Two Point Norms for Massless Fields

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In **TN 39**, I described the action of the *quantum complex structure* J on any field or potential satisfying the wave equation in flat spacetime, as an integral over an arbitrary curved spacelike hypersurface. In the present article I use this result to demonstrate the relationship between the twistor scalar product for massless fields, with 2-point norms in the bosonic cases of a massless scalar field, electromagnetism and linear gravity. The 2-point norms are automatically positive definite for fields of *mixed* frequency, and invariant under the 15-parameter conformal group. They have the added advantage that their computation involves no extraction of potentials or positive/negative frequency parts from the principal dynamical fields.

In the case of gravity I give an interpretation in terms of coherent states and superpositions of spacetime geometries.

I also describe the corresponding two point integral for the fermionic case of a massless neutrino field.

The Hermitian, Symplectic, Complex and Metric Structures

To begin with we introduce $\mathcal{F}$ as the standard Hermitian Fierz scalar product for a spin $\frac{1}{2}n$ field ϕ, given by

$$\mathcal{F}(\chi,\phi) := \kappa\int_\Sigma \overset{(n-1)}{\phi}{}_{L}^{A'B'\ldots K'}\bar{\chi}_{A'B'\ldots L'}d^3\Sigma^{LL'}$$

or for two scalar positive frequency fields ϕ and χ

$$\kappa\int_\Sigma \phi\nabla_a\bar{\chi}-\bar{\chi}\nabla_a\phi,\qquad \kappa=i.$$

Here $\overset{(n)}{\phi}$ is the n^{th} Dirac potential for ϕ, and κ is equal to 1 or i according as n is odd or even. Now $\mathcal{F}$ is the Hermitian structure. Denoting the symplectic, quantum complex and metric structures by Ω, J and $\langle\ ,\ \rangle$ respectively, the scheme for these three basic structures is

$$\mathcal{F}(\underset{1}{\varphi},\underset{2}{\varphi})\to\Omega(\underset{1}{\varphi},\underset{2}{\varphi}) := -i[\mathcal{F}(\underset{1}{\varphi},\underset{2}{\varphi})-\mathcal{F}(\underset{1}{\varphi},\underset{2}{\varphi})]$$

and the positive definite *metric* or norm, for a *mixed* frequency bosonic state is now,

$$\langle\underset{1}{\varphi}|\underset{2}{\varphi}\rangle=\Omega(\underset{1}{\varphi},J[\underset{2}{\varphi}]).$$

Remark on locality and complexification. It is worth noting the two distinct types of possible complexification for spacetime fields, with regard to locality. The complexification of a field itself (to allow for a distinct antiparticle) is *local*, as is the splitting of a field into its self-dual and anti-self-dual parts. However, as the results above show, the splitting of any field into its positive and negative frequency parts is *non-local* – that is to say, J is a *non-local operator*. It is non-local to the extent that it requires data over an entire Cauchy spacelike hypersurface.

The Scalar Case

We consider a real Klein-Gordon field of mixed frequency satisfying the massless wave equation. Then the symplectic form is given by

$$\Omega(\phi,\psi)=\int_\Sigma(\psi\nabla_a\phi-\phi\nabla_a\psi)d^3\Sigma^a$$

which is compatible with the complex structure in that $\Omega(\phi,\psi) \equiv \Omega(J\phi, J\psi)$. Now the Hermitian scalar product $\mathcal{F}$ is

$$\mathcal{F}(\phi,\psi) = \Omega(\phi, J[\psi]) + i\Omega(\phi,\psi)$$

and the symmetric positive definite metric is

$$\langle\phi,\psi\rangle \equiv \langle\psi,\phi\rangle = \Omega(\phi, J[\psi]) \propto \int\int_{\Sigma} [2\frac{(x-x')_a}{(x-x')^4}\phi(x')\nabla_b\psi(x) - \frac{\nabla'_a\phi(x')\nabla_b\psi(x)}{(x-x')^2}]\, d^3x'^a \cdot d^3x^b$$

where Σ is a once differentiable spacelike hypersurface. The 2-point integrand above represents the probability current in the *product* of two configuration spaces. For bosonic fields it is *not possible* in the relativistic theory to construct a single point current vector $j_a(x)$, which is future pointing, so that $j_a \cdot d^3\Sigma^a$ would represent the *positive* probability flux. (This is in contrast to the standard non-relativistic Schrödinger prescription of forming the symplectic probability 3-current $\mathbf{j} = \psi\nabla\psi^* - \psi^*\nabla\psi$ for some wave function ψ.) Thus we see that the particle number concept, which we obtain from the metric $\langle\ ,\ \rangle$, is non-local in the single configuration space but *local in* $\Sigma(x)\times\Sigma(y)$ – two *copies* of a single configuration space. This important idea as we shall see carries over to the electromagnetic and linear gravity cases also.

