New Ideas for the Googly Graviton

In my articles in $\mathbb{T}$N 31(pp.6–8), 32(1–5), 38(1–9), 39(1–5), in Gravitation and Modern Cosmology (eds. A. Zichichi, N. de Sabbata, & N. Sánchez; Plenum 1991), in Twistor Theory (ed. S. Huggett; Marcel Dekker 1995); and in L.J.M. & R.P. $\mathbb{T}$N 37(1–6), J.F. $\mathbb{T}$N 37(7–9), J-P.N. $\mathbb{T}$N 39(6–10), and elsewhere, it has been suggested that the appropriate twistor-space procedure for encoding the structure of an arbitrary vacuum space-time $\mathcal{M}$ would be to find the space $\mathcal{T}$ of "charges" for helicity $\frac{3}{2}$ massless fields — or, rather, for potentials $\sigma_{A'B'}^{\ \ \ C}$ for such fields. This suggestion was based on the following two observations:

(1) the field equations (Dirac or Rarita-Schwinger) for $\sigma_{A'B'}^{\ \ \ C}$ are consistent iff $\mathcal{M}$ is Ricci-flat,

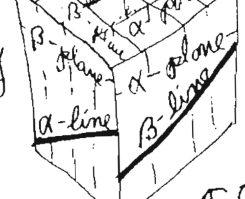
(2) in Minkowski space $\mathbb{M}$, the space of charges for $\sigma_{A'B'}^{\ \ \ C}$ is standard twistor space $\mathbb{T}$.

The programme, then, is to try to find the appropriate notion of "charge" for such fields for a general Ricci-flat $\mathcal{M}$, and then to try to see how to reconstruct $\mathcal{M}$ (with vacuum equations automatically satisfied) from the "twistor space" $\mathcal{T}$ of such charges.

In general, there appear to be severe difficulties standing in the way of constructing an appropriate twistor space in accordance with these ideas (cf. particularly R.P. in $\mathbb{T}$N 33(1–6) and the above ref. in Grav. & Mod. Cosmol. 1991). The most plausible procedure for an asymptotically flat (analytic) $\mathcal{M}$ seems to be to go to infinity, say to $\mathscr{I}^+$, and see how best to define the space of helicity 3/2 charges in terms of "glitches" in the potentials at $\mathscr{I}^+$ or $\mathbb{C}\mathscr{I}^+$. These glitches would have to be not only in the first potential $\sigma_{A'B'}^{\ \ \ C}$ but also in a second potential $\rho_{A'}^{\ BC}$, all restricted to $\mathscr{I}^+$ (or $\mathbb{C}\mathscr{I}^+$). Recent work by Jorge Frauendiener, Ted Newman, and Joy Ghose has explored the nature of $\mathcal{T}$ constructed in this way, and seems to confirm my general expectation that $\mathcal{T}$ ought to turn out to be the asymptotic twistor space of $\mathcal{M}$.

However, according to the understanding that has existed to date, the asymptotic twistor space of $\mathcal{M}$ encodes only the anti-self-dual (ASD) part of the radiation field of $\mathcal{M}$ — according to the $\mathcal{H}$-space/non-linear graviton construction — and not the self-dual (SD) part. (It should be borne in mind that all this information is to be encoded holomorphically, the operation of complex conjugation, or reality structure, not being something that one may call upon. Thus, we are, for the time being at least, operating within the framework of <u>complex</u> general relativity.) Accordingly, it appears to be essential to re-examine the <u>googly programme</u> for ascertaining how the SD part of the radiation field may also be holomorphically encoded in the structure of $\mathcal{T}$ (and not simply in that of the dual asymptotic twistor space $\mathcal{T}^*$). Ultimately, some information about sources for the gravitational field would also have to be encoded somehow. This article describes various ideas that, I believe, point to some significant new progress towards a solution of the googly problem.

