principle for the Cohomological Inner Product on Twistor Space

Franz Müller (e-mail: muller@ihes.fr)

In this note we describe how, in the Dolbeault framework, one can obtain the $SU(2,2)$-invariant inner product on the cohomology groups $H^1(\tilde{\mathcal{P}}_\pm\,,\ \mathcal{O}(-2-n))$ ($\tilde{\mathcal{P}}_\pm$ a small nbh. of $\overline{\mathcal{P}}_\pm \subset \mathcal{P}T$), $n \geq 0$, starting from a pos. def. inner product on representative $(0,1)$-forms and evaluating it for representatives with *minimal norm.* In the standard approach one puts a $SU(4)$-invariant Fubini-Study metric $g$ on $\mathcal{P}T$ which gives a pos. def. $SU(4) \cap SU(2,2)$-invariant inner product on forms [R. et al.], [S]. But the fact that full $SU(2,2)$-invariance is recovered by going to representatives with minimal $g$-norm might have been overlooked.

It follows, at least for $n > 0$, from the general theory [FK] (as referred to in [S]) that the minima are attained by unique g-harmonic representatives satisfying $\overline{\partial}$-Neumann boundary conditions (3.2). Thus, one can just *calculate* to see that these minima agree with the $SU(2,2)$-invariant cohomological norm (for sufficiently many states (3.8)). We shall perform such a calculation (3.6), (3.12) for the case $n = 0$ where in fact one integrates over $\mathcal{P}_0$ generalising (1.1).

**1. The case of $SU(1,1)$ as an Introduction:** We consider the analogue of $H = \mathcal{L}^2(S^1)$ for the hypersurface $\mathcal{P}_0$ in $\mathcal{P}T$. The Hermitian inner product

$$\langle f \,|\, g\rangle = \frac{1}{2\pi i}\int_{|z|=1} \overline{f(z)}g(z)\frac{dz}{z} \quad , \quad z \in \mathbb{C}\,, \tag{1.1}$$

on $H$ is positive definite and induces an orthogonal splitting $H = H_+ \oplus H_-$ into functions extending holomorphically over the upper ($S_+$) and lower ($S_-$) hemisphere of $\mathbb{C}P^1 \sim S^2$. It is furthermore invariant under the action

$$(g * f)(z) = \frac{1}{\bar{b}z+\bar{a}}\, f\Big(\frac{az+b}{\bar{b}z+\bar{a}}\Big) \quad , \quad g^{-1} = \begin{pmatrix} a & b \\ \bar{b} & \bar{a}\end{pmatrix} \in G = SU(1,1)\,. \tag{1.2}$$

This is best seen by thinking of $f$ as a section of $\mathcal{O}(-1)$ over $S_+$ or $S_-$ , represented by a function of homogeneity $-1$ in the homogeneous coordinates $[z_0\,,\ z_1]$ , e.g.

$$f_a = 1/(a_0 z^0 + a_1 z^1) \quad , \quad a = [a_0\,,\ a_1] \in S_\pm \quad . \tag{1.3}$$

The action of $G$ is then induced by the natural action on points $a \in S^2 \backslash S^1$ and the inner product is

$$\langle f_a \,|\, g_b\rangle = \pm(\bar{a}_0 b_0 - \bar{a}_1 b_1)^{-1} \ \text{ or zero} \tag{1.4}$$

depending on whether $a, b \in S_+$ or $S_-$ . The sign is such that the expression is positive for $a = b$. In the same way one gets realisations on $\mathcal{O}(-1-n)$ for all the (holomorphic) discrete series representations of $SU(1,1)$ ($n \in \mathbb{N}$), an invariant inner product for $n > 0$ being [BE]

$$\langle f \,|\, g\rangle_n = \frac{1}{2\pi i}\int_{|z|<1} (1-z\bar{z})^{n-1}\overline{f(z)}g(z)\, d\bar{z}\wedge dz \quad , \quad z \in \mathbb{C}\,. \tag{1.5}$$