The Photon Case

We consider a generally complex electromagnetic field,

$$F_{ab} = \varphi_{AB}\varepsilon_{A'B'} + \psi_{A'B'}\varepsilon_{AB}$$

(so that in an alternative frequently adopted notation $\tilde{\varphi} \equiv \psi$). The splitting of F into its self-dual and anti-self-dual parts is given simply by,

$$^{(\mp)}F_{ab} = \frac{1}{2}(F_{ab} \pm i^*F_{ab})$$

where $*$ denotes the duality transformation, and the splitting into positive and negative frequency parts is given by,

$$F^{\pm} = \frac{1}{2i}(i \pm J)[F].$$

Remark on photon states. A physical photon state can be given in two ways: (1) as a real electromagnetic field with the relevant complex structure J at hand, or equivalently (2) a positive frequency field with independent left and right handed parts.
Remark on self-duality and helicity. The (anti-)self-dual part of a massless field is given always by its (un)primed spinor part – it is important to realise that the right (left) handed part of the field is given by the (un)primed spinor part *only* if the field is of *positive* frequency.
Remark on complex fields. These allow for the case that the particles associated to the field are *not* identical with their antiparticles. In the electromagnetic case the photon is its own antiparticle, so that F_{ab} is real for a physical field. However for its mathematical interest we study the general case of a complex field tensor.
Now the conserved Fierz scalar product is given by,

$$\mathcal{F}(\varphi,\varphi) = \int_{\Sigma} \bar{\varphi}_{A'B'}\eta^{A'}_{A}\, d^3x^{AB'}$$

where η is the vector potential, subject to the Lorenz gauge condition $\nabla^A_{B'}\eta^{A'}_A = 0$ (GL), and Σ is a spacelike hypersurface.

Note on the gauge condition: the spinor gauge GL implies that η satisfies the wave equation $\nabla^2\eta = 0$ and that the divergence of η vanishes $\nabla_a\eta^a = 0$. Later on we will need to assume the *Coulomb* gauge: $\eta^a = (0,\mathbf{A}) = \int(0,\tilde{\mathbf{A}}(k))e^{ikx}d^4k$ – such η clearly satisfies the wave equation since the integration is over $\{k^2 = 0\}$ – to obtain consistency with GL we need that $\mathbf{k}\cdot\mathbf{A} = 0$ – the 3-vector potential $\mathbf{A}$ and the electric field are parallel, and the electric, magnetic and momentum 3-vectors form a right-handed set.

Now from **TN 39** the repolarised potential is given by,

$$J[\eta_A^{A'}](x) = \frac{1}{2\pi^2} \int_{\Sigma' \ni x} \frac{1}{K^2} (\overleftarrow{\nabla}^b - \overrightarrow{\nabla}^b)\, \eta_A^{A'}(x')\, d^3x'_b.$$

In the general case,

$$\mathcal{F}(\varphi, J[\varphi]) \propto \iint_{\Sigma \times \Sigma'} \bar{\varphi}_{A'B'}(x) [\frac{\nabla'_{CC'} \eta_A^{A'}(x')\, d^3x'^{CC'}}{K^2(x, x')}]\, d^3x^{AB'}$$

$$+ \iint_{\Sigma \times \Sigma'} \bar{\varphi}_{A'B'}(x) [(\nabla'_{CC'} \frac{1}{K^2}) \eta_A^{A'}(x')\, d^3x'^{CC'}]\, d^3x^{AB'}$$

where Σ, Σ' are identical, but in general curved spacelike hypersurfaces. In the case that Σ is a flat 3-plane the second term above vanishes since this contains the factor $(x - x')^a$ orthogonal to the integration measure. The first term above upon interchanging the upper indices A', C' is equal to

$$-\iint_{\Sigma \times \Sigma'} \frac{\bar{\varphi}_{A'B'}(x)}{K^2} \nabla'_{CC'} \eta_A^{C'}(x')\, d^3x'^{CA'}\, d^3x^{AB'} \quad \text{(term 1)}$$

$$-\iint_{\Sigma \times \Sigma'} \frac{\bar{\varphi}_{A'B'}(x)}{K^2} \nabla'_{CC'} \varepsilon^{A'C'} \eta_{D'A}(x')\, d^3x'^{CD'}\, d^3x^{AB'} \quad \text{(term 2)}.$$