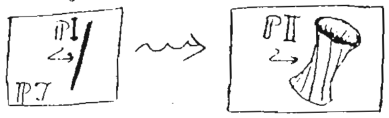
Let us first recall the way in which asymptotic twistor space arises in terms of $\mathbb{C}\mathscr{I}^+$. We may regard $\mathbb{C}\mathscr{I}^+$ as a union of α-planes, or alternatively as a union of β-planes. There are null geodesics on $\mathbb{C}\mathscr{I}^+$ of three types: <u>α-lines</u>, <u>β-lines</u>, and <u>generators</u>, the generators being limiting cases of α-lines and also of β-lines. Each α-line lies on a unique β-plane and each β-line on a unique α-plane (all on $\mathbb{C}\mathscr{I}^+$), each generator lying on one of each. The "finite" part of projective asymptotic twistor space $\mathbb{P}\mathcal{T}^b$ is simply the space of α-lines (and the corresponding dual twistor space $\mathbb{P}\mathcal{T}^{*b}$ is the space of β-lines).



$\mathbb{C}\mathscr{I}^+$

In the flat case of $\mathbb{M}$, $\mathbb{P}\mathcal{T}^b = \mathbb{P}\mathbb{T}^b$, which is the ordinary twistor space $\mathbb{P}\mathbb{T}$, but with the line $\mathbb{P}\mathrm{I}$ (representing infinity) removed. The entire space $\mathbb{P}\mathbb{T}$ arises if we include the α-planes on $\mathbb{C}\mathscr{I}^+$ in addition to the α-lines, these various α-planes providing the points of $\mathbb{P}\mathrm{I}$. This procedure could be adopted in the case of a general $\mathbb{C}\mathscr{I}^+$ also, but then it turns out that the resulting space $\mathbb{P}\mathcal{T}$ is not a smooth complex manifold, being singular along $\mathbb{P}\mathrm{I}$. In some appropriate sense, it is the nature of this singularity along $\mathbb{P}\mathrm{I}$ that contains all the googly gravitational information.

One standard way to proceed is to <u>blow up</u> the twistor space, so the line $\mathbb{P}\mathrm{I}$ becomes a quadric surface $\mathbb{P}\mathrm{II}$: In the flat case, this is achieved by $\frac{1}{z} \rightsquigarrow \frac{1}{z\bar{z}}$,



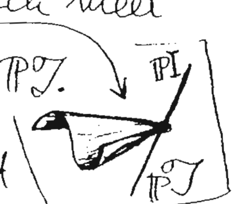
i.e. by

$$(\omega^A, \pi_{A'}) \rightsquigarrow (\omega^A \bar{\pi}^{B'}, \pi_{A'}\bar{\pi}^{B'}) = (\omega^{AB'}, \pi_{A'}{}^{B'})$$

where $\omega^{AA'}\pi_{A'}{}^{B'} = 0$, $\pi_{A'}{}^{A'} = 0$.

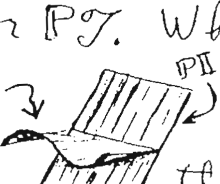
The space $\mathbb{T}^\#$ of blown up twistors is the space of such $(\omega^{AB'}, \pi_{A'}{}^{B'})$. For the space $\mathbb{PT}^\#$ of projective blown up twistors, we factor out by an overall factor. $\mathbb{PT}^\#$ is a (non-singular) complex 3-manifold in $\mathbb{CP}^6$. The projective blown-up twistors (elements of $\mathbb{PT}^\#$) represent α-lines on $\mathbb{C}\mathscr{I}^+$ together with their limits, the generators of $\mathbb{C}\mathscr{I}^+$. This limiting procedure is achieved by applying $\gamma \to 0$ to $(\gamma^{-1}\omega^A, \gamma\pi_{A'})$, which leaves us with blown-up twistors of the form $(\omega^A\pi^{A'}, 0)$, where $\pi^{A'}$ specifies a β-plane and ω^A an α-plane on $\mathbb{C}\mathscr{I}^+$ these intersecting in the generator in question. The α-lines and generators are scaled in terms of parallelly propagated covectors "pointing along" them (a conformally invariant notion), which gives the scalings needed to define non-projective blown-up twistors (elements of $\mathbb{T}^\#$).