All of this has analogues for projective (flat) twistor space $\mathcal{PT} \sim \mathbb{C}P^3$ although one has to work with cohomology groups. $S^1$ is replaced by the five-dimensional real hypersurface

$$\mathcal{P}_0 = \{\,[z]\,|\,h(z,z) = 0\,\} = (SU(2)\times SU(2))/U(1) \quad , \tag{1.6}$$

$$\text{where } h(z,z) = z^0\overline{z}^0 + z^1\overline{z}^1 - z^2\overline{z}^2 - z^3\overline{z}^3 \quad , \tag{1.7}$$

which divides $\mathcal{PT}$ into $\mathcal{P}_+$ and $\mathcal{P}_-$ . We have "elementary states" (sections of $\mathcal{O}(-2)$)

$$f_{ab} = \frac{1}{(a_i z^i)(b_i z^i)} \quad , \quad a \wedge b \in \mathcal{P}_\pm \tag{1.8}$$

which generate 1-cocycles in Čech cohomology via the connecting homomorphism in the Mayer-Vietoris long exact sequence on a cover with two Stein sets [EH]. The positive definite $SU(2,2)$-invariant inner product on these is

$$\langle [f_{ab}]\,|\,[f_{cd}]\rangle_{h,-2} = (h(a,c)h(b,d) - h(a,d)h(b,c))^{-1} \text{ or zero,} \tag{1.9}$$

depending on the respective position of the lines $a \wedge b$, $c \wedge d$.

**2.** **(0, 1)-forms on $\mathcal{P}_0$ :** In a fixed basis where $h$ has the form (1.7) we fix

$$g(z,z) = z^0\overline{z}^0 + z^1\overline{z}^1 + z^2\overline{z}^2 + z^3\overline{z}^3 \quad . \tag{2.1}$$

Simultaneously, $g$ denotes the Fubini-Study metric induced on $\mathcal{PT}$. We use affine coordinates

$$(\zeta, \eta, \xi) = (z^1/z^0\,,\, z^2/z^0\,,\, z^3/z^0) \tag{2.2}$$

on a dense open cell $\mathbb{C}P^3 \backslash \mathbb{C}P^2$. A $g$-normal to $\mathcal{P}_0$ of constant $g$-length is given by

$$n = \eta\partial_\eta + \xi\partial_\xi + \overline{\eta}\partial_{\overline{\eta}} + \overline{\xi}\partial_{\overline{\xi}} \quad . \tag{2.3}$$

We then naturally define (0, 1)-forms $\omega$ on $\mathcal{P}_0$ to be those ($C^\infty$) 1-forms which vanish on holomorphic tangent vectors. If such a (0, 1)-form $\omega$ is the restriction of a $\overline{\partial}$-*closed* form on a nbh. $\mathcal{U}$ of $\mathcal{P}_0$ then it satisfies the boundary CR-equations

$$\overline{\partial}_b\omega = 0 \iff v \lrcorner d\omega = 0 \ \forall v \in T^{0,1}\mathcal{PT} \cap T_{\mathbb{C}}\mathcal{P}_0 \tag{2.4}$$

where $d$ is the exterior derivative on $\mathcal{P}_0$. Now we can complete to a Hilbert space of (0, 1)-forms on $\mathcal{P}_0$ with the inner product

$$\langle \omega\,|\,\eta\rangle_g = \int_{\mathcal{P}_0} \eta \wedge \overline{*}_{\tilde{g}}(\omega) \tag{2.5}$$

where $\overline{*}_{\tilde{g}}$ is the Hodge star with respect to the metric $\tilde{g}$ induced from $g$ on $\mathcal{P}_0$. This is obviously a positive definite inner product on arbitrary (square-integrable) 1-forms on $\mathcal{P}_0$, invariant under the transformations in $SU(2,2)_h \cap SU(4)_g$ . It is however *not* invariant under general transformations in $SU(2,2)$ . ¿From (1.1) it is clear that one should try to construct a fully invariant — on the level of cohomology — inner product from the "boundary values" of elements in the cohomology groups $H^1(\mathcal{P}_\pm\,,\,\mathcal{O}(-2))$ of $\mathcal{P}_\pm$ . It is surprising that this can be achieved with a formula (2.5), which depends upon the choice of $g$, by picking unique $g$-harmonic representatives satisfying the $\overline{\partial}$-Neumann conditions on the boundary $\mathcal{P}_0$.