We claim that (term 2) above is zero : this term contains $\nabla'^{A'}_C \eta_{D'A}(x') d^3x'^{CD'}$ which by GL equals $\nabla'^{A'}_A \eta_{D'C}(x') d^3x'^{CD'}$ which vanishes in the Coulomb gauge. Since the original Fierz expression was gauge independent subject to GL holding, and since GL is consistent with the Coulomb gauge, we may assume the latter gauge to hold when we gauge fix Σ to be a flat 3-plane. We are left then with (term 1) which from the Dirac chain is equal to

$$-\iint_{\Sigma \times \Sigma'} \frac{\bar{\varphi}_{A'B'}(x) \varphi_{AC}(x')}{K^2} d^3x'^{CA'}\, d^3x^{AB'}.$$

Now define

$$P(x, y) = \mathbf{E}(x) \cdot \mathbf{E}^*(y) + \mathbf{B}(x) \cdot \mathbf{B}^*(y).$$

Then $P^*(x, y) = P(y, x)$ and $P(x, x)$ is real. Define

$$Q(x, y) = \mathbf{E}(x) \cdot \mathbf{B}^*(y) - \mathbf{B}(x) \cdot \mathbf{E}^*(y).$$

Then $Q(y, x) = -Q(x, y)^*$ and $Q(x, x)$ is purely imaginary. Also define,

$$\mathbf{C} = \mathbf{E} - i\mathbf{B}.$$

Now introduce the following generalised *two-point* stress-energy tensor for electromagnetism:

$$T_{ab}(x, y) = \varphi_{AB}(x) \bar{\varphi}_{A'B'}(y).$$

Note that, even in the 2-point case this tensor retains its symmetry and its trace-free property, since the field spinors φ and ψ are symmetric. Now,

$$\begin{aligned}
\varphi_{AB}(x) \bar{\varphi}_{A'B'}(y) n^a n^b &= \varphi_{00}(x) \bar{\varphi}_{0'0'}(y) + 2\varphi_{01}(x) \bar{\varphi}_{0'1'}(y) + \varphi_{11}(x) \bar{\varphi}_{1'1'}(y) \\
&= \frac{1}{2} \mathbf{C}(x) \cdot \mathbf{C}^*(y) \\
&= \frac{1}{2} [P(x, y) + iQ(x, y)]
\end{aligned}$$

Now to calculate the corresponding ψ part: we have

$$\psi_{0'0'} = \frac{1}{2}(D_1 + iD_2)$$

$$\psi_{0'1'} = -\frac{1}{2}D_3$$

$$\psi_{1'1'} = -\frac{1}{2}(D_1 - iD_2)$$

where now $\mathbf{D} = \mathbf{E} + i\mathbf{B}$, noting the change in the sign in front of i (of course the electric and magnetic vectors here are the same as in the φ case – there is only a single complex electromagnetic field under consideration).

Now, introducing new variables $\{X, Y\} = \{x, y\}$ but with a free ordering,

$$\psi_{A'B'}(Y)\bar{\psi}_{AB}(X)n^a n^b$$

$$= \bar{\psi}_{00}(X)\psi_{0'0'}(Y) + 2\bar{\psi}_{01}(X)\psi_{0'1'}(Y) + \bar{\psi}_{11}(X)\psi_{1'1'}(Y)$$

$$= \frac{1}{2}\mathbf{D}^*(X) \cdot \mathbf{D}(Y)$$

$$= \frac{1}{2}[P^*(X,Y) + iQ^*(X,Y)].$$

Then,

$$[\varphi_{AB}(x)\bar{\varphi}_{A'B'}(y) + \psi_{A'B'}(Y)\bar{\psi}_{AB}(X)]n^a n^b \quad (\dagger)$$