In the curved case, this definition in terms of scaled α-lines and scaled generators works just as well as in the flat case, and provides us with the asymptotic (blown-up) twistor space $\mathscr{T}^\#$ (provisional definition). Dropping the scaling, we get $\mathbb{P}\mathscr{T}^\#$. It is not hard to see that $\mathbb{P}\mathscr{T}^\#$ is a non-singular complex manifold. The trouble, however, is that now the (googly) information of the SD radiation field seems to have got lost. This information lies in the location of the β-lines on $\mathbb{C}\mathscr{I}^+$. A β-line can be represented by the family of α-lines which meet it, i.e. by the corresponding "crinkly cone" in $\mathbb{P}\mathscr{T}$.



$\mathbb{C}\mathscr{I}^+$

The fact that these crinkly cones cannot be consistently represented as having flat tangent planes at their vertices on $\mathbb{PI}$ is a manifestation of the fact that $\mathbb{PI}$ is singular for $\mathbb{P}\mathscr{T}$. When we blow-up, we get smooth surfaces looking like



What is being proposed here is that if we specify the appropriate structure on $\mathbb{P}\mathscr{T}$ or $\mathbb{P}\mathscr{T}^\#$, then these surfaces will be singled out. It is not expected that this structure will manifest itself on $\mathscr{T}^\flat$, but rather in the way that $\mathbb{I}$ (or whatever is appropriate) is attached to $\mathscr{T}^\flat$ to yield $\mathscr{T}^\#$.

In TN 23 (1-4), the forms

$$\delta = \overline{Z\,dZ} \qquad \text{and} \qquad \mathcal{D} = \frac{1}{6}\,\overline{Z\,dZ\wedge dZ\wedge dZ},$$

both well defined on $\mathcal{T}^b$, with

$$\delta_\wedge \mathcal{D} = 0, \quad \delta_\wedge d\delta = 0,$$

were considered. It was noted that in the flat case the quantity

$$\Gamma = \delta \otimes \mathcal{D}$$

(a $\boxplus$-tableau quantity) is regular (and non-zero) on $\mathbb{T}^\#$, whereas $\mathcal{D}$ blows up at $\mathbb{I}$ and δ goes to zero there. We can see this by choosing local coordinates ($w = 0$ giving $\mathbb{I}$):

$$w = (Z^2)^2 = \pi_{0'}\pi_{0'}, \quad x = \frac{Z^1}{Z^0} = \frac{\omega^1}{\omega^0}, \quad y = Z^0 Z^2 = \omega^0 \pi_{0'}, \quad z = \frac{Z^3}{Z^2} = \frac{\pi_{1'}}{\pi_{0'}}$$

whence

$$\delta = w\,dz, \quad \mathcal{D} = \frac{y}{w}(w\,dy - y\,dw)_\wedge dx_\wedge dz,$$

$$\Gamma = dz \otimes dz_\wedge (w\,dy - y\,dw)_\wedge y\,dx_\wedge dz.$$

In TN 23, the hope was expressed that some differential operator might be found, which is coordinate independent for objects of the nature of Γ, and whose kernel would, in some appropriate sense, be objects defined by twistor functions homogeneous of degree -6. In fact, there *is* such an operator, defined as follows. I shall use the notation $\textcircled{\scriptsize$\wedge$}$ for a product of a 2-form with an n-form defined by

$$(dp_\wedge dq)\,\textcircled{\scriptsize$\wedge$}\,\alpha = dp \otimes dq_\wedge \alpha - dq \otimes dp_\wedge \alpha$$

and extended by linearity to arbitrary 2-forms. Then the required operator is given by