**3. The $SU(2,2)$-invariance on normalised representatives :** An orthogonal basis of the Hilbert space completion of $H^1(\widetilde{\mathcal{P}}_+, \mathcal{O}(-2))$ with respect to the cohomological inner product is given by the classes generated by the following $g$-harmonic $(0,1)$-forms [EP],[S], written in homogeneous coordinates

$$\omega_{n,ij} = a_{n,ij}(\overline{z}_0 d\overline{z}_1 - \overline{z}_1 d\overline{z}_0) \quad , \quad a_{n,ij} = \frac{\overline{z}_0^{n-i}\overline{z}_1^{i}\, z_2^{n-j}\, z_3^{j}}{(z_0\overline{z}_0 + z_1\overline{z}_1)^{2+n}} \quad , \quad n \in \mathbb{N}\, ;\ i,j = 0,\ldots,n\ . \tag{3.1}$$

Furthermore, they satisfy the $\overline{\partial}$-Neumann boundary conditions [FK]

$$\sigma(\overline{\partial}_g^*, dr)\omega = -\pi_{0,1}(n)\lrcorner\omega = -n\lrcorner\omega = 0 \quad \text{at } \mathcal{P}_0 \tag{3.2}$$

where $\sigma$ is the symbol of an operator and $\pi_{0,1}$ is the antiholomorphic projection of a vector field. Notice that the $\omega_{n,ij}$ are also $h$-harmonic in the sense

$$\overline{\partial}\omega_{n,ij} = \overline{\partial}_h^*\omega_{n,ij} = 0 \quad \forall n,i,j \tag{3.3}$$

but the associated Laplacian is not elliptic. If we are presented with $(0,1)$-forms $\omega$, $\eta$ on some nbh. $\mathcal{U}$ of $\mathcal{P}_0$ which extend $\tilde{\omega}$, $\tilde{\eta}$ and one of which satisfies (3.2) then (2.5) simplifies to

$$\langle\tilde{\omega}\,|\,\tilde{\eta}\rangle_g = \int_{\mathcal{P}_0} g^*(\omega\,,\,\eta)\, n\lrcorner\Omega_g \tag{3.4}$$

where $g^*$ is the dual of $g$ and $\Omega_g$ is its determinant. (This is probably the "better" definition anyway although seemingly less intrinsic.) We calculate

$$\langle\omega_{n,ij}\,|\,\omega_{m,kl}\rangle_{g,-2} = \int_{\mathcal{P}_0} \frac{\overline{a}_{n,ij}a_{m,kl}(z^0\overline{z}^0 + z^1\overline{z}^1)\, g(z,z)}{g^{-2}(z,z)}\, n\lrcorner\Omega_g \quad \text{with}$$

$$n\lrcorner\Omega_g = n\lrcorner\left(\frac{1}{2i}\right)^3\frac{d\zeta\wedge d\overline{\zeta}\wedge d\eta\wedge d\overline{\eta}\wedge d\xi\wedge d\overline{\xi}}{(1+\zeta\overline{\zeta}+\eta\overline{\eta}+\xi\overline{\xi})^4} = \frac{2}{(2i)^3}\,\frac{(\eta\overline{\eta}+\xi\overline{\xi})d\zeta\wedge d\overline{\zeta}\wedge d\eta\wedge d\overline{\eta}\wedge d\xi/\xi}{(1+\zeta\overline{\zeta}+\eta\overline{\eta}+\xi\overline{\xi})^4}\ , \tag{3.5}$$

and thus

$$\begin{aligned}\langle\omega_{n,ij}\,|\,\omega_{m,kl}\rangle_{g,-2} &= \frac{1}{(2i)^3}\int_{\mathcal{P}_0}\frac{\zeta^i\overline{\zeta}^k\overline{\eta}^{n-j}\eta^{m-l}\overline{\xi}^j\xi^l d\zeta\wedge d\overline{\zeta}\wedge d\eta\wedge d\overline{\eta}\wedge d\xi/\xi}{(1+\zeta\overline{\zeta})^{3+m+n}}\\ &= \pi^3\delta_{nm}\delta_{ik}\delta_{jl}\left((1+n)^2\binom{n}{i}\binom{n}{j}\right)^{-1}\ .\end{aligned} \tag{3.6}$$