$$= \frac{1}{2}(P + iQ)(x,y) + \frac{1}{2}(P^* + iQ^*)(X,Y)$$

which in the case of ordering $(X,Y) = (x,y)$

$$= \frac{1}{2}[P(x,y) + P(y,x)] + \frac{1}{2}i[Q(x,y) - Q(y,x)]$$

which integrates against symmetric K to

$$\int \frac{P(x,y)}{K(x,y)}.$$

Alternatively in the case $(X,Y) = (y,x)$, $(\dagger) = P(x,y)$. Thus both orderings give the same result for the photon norm $\langle\varphi|\varphi\rangle$ below. This is a reflection of the fact that the photons obey *Bose statistics.* More precisely the 2-photon state represented by $F_{ab,cd}(x,y)$ satisfies Maxwell's equations on the first two indices with respect to x and on the second two indices with respect to y, and Bose statistics implies $F_{ab,cd}(x,y) = F_{cd,ab}(y,x)$. Our norm is in fact the Hilbert space norm for Quantum Electrodynamics, which is closely related to the expectation of the number operator $\hat{N}$ in the Fock space state coherent to the classical state (φ, ψ). In summary we have shown

$$\langle\varphi|\varphi\rangle \propto \iint \frac{\mathbf{E}(x) \cdot \mathbf{E}^*(y) + \mathbf{B}(x) \cdot \mathbf{B}^*(y)}{|\mathbf{x} - \mathbf{y}|^2} \geq 0 \ \forall\, \mathbf{E}, \mathbf{B}$$

and that this is identical to the Fierz/Twistor scalar product with the relevant complex structure inserted. (I have shown that the generic 2-point 3-space integral $\int \phi(x)\phi(y)/|x-y|^2$ is convergent if and only if $\phi \sim 1/r^{1+\gamma}$ for $\gamma > 1$ at spatial infinity, though I omit the proof of this here.)

Remarks. The invariance under the restricted conformal group $C_+^\uparrow$

$$SU(2,2) \xrightarrow{2-1} SO(2,4) \xrightarrow{2-1} C_+^\uparrow$$

follows automatically from the twistor translation of the Fierz scalar product, and that the action of J commutes with any conformal transformation. In particular the 2-point norm is Lorentz and translation invariant, as in fact follows immediately from the above argument – the current vector in the Fierz scalar product is divergence free so one is free to boost and translate Σ, before J is performed. The photon norm is thus a property of a given complex electromagnetic field – not of the particular time slice. In this sense it contains information about the time evolution of the field, *via* the requirement that it be constant. It is also worth noting that the 2-point integral enables one to compute the norm of the field *without* having to extract any potentials or positive/negative frequency parts.

The remarks above carry over naturally to the spin-2 case, that is linear gravity, as we demonstrate below.

The Gravitational Case

To begin with we integrate by parts once in the Fierz scalar product given earlier to obtain,

$$\mathcal{F}(\psi,\psi) = -i\int_\Sigma \bar{\Gamma}^A_{B'C'D'}(x)\chi^{B'C'}_{AD}(x)\, d^3x^{DD'}$$

where Γ and χ are the first and second potentials for the linearised Weyl spinor respectively. It follows that, with $h = \chi + \bar{\chi}$

$$\Omega(\psi, J[\psi]) \propto \iint_{\Sigma\times\Sigma'} \bar{\Gamma}^A_{B'C'D'}(x)\cdot[\frac{1}{(x-x')^2}\cdot\nabla'_{EE'}h^{B'C'}_{AD}(x') - 2\frac{(x-x')_{EE'}}{(x-x')^4}h^{B'C'}_{AD}(x')]d^3x'^{EE'}\cdot d^3x^{DD'}$$

where Σ, Σ' are in general curved but coincident. In the special case where Σ is flat this simplifies to

$$\Omega(\psi, J[\psi]) \propto \iint_{\Sigma\times\Sigma'} \bar{\Gamma}^A_{B'C'D'}(x)\cdot\frac{1}{(x-x')^2}\cdot\nabla'_{EE'}h^{B'C'}_{AD}(x')d^3x'^{EE'}\,d^3x^{DD'}.$$

Here ψ represents a real spin-2 field of *mixed* frequency, and we assume the transverse gauge $h_{ab}\cdot d^3x^a = 0$. Then we have

$$\nabla'_{EE'}h^{B'C'}_{AD}(x')d^3x'^{EE'}\,d^3x^{DD'}$$

$$= \nabla'_{EE'}h^{B'E'}_{AD}(x')d^3x'^{EC'}\,d^3x^{DD'} + \nabla'_{EE'}\epsilon^{C'E'}h^{B'}_{F'AD}(x')d^3x'^{EF'}\,d^3x^{DD'}$$

in which the second term vanishes since we can interchange the lower indices A and E by the spinor gauge condition, leaving a term involving $h^{B'}_{DEF'}(d')^{EF'}$ which vanishes in the transverse gauge. Then using the Dirac chain we are left in the transverse gauge simply with

$$\mathcal{F}(\psi, J[\psi]) \propto \iint_{\Sigma\times\Sigma'}\frac{\bar{\Gamma}^A_{B'C'D'}(x)\Gamma^{B'}_{EAD}(x')(d')^{EC'}(d)^{DD'}}{(x-x')^2}.$$