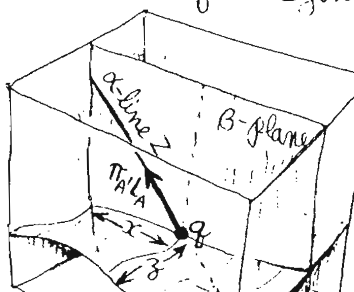
$$D(\delta \otimes \mathcal{D}) = d\delta\,\textcircled{\scriptsize$\wedge$}\,\mathcal{D} - \delta \otimes d\mathcal{D}$$

whence (by $\delta_\wedge \mathcal{D} - 0$)

$$D((\lambda\delta)\otimes\mathcal{D}) = D(\delta \otimes (\lambda \mathcal{D})).$$

It follows that D does indeed operate on Γ itself, not depending on the particular way in which it is represented. $D\Gamma$ is a $\boxplus$-tableau tensor. We also have $D(f\Gamma) = 0$ iff f has homogeneity degree -6. We might hope to see a role for this in the encoding of googly information into the structure of $\mathcal{T}^\#$, but this has so far remained somewhat elusive. (For example, we might have found that there is some ambiguity in Γ in its natural extension to $\mathbb{I}$, but that $D\Gamma$ is defined unambiguously there.) However, Γ appears also to be unambiguously regular on $\mathscr{I}^{\#}$ (as here defined) in the curved case.

To see this, let us go to $\mathbb{C}\mathscr{I}^+$, a suitable conformal factor Ω having been chosen to make the metric and ε-spinors finite. We can (locally) make the induced metric on $\mathbb{C}\mathscr{I}^+$ flat and (locally) choose spinors ι^A, $\tilde{\iota}^{A'}$ constant on $\mathbb{C}\mathscr{I}^+$, where the vectors $\iota^A\tilde{\iota}^{A'}$ are tangent to the generators of $\mathbb{C}\mathscr{I}^+$. However, the spinors o^A, $\tilde{o}^{A'}$, needed to complete a spinor basis at each point of $\mathbb{C}\mathscr{I}^+$, cannot be chosen constant unless the Bondi-Sachs news function vanishes. There is curvature in the connection along α-planes on $\mathbb{C}\mathscr{I}^+$ if there is SD gravitational radiation and along β-planes if there is ASD gravitational radiation. Choose (locally) an arbitrary cut $\mathscr{C}$ of $\mathbb{C}\mathscr{I}^+$ (smooth cross-section of the generators) and mark the point q where an α-line or generator intersects $\mathscr{C}$. We adopt a local twistor description of the α-lines, so that along each α-line a twistor corresponding to that α-line has the description



$$Z^\alpha = (0, \pi_{A'}),$$

each tangent vector the α-line being proportional to $\iota^A\pi^{A'}$. (It is important to bear in mind that, because of the conformal rescaling, this "$\pi_{A'}$" is quite different from the "$\pi_{A'}$" used as a flat-twistor-space coordinate above, where we had (old $\pi_{A'}$) $Z = (0, \pi^{A'})$.) We recall (R.P. & W.R. Spinors & Space-Time Vol. 2, pp. 376, 65; TN 23, p2) that on $\mathbb{C}\mathscr{I}^+$ the infinity twistor takes the form

$$\Pi = I_{\alpha\beta} = \begin{bmatrix} 0 & -i\iota_A\tilde{\iota}^{A'} \\ i\tilde{\iota}^{A'}\iota_B & 0 \end{bmatrix}$$