The space generated by the $g$-harmonics $\{\omega_{n,ij}\}$ is not invariant under $SU(2,2)$. Rather, we obtain an basis of an $sl_{\mathbb{C}}(4)$-module, orthogonal in cohomology, from the lowest weight vector $\omega_0 := \omega_{0,00}$ by the application of step operators

$$X_+^i\, Y_+^j\, E_+^n\,\omega_0 \quad , \quad n \in \mathbb{N}\, ;\ i,j = 0,\ldots,n \tag{3.7}$$

where $X_+$, $Y_+$ are the step operators in $su_{\mathbb{C}}(2)\otimes\mathbb{I}$ and $\mathbb{I}\otimes su_{\mathbb{C}}(2)$ complemented by $E_+$ (specified below) to generate the whole action of $sl_{\mathbb{C}}(4)$. Since $\langle\ |\ \rangle_g$ is invariant under $SU(2)\times SU(2)$ we only have to compare

$$\langle [E_+^n\,\omega_0]\,|\,[E_+^n\,\omega_0]\rangle_h \leftrightarrow \langle \omega_{n,00}\,|\,\omega_{n,00}\rangle_g \tag{3.8}$$

in order to ascertain the full invariance of $\langle\ |\ \rangle_g$ on normalised representatives, because we claim

**Lemma:** $E_+^n\,\omega_0$ and $c_n\,\omega_{n,00}$ are *cohomologous* for the right choice of $c_n\in\mathbb{Z}$ and $E_+\in sl_{\mathbb{C}}(4)$. For the $\bar\partial$-closed (0,1)-forms

$$\omega_{ab} := \frac{(a_i\hat z^i)(b_i d\hat z^i)-(b_i\hat z^i)(a_i d\hat z^i)}{[(a_i z^i)(b_i\hat z^i)-(b_i z^i)(a_i\hat z^i)]^2} \tag{3.9}$$

which correspond to the Čech representatives $f_{ab}$ of (1.8) one obtains the cohomological inner product as in (1.9). Here $z\to\hat z$ is the map

$$(z^0,\,z^1,\,z^2,\,z^3)\to(\hat z^0,\,\hat z^1,\,\hat z^2,\,\hat z^3)=(-\bar z^1,\,\bar z^0,\,-\bar z^3,\,\bar z^2) \tag{3.10}$$

which is invariant under $SU(2)\times SU(2)$ and can be viewed as the antipodal map on the fibres of a fibration [A] $S^2\longrightarrow \mathcal{P}T\longrightarrow S^4$ which restricts to a fibration $S^2\to\mathcal{P}_0\to S^3$ of $\mathcal{P}_0$. We choose $E_+=\partial_{a_2}(.)|(a_i=\delta_i^0)$ such that

$$E_+^n\,\omega_0=(c_i\partial_{a_i})^n\,\omega_{ab}\big|_{a_i\,=\,\delta_i^0\,,\;b_i\,=\,\delta_i^1\,,\;c_i\,=\,\delta_i^2\,,\;d_i\,=\,\delta_i^3}\ , \tag{3.11}$$

(similarly, we could choose $X_+=b_i\partial_{a_i}(.)|_{\dots}$ and $Y_+=d_i\partial_{c_i}(.)|_{\dots}$). See [BE] §11.4 for a similar notation. From (1.9) one quickly computes

$$\langle [E_+^n\,\omega_0]\,|\,[E_+^n\,\omega_0]\rangle_h=\partial_{\bar a_2}^n\,\partial_{a_2}^n\,\langle[\omega_{ab}]\,|\,[\omega_{ab}]\rangle_h|_{\dots}=(n!)^2\ . \tag{3.12}$$

**Proof of the Lemma:** In affine coordinates one computes that on $\mathcal{O}(-2)$ one has

$$E_+=\mathcal{L}_v-2\eta \text{ where } v=\bar\xi\partial_{\bar\zeta}-\eta(\zeta\partial_\zeta+\eta\partial_\eta+\xi\partial_\xi) \tag{3.13}$$

and

$$(\mathcal{L}_v-2\eta)\frac{\eta^n\,d\bar\zeta}{(1+\zeta\bar\zeta)^{2+n}}=-(n+2)\frac{\eta^{n+1}\,d\bar\zeta}{(1+\zeta\bar\zeta)^{3+n}}+\bar\partial\{\frac{\eta^n\bar\xi}{(1+\zeta\bar\zeta)^{2+n}}\}\ . \tag{3.14}$$