A tensorial version of this expression can be obtained as follows. In the transverse gauge the *linearised* extrinsic curvature is given by

$$\Pi_{ab} = \frac{1}{2}q^c_a q^d_b n^e\nabla_e h_{cd}$$

in which the derivative of h arises from the perturbed covariant derivative $\hat{\nabla}$.

We now proceed to demonstrate that,

$$\bar{\Gamma}^A_{B'C'D'}(x)\Gamma^{B'}_{EAD}(x')\,(d')^{EC'}(d)^{DD'} = \Pi_{ab}(x)\Pi^{ab}(x')\,d^3x\,d^3x'.$$

Clearly,

$$4\Pi_{ab}(x)\Pi^{ab}(x') = (g^{cs}-n^cn^s)(g^{dt}-n^dn^t)n^en^r(\nabla_e h_{cd})(x)(\nabla_r h_{st})(x')$$

which by the gauge choice reduces to,

$$n^en^r(\nabla_e h_{ct})(x)(\nabla_r h^{ct})(x').$$

The spinor part of our assertion is,

$$\nabla_{BB'}\bar{\chi}^{AB}_{C'D'}(x)\nabla_{EE'}\chi^{B'E'}_{AD}(x')n^{EC'}n^{DD'}.$$

By the spinor gauge condition on χ (from the Dirac chain) we have,

$$\nabla_{EE'}\chi^{B'E'}_{AD}n^{EC'} = \nabla_{AE'}\chi^{B'E'}_{ED}n^{EC'} = \nabla_{AE'}\chi^{B'C'}_{ED}n^{EE'} = \nabla_{EE'}\chi^{B'C'}_{AD}n^{EE'}$$

and also,

$$\nabla_{BB'}\bar{\chi}^{AB}_{C'D'}n^{DD'} = \nabla_{BD'}\bar{\chi}^{AB}_{C'B'}n^{DD'}$$

which together give a contribution

$$(\nabla_{EE'}\chi^{B'C'}_{AB})(x')n^{EE'}(\nabla_{DD'}\bar{\chi}^{AD}_{C'B'})(x)n^{BD'}.$$

Now we can interchange the upper indices B and D since this just gives an extra term

$$(\nabla_{EE'}\chi^{B'C'}_{AB})(x')n^{EE'}(\nabla_{DD'}\bar{\chi}^{A}_{C'B'F})(x)n^{FD'}\varepsilon^{DB}$$

in which we may interchange the lower indices D' and C' by the spinor gauge condition on $\bar{\chi}$, and then this vanishes since $h \cdot n = 0$. Up to an overall numerical factor therefore, our assertion is established.

The conformal invariance of this 2-point gravity norm follows from the same arguments given for the photon case above.

In terms of Laplacians we have,

$$\langle \underset{1}{\psi} \,|\, \underset{2}{\psi} \rangle \propto \int \underset{1}{\Pi}^{ab}(x)\Delta^{-1/2}\underset{2}{\Pi}_{ab}(x) = \frac{i}{2\pi^2}\iint \frac{\underset{1}{\Pi}^{ab}(x)\underset{2}{\Pi}_{ab}(x')}{(x-x')^2} \quad (\dagger)$$

in which $\Delta^{-1/2}$ acts as a non-local operator. In the vacuum case invariance under the conformal group holds so that in particular the expression is conserved under time translations. It is noteworthy that one could calculate this integral *in the presence of matter* where the Einstein vacuum field equations fail to hold. For the integrand only requires a foliation of the spacetime. However in the non-vacuum case the norm is certainly not conserved in general. Only in the special case that there exists a timelike Killing vector field $\partial/\partial t$ along which the Lie derivative of Π vanishes is the above norm guaranteed to be conserved in the presence of matter.