($p'=0$, $A=1$, being chosen here, in the notation of S.&S.-T.). Thus $\frac{1}{2}\mathcal{Z}$ has the local twistor description

$$Z^\alpha I_{\alpha\beta} = (L_B(i\tilde{L}^{A'}\pi_{A'}),\ 0) = i\pi_{1'}(L_B,\ 0),$$

so the covector associated with $\frac{1}{2}\mathcal{Z}$ is

$$p_a = (i\pi_{1'})\, L_A \pi_{A'}\,.$$

Local coordinates for $\mathcal{T}^\#$ (for the α-line non-tangent to $\mathcal{C}$) can be taken to be the two independent components of p_a and the two coordinates (x, y) specifying the point q on $\mathcal{C}$. (See figure on previous page). Indeed, we obtain local coordinates (w, x, y, z), closely analogous to (and direct generalizations of) those introduced for $\mathbb{T}^\#$ above, by putting

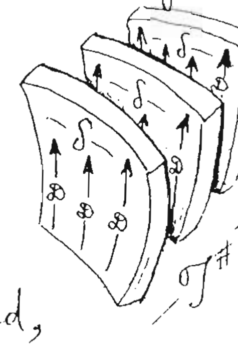
$$w = i\pi_{1'}\pi_{1'}\,, \quad y = i\pi_{1'}\pi_{0'}\,,$$

the ratio $w/y = \pi_{1'}/\pi_{0'}$ of these two quantities being a coordinate defining the direction of the α-line through q (in the β-plane on $\mathbb{C}\mathscr{I}^+$ through q). Provided that the scalings for z and x are chosen appropriately in relation to L^A and $\tilde{L}^{A'}$, we find that, just as before,

$$\delta = y\,dz \quad \text{and} \quad \mathcal{D} = \frac{y}{w}(w\,dy - y\,dw)\wedge dx \wedge dz\,.$$

It follows that there can be no "googly information" solely in the behaviour of the forms δ and $\mathcal{D}$ on $\mathcal{T}^\#$, with the definition of $\mathcal{T}^\#$ so far given. (It is of interest to note that the ratio y/w of the $\pi_{A'}$ components used here define the slope of the α-line, with $\pi_{1'}=0$ for a generator. This may be contrasted with the "standard" $\pi_{A'}$ used earlier, for flat twistor space $\mathbb{T}$, the ratio of whose components fixed the choice of β-plane, this ratio being essentially the quantity z.

The geometrical meaning of the tensor Γ up to proportionality — equivalently, the pair of forms δ and $\mathcal{D}$ up to proportionality — is that it defines a structure on $\mathcal{T}^\#$ that is the same at each point. This structure is a foliation by curves (the integral curves of the Euler vector field Υ, defined as the dual of $\mathcal{D}$, dualized with respect to $\frac{1}{4}d\mathcal{D}$) which lie in 3-surfaces (whose tangent spaces are duals to δ). The Frobenius relation $\delta \wedge d\delta = 0$ ensures that these 3-surfaces are locally integrable. Moreover $\delta \wedge \mathcal{D} = 0$, as we have here, is the condition that the curves lie on the 3-surfaces.



As a point of twistor philosophy, one might take the view that all information should be stored in global structure, the neighbourhood of any point being indistinguishable from that of the neighbourhood of any other. It is not clear how closely such a viewpoint can be followed, but the use of the space $\mathcal{T}^\#$ is in accordance with its spirit. We note that even in the flat case the non-blown-up space $\mathbb{T}$ does not qualify, since the points of I do not have neighbourhoods with the above structure, whereas the blown-up space $\mathbb{T}^\#$ does.

In the standard nonlinear graviton construction, one uses slightly more than the above local structure since the space-time metric itself, as opposed to merely the conformal metric, uses the forms δ and $\mathcal{D}$ themselves and not just their proportionality classes. More precisely, it is the ratio $\mathcal{D}/\delta\otimes\delta$ that comes in to define the metric (for δ defines $I_{\alpha\beta}$ which is, in effect, $\varepsilon^{A'B'}$ and $\delta/\mathcal{D}$ defines $I^{\alpha\beta}$ which is, in effect, ε^{AB}; thus $\delta\otimes\delta/\mathcal{D}$ in effect defines g^{ab}). We may imagine assigning a new $\hat{\delta}$ and $\hat{\mathcal{D}}$ in the neighbourhood of some point of $\mathcal{T}^b$, compatible with the foliation structure, so $\hat{\delta}$ and $\hat{\mathcal{D}}$ are scalar multiples of δ and $\mathcal{D}$. For the metric of $\mathcal{M}$ to be preserved, we must have