Thus, by induction

$$[E_+^n\,\omega_0]=(-1)^n(n+1)![\omega_{n,00}]\ . \tag{3.15}$$

A comparison of (3.6) and (3.12) now yields

**Theorem:** Let $\langle\ |\ \rangle_h$ denote the positive definite inner product on the cohomology groups $H^1(\widetilde{\mathcal{P}}_\pm,\,\mathcal{O}(-2))$ invariant under $SU(2,2)=SU_h$ (for example given by linear extension of (1.9) but properly defined by the twistor transform [DE]), where $h$ is an indefinite Hermitian form of signature $(+,+,-,-)$ on $\mathbb{C}^4$ which defines $\mathcal{P}_\pm$ and $\mathcal{P}_0$. Let $g$ be the positive definite

Hermitian form obtained from $h$ by a flip of signs (in some $h$-diagonal basis) with associated unitary group $SU_g$ and Fubini-Study metric $g_{-2}$ on $\mathcal{O}(-2)$ and define

$$\langle \omega \,|\, \eta \rangle_{g,-2} = \int_{\mathcal{P}_0} \eta \wedge \bar{*}_{\tilde{g}_{-2}}(\omega) \tag{3.16}$$

where $\tilde{g}_{-2}$ is the restriction of $g_{-2}$ to $\mathcal{P}_0$ invariant under $SU_h \cap SU_g$ and $\omega$, $\eta$ are square integrable (0,1)-forms on $\mathcal{P}_0$. Let $c_i$ $(i = 1, 2)$ be in the above cohomology groups. One has, for some $g$-independent constant $k \in \mathbb{R}_+$

$$\langle c_1 \,|\, c_2 \rangle_h = k \, \langle \omega_1^g \,|\, \omega_2^g \rangle_{g,-2} \tag{3.17}$$

where $\omega_i^g \in c_i$ are the (restrictions of the) unique g-harmonic representatives which satisfy the $\bar{\partial}$-Neumann boundary conditions. They are characterised by having minimal $g$-norm within their class. □

The last statement follows from the fact that the $\bar{\partial}$-Neumann conditions (3.2) on $\omega$ are equivalent to

$$\langle \omega \,|\, \bar{\partial}\phi \rangle_{g,-2} = 0 \quad \forall \phi \in \Gamma(\mathcal{U}(\mathcal{P}_0), \mathcal{O}(-2)) \ . \tag{3.18}$$

**Remarks**: We can consider the two spaces $H^1(\tilde{\mathcal{P}}_\pm, \mathcal{O}(-2))$ simultaneously and obtain an orthogonal splitting of "boundary values" as in the case of $SU(1,1)$. A weak form of (2.4) should be enough to characterise representatives in this total space.
We also remark that (3.6) makes it very easy to sum the $g$-harmonics to a (Szegö-type) kernel which is in fact different from the usual one [W] appearing in the twistor transform, the latter being a *g-unbounded* operator ! One would expect this kernel $K$ to fit naturally into an integral representation (a so-called homotopy formula [HP]) of an arbitrary $(0, 1)$-form on $\mathcal{P}_+$, say, viz.

$$\omega(z) = c\Big(\int_{\mathcal{P}_0} \omega(\zeta) \wedge K(\zeta, z) - \int_{\mathcal{P}_+} \bar{\partial}\omega(\zeta) \wedge K_1(\zeta, z)\Big) + \bar{\partial}_z \int_{\mathcal{P}_+} \omega(\zeta) \wedge K_2(\zeta, z) \,\Big) \ . \tag{3.19}$$

One can ponder the themes in this article from a physical point of view: Broken symmetry, normalisation, minimum principle, ...

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# COMPLEX NUMBERS

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# QUANTUM MECHANICS

A One-Day Seminar will be held at the Sub-Faculty of Philosophy, University of Oxford, 10 Merton Street, Oxford on

## Saturday, 3 June, 1995

The aim of the seminar is to survey the uses of complex numbers and complex structures in existing quantum theory and to consider the role they might play in quantum gravity. Some abstracts are given overleaf.