For two distinct fields the above expression could be interpreted as the 'overlap' of two neighbouring spacetime geometries, where the constituent fields themselves are normalised. It is this overlap which one could interpret as a measure of the probability of a transition $|\underset{1}{\Pi}\rangle \mapsto |\underset{2}{\Pi}\rangle$ where the Dirac kets denote the *coherent* states in the projective Fock space (state space) PF. Now any state $|\psi\rangle$ in PF has an unique decomposition into *coherent states*, and thus the submanifold C of coherent states in PF is *non-linear*. That is to say the complex projective line L in state space joining the coherent states $|\underset{1}{\Pi}\rangle$ and $|\underset{2}{\Pi}\rangle$ lies entirely off the coherent state submanifold apart from at the points $|\underset{1}{\Pi}\rangle$ and $|\underset{2}{\Pi}\rangle$ where it intersects C transversally. Thus we think of the superposition of two neighbouring spacetime geometries as a point $|\psi\rangle$ on $L\backslash C$ and then the probability of transition to $|\underset{i}{\Pi}\rangle$ is the distance $d(\psi, \underset{i}{\Pi})$ with respect to the standard Fubini-Study metric restricted to the projective line joining $|\psi\rangle$ to $|\underset{i}{\Pi}\rangle$. It is important to realize that the nonlinearity of C is fully present even in the *linear* gravity. This is a desired feature of our description, since in the presence of weak gravitational fields nature in the classical domain does not support superpositions of neighbouring well defined classical geometries. It is precisely the space C which corresponds under exponentiation to these classical geometries – the solutions $\underset{i}{\Pi}$ of the classical field equation. Moreover, as the early work of Schrödinger shows, C is unitarily invariant under a wide class of Hamiltonians, in particular all harmonic oscillator Hamiltonians. Thus a free coherent state of the gravitational field remains coherent under its time evolution and is associated at each time to an unique classical geometry. Since the neighbouring coherent states overlap, according to (†), the coherent state is free to wander within C under unitary evolution, and so under time evolution the geometry may change, but *retain* its classical nature.

The Fermionic Case

Let us now consider the corresponding situation for the case of a massless neutrino field ν_A. The Fierz scalar product is now

$$\mathcal{F}(\nu,\bar{\nu}) = \int_{\Sigma} \bar{\nu}_A \nu_{A'} d^3\Sigma^{AA'} \quad (*).$$

Note here that the current 4-vector $j = \nu\bar{\nu}$ is automatically a *future* pointing null vector so that the expression above is positive definite for all ν. We can apply the same repolarisation procedure as before to give

$$\mathcal{F}(\bar{\nu}, J[\nu]) \propto \iint_{\Sigma^2} -2\frac{(x-y)^{BB'}}{(x-y)^4}\bar{\nu}_A(x)\nu_{A'}(y) - \frac{1}{(x-y)^2}\bar{\nu}_A(x)\nabla^{BB'}_{(y)}\nu_{A'}(y)d^{3AA'}_{(x)} \cdot d^3_{(y)BB'} \quad (**)$$

where Σ is a single once differentiable spacelike hypersurface, which is in general curved. In the case that Σ is flat we are left simply with

$$-\iint_{\Sigma^2} \frac{\bar{\nu}_A(x)\nabla^{BB'}_{(y)}\nu_{A'}(y)}{(x-y)^2} d^{3AA'}_{(x)} \cdot d^3_{(y)BB'}.$$

We remark on the physical significance of the above formula. Suppose we have a neutrino field consisting of a *mixture* of both left and right-handed particles. Firstly note that for such a general mixture we only use a *single* spinor field ν. More precisely both ν_A and $\bar{\nu}_{A'}$ have positive frequency parts corresponding to the left and right-handed fields respectively. The standard Fierz 1-point integral (*), because of the Fermi statistics which ensures the current vector is always future pointing, represents for all ν the positive definite norm which corresponds to the *sum* of the numbers L and R of left and right-handed neutrinos. The effect of the quantum complex structure in (**), as one can see from the repolarising action of J in (*), is simply to produce the *difference* $L - R$. The situation for Bose statistics is reversed in the sense described earlier.

Concluding remarks

I have been able to show that these two-point norms can be expressed in terms of Dirac's CP^5 calculus, and also as twistor diagrams. In the latter case I have shown using RP's article in **TN 27** that the action of the quantum complex structure J is expressible as an holomorphic link integral in twistor space. Thus also, from the integral representation of J, is the scalar product of a field with the Green's function for the wave operator, a suggestion of Atiyah which is alluded to in RP's article.

Details of these issues will appear later.

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