$$\hat{\delta} = \chi\delta, \quad \hat{\mathcal{D}} = \chi^2\mathcal{D},$$

whence

$$\hat{\Gamma} = \chi^3\,\Gamma.$$

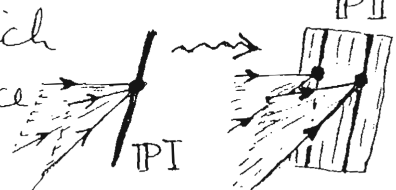
If we allow for a conformal factor (needed for $\mathbb{C}\mathscr{I}^+$, in any case), so that

$$\hat{g}_{ab} = \Omega^2 g_{ab},$$

we have
$$\hat{\delta} = \Omega^{-1}\chi\,\delta, \quad \hat{\mathcal{D}} = \chi^2\mathcal{D}, \quad \hat{\Gamma} = \Omega^{-1}\chi^3\,\Gamma$$
corresponding to $\hat{\varepsilon}_{AB} = \Omega\chi\,\varepsilon_{AB}$, $\hat{\varepsilon}_{A'B'} = \Omega\chi^{-1}\varepsilon_{A'B'}$. However, we should bear in mind that these rescalings apply at points of $\mathcal{T}^{\#}$ rather than at individual points of $\mathcal{M}$. (A conformal rescaling of $\mathcal{M}$ does not preserve δ, even up to proportionality; the rescaled $\mathcal{M}$ would not be Ricci-flat.)

Now suppose that a rescaling is introduced which makes both of $\hat{\delta}$ and $\hat{\mathcal{D}}$ regular and non-zero at $\mathbb{I}$. As things stand, there is too much choice for such scalings. We want somehow to restrict this choice so that the googly information is encoded. It may be noted that some choices are distinguished by the fact that they yield a 2-form δ that is closed, so locally
$$\hat{\delta} = d\zeta.$$
In the flat case, we may take
$$\Omega\chi^{-1} = \omega, \quad \zeta = z$$
and we recall that $z = \pi_{1'}/\pi_{0'}$ where $\pi_{0'}, \pi_{1'}$ are the ordinary flat-space $\pi_{A'}$-components. $SL(2,\mathbb{C})$ transformations of these coordinates are generated by the two types of replacement
$$\zeta \mapsto \zeta + \text{const.}, \quad \zeta \mapsto -\zeta^{-1}.$$
For the first of these, the rescaling factor $\Omega^{-1}\chi$ is unchanged; for the second, it is replaced by $\Omega^{-1}\chi\,\zeta^{-2}$. This procedure applies to any rescaling factor for which $\hat{\delta}$ is closed, so each such choice provides us with a local (primed) spin-space. As described in TN 23, this family of primed spin-spaces encodes, in a sense, the entire googly information. Thus, we

require to know the appropriate "preferred" rescaling factors. Each such factor would have to depend upon the actual direction in which a point of $\mathbb{P}\mathrm{I}$ is approached, from within $\mathbb{P}\mathcal{T}^b$, which still leaves some directional dependence at individual points of $\mathbb{P}\mathrm{II}$: ⟶