### Programme

10:00 Registration and coffee (Lunch 12:45, Tea 16:30)

10:30 L. P. Hughston: "Complex Geometry and Quantum Probability"

11:30 J. S. Anandan: "A Geometric View of Quantum Theory"

14:15 L. J. Mason: "Generating Spacetime from Complex Numbers"

14.45 A. P. Hodges: "The Twistor Diagram Program"

15:15 R. F. Streater: "Complex Numbers and Complex Structures in Quantum Mechanics"

17:00 J. B. Barbour: "Why *i* and Is It in Quantum Gravity?"

17:15 G. W. Gibbons: "How the Complex Numbers Got into Physics"

Anyone interested is welcome to attend; time has been included in the programme for questions and comments from participants. Tea, coffee, and biscuits will be available at the Sub-Faculty from 10:00 and during the lunch and tea breaks. There are several restaurants near the Sub-Faculty, but participants are encouraged to bring sandwiches for a picnic lunch in the nearby Botanical Gardens. There is no registration fee but please advise either of the organizers if you plan to attend. **Organizers:**
Julian B. Barbour, College Farm, South Newington, Banbury, Oxon, OX15 4JG (Tel: 01295-720492; fax: 01295-721851). Dr Harvey Brown, Sub-Faculty of Philosophy, University of Oxford, 10 Merton Street, Oxford, OX1 4JJ (Tel: 01865-276930; fax: 01865-276932; e-mail: harvey.brown@philosophy.oxford.ac.uk.).

## Abstracts and References

**L. P. Hughston:** "Complex Geometry and Quantum Probability." A survey of the basic constructions of ordinary quantum mechanics is presented by the use of the formalism of complex projective spaces and the Fubini–Study metric. In this way the various probabilistic assumptions that go into the interpretation of quantum mechanics can be formulated in a geometric language. The geometric view helps to clarify many aspects of the ordinary theory, but also highlights some of its problems, and forms a basis for possible generalisations. References:
L. P. Hughston (1995) "Geometric aspects of quantum mechanics," in *Twistor Theory* (S. A. Huggett, ed.), Marcel Dekker; L. P. Hughston (1995) "Geometry of stochastic state vector reduction" (preprint); L. P. Hughston, R. Josza, and W. K. Wooters (1993) "A complete classification of quantum ensembles having a given density matrix," *Phys. Letts.* A 183, 14–18.

**J. S. Anandan:** "A Geometric View of Quantum Theory." A new method of observing the wave function of a single quantum system and the geometric phase are used to justify operationally a geometric formulation of quantum theory in the quantum state space (the set of rays of the Hilbert space or the projective Hilbert space). Some generalizations and modifications of quantum theory are considered from this point of view.

**J. B. Barbour:** "Why *i* and is it in quantum gravity." My interest is this: Is there *one specific way* in which complex numbers enter quantum mechanics? If so, what is the corresponding physical phenomenon that makes this necessary? If, as Pauli argued, complex wave functions are needed to ensure unitarity, what happens in quantum gravity, in which there may be neither time nor unitarity? Does some other effect enforce the appearance of complex structures in quantum gravity? References: Chen Ning Yang, "Square root of minus one, complex phases and Erwin Schrödinger," in *Schrödinger: Centenary Celebration of a Polymath* (C. W. Kilmister, ed), CUP (1987); W. Pauli, Z. *Physik* **80**, 573 (1933); J. B. Barbour, *Phys. Rev.* D, 5422 (1993).

**G. W. Gibbons:** "How the Complex Numbers Got into Physics." I review the role that complex numbers play in setting up the usual quantum mechanical formalsim and I argue that there may be circumstances under which this is not possible, for example, if spacetime does not admit a global time orientation. I also argue that in currently popular models of quantum cosmology it makes sense to say that the complex numbers emerged contemporaneously with the emergence of a classical notion of time. References: G. W. Gibbons, *Nucl. Phys.* **B410** (1993) 117–142; *Nucl. Phys.* **B271** (1986) 495–508; *Int. J. Mod. Phys.* **D3** (1994) 61–70.

# TWISTOR NEWSLETTER No. 39

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Short contributions for **TN 40** should be sent to


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