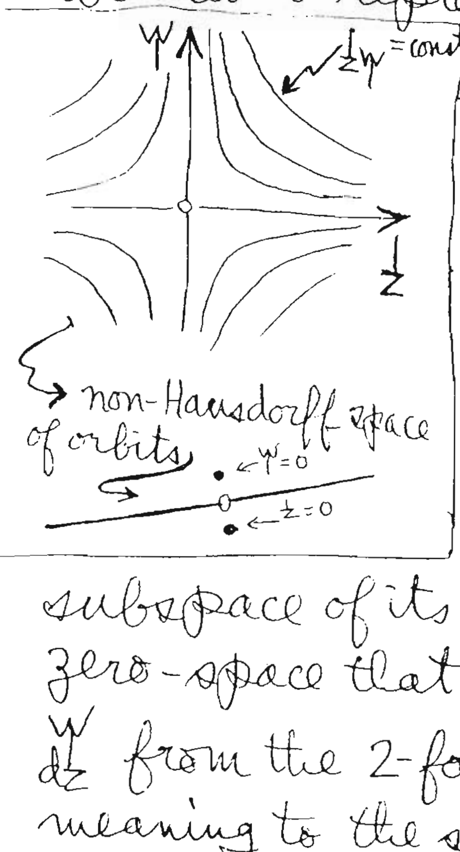
The rescaling factor $\Omega^{-1}\chi$ has to have homogeneity -2 in order for $d\hat{S}=0$ to hold. We are not at liberty to incorporate twistor scale-dependence into the conformal factor Ω, so taking that dependence to lie in χ, we find that the homogeneity degree for the scale factor for $\hat{\Gamma} = \Omega^{-1}\chi^3\Gamma$ is -6. This has some suggestiveness for a role for the googly twistor function, but the matter is not entirely clear, as yet.

In any case, we still have the problem of fixing the appropriate scale factors in terms of the complex geometry of $\mathcal{T}^\#$. As specified so far, the geometry of $\mathcal{T}^\#$ does not seem to contain this information, and a further idea is needed. From the twistor point of view, there is something a little unnatural about forming the space of products $Z\bar{Z}$ to replace the space of twistors Z. I wish to adopt a subtly different viewpoint here. We are to think of the product $Z\bar{Z}$ as a particular case of a product ZW, where the dual twistor W specifies a covector $\frac{W}{dZ}$. The covector bundle $T^*\mathcal{T}^b$ of $\mathcal{T}^b$ is, of course, well defined; and one can easily specify a covector at a point Z of $\mathcal{T}^b$ in terms of $\mathbb{C}\mathscr{I}^+$, where W is represented as a local dual twistor which is local-twistor constant along the α-line representing $\mathbb{P}Z$. The exterior derivative $d(W\wedge dZ)$ of $\frac{W}{dZ}$ is the

2-form defining the symplectic structure of $T^*\mathcal{T}^b$. We are interested in the subspace of $T^*\mathcal{T}^b$ defined by $\overset{W}{Z} = 0$ (or possibly $\overset{W}{Z} = k$ — of relevance to APH's work?). Taking $\overset{W}{Z}$ as a Hamiltonian, we obtain the flow whose orbits are to be factored out by in order to obtain the relevant "reduced phase space", where

$$(\lambda W, Z) \equiv (W, \lambda Z)$$

(integral curves of $\overset{W}{\partial_W} - \overset{Z}{\partial_Z}$). The product ZW can be used to represent these equivalence classes, for the most part, but there is a subtle difference. There are those particular orbits for which either $Z = 0$ or $W = 0$, in addition, giving a reduced phase space which is non-Hausdorff. In fact, for an ordinary cotangent bundle, one may not take $Z = 0$, whereas the space given by $W = 0$ is the canonical copy of the original (twistor) space as a subspace of its cotangent bundle. Indeed, it is this zero-space that enables us to reconstruct the 1-form $\overset{W}{dZ}$ from the 2-form $dW \wedge dZ$ in a unique way. To give meaning to the subspace $Z = 0$, it would be necessary to have some identification between covectors at different Z-points as being, in an appropriate sense, the same W.



One "solution" to the googly problem, therefore, would be to attach a "$Z=0$ zero-set" to the cotangent bundle $T^*\mathcal{T}^b$. It seems to me, however, that this procedure would be more in the nature of "begging the question" than in providing a solution to the googly problem. It would really be stretching a point to consider this procedure to be providing us with an extended "twistor space". The resulting space would be completely symmetrical with respect to twistors and dual twistors and is really much more related to ambitwistor space than to either of $\mathcal{T}$ or $\mathcal{T}^*$ individually.

Nevertheless, I believe that the above ideas provide an appropriate setting for what one should really be doing to construct $\mathcal{T}^\#$. What is required, initially, is the sub-bundle of $T^*\mathcal{T}^\flat$ for which the covector Ψ has the special form $\bar{x}\rceil$. The "Hamiltonian" $\underset{Z}{W}$ now becomes $\bar{x}\underset{Z}{\rceil}$; accordingly we appear to have to factor out by

$$(\lambda \bar{x}\rceil, \underset{Z}{\perp}) \equiv (\bar{x}\rceil, \lambda \underset{Z}{\perp})$$

where $\bar{x}\underset{Z}{\rceil} = 0$, i.e. where $\bar{x}\rceil$ and $\bar{Z}\rceil$ are proportional. To a first approximation, this indeed gives the blown-up twistor space $\mathcal{T}^\#$, as defined above. Ignoring the issue of zero-sets, for the moment, we can think of a point of this space as having elements of the form

$$\underset{Z}{\perp}\bar{x}\rceil,$$

with $\bar{x}\rceil$ proportional to $\bar{Z}\rceil$, where $\bar{x}\,\underset{dZ}{\rceil}$ is a covector at the point $\underset{Z}{\perp}$. We have the freedom to choose the proportionality factor by putting it equal to unity, so our space (the $\mathcal{T}^\#$ defined above) consists of objects

$$\underset{Z}{\perp}\bar{Z}\rceil,$$

where $\bar{Z}\rceil$ defines a (restricted) covector:

$$\delta = \bar{Z}\,\underset{dZ}{\rceil}.$$

However, we need to consider the zero sets, and it seems to me that the googly information ought to be contained therein. Rather than merely considering the two alternatives $X\rceil = 0$ and $\overset{|}{Z} = 0$ in $(\lambda X\rceil, \overset{|}{Z}) \equiv (X\rceil, \lambda\overset{|}{Z})$ with $\overline{X\,Z} = 0$, it may be that we ought to view all this slightly differently. The space of $(X\rceil, \overset{|}{Z})$ is not, after all, a non-degenerate symplectic manifold, the 2-form $d\overline{X \wedge dZ}$ having rank 4 on a 6-dimensional space. For a start, we may examine the zero-sets in relation to the "π-spaces" $(X\rceil, Z\rceil)$ alone. Putting (flat case) $\Psi = X\rceil = (0, \xi^{A'})$, $\overset{|}{Z} = (\omega^A, \pi_{A'})$, we have

$$dW \wedge dZ = d\overline{X \wedge dZ} = d\xi^{A'} \wedge d\pi_{A'}.$$

Taking the Hamiltonian to be $\xi^{A'}\pi_{A'}$ $(= \overline{X\,Z})$, we are led to contemplate the equivalence

$$(\lambda\xi^{A'}, \pi_{A'}) \equiv (\xi^{A'}, \lambda\pi_{A'})$$

with zero-sets given by $\xi^{A'} = 0$ and $\pi_{A'} = 0$. However, all of this is parameterized by ω^A in some way. It should be noted that the limits like

$$(\omega^A, \lambda\pi_{A'}) \to (\omega^A, 0)$$

given when $\lambda \to 0$ are just those of the kind in which an α-line rotates about a point of $\mathbb{C}\mathscr{I}^+$ to become a generator, as considered earlier. More work needs to be done to discover the detailed nature of the appropriate zero-sets and how quantities like Γ are to be defined and extended to these zero-sets. The condition that such extensions to the zero-set regions exist could well provide the necessary restrictions on, say, Γ that encode the full googly information — and perhaps, if this is successful, the required source information also. Work very much in progress....

Thanks to L.M.J., E.T.N., J.F., ...

Roger