In search of a twistor correspondence for the KP equations

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1 Introduction

This article summarises work done in an attempt to develop a twistor correspondence for the Kadomtsev-Petviashvili equations,

$$(4u_3 + 6uu_1 - u_{111})_1 - 3\sigma^2 u_{22} = 0,$$

where $\sigma^2 = -1$ for the KPI equation and $\sigma^2 = +1$ for the KPII equation, and other related equations as yet unattainable as reductions of the anti-self-dual Yang-Mills equations. The motivation comes from [2] and the work done on developing the Dirac operator and the nonlocal Riemann-Hilbert problem is adapted from work in [1] and [5].

We shall take as our twistor space $\mathcal{O}(n)$, the twisted line bundle of Chern class n, fibred over $\mathbb{CP}^1$, which is viewed stereographically as $\mathbb{C} \cup \infty$. The Riemann sphere will be given coordinates λ on $\mathbb{C}$ and $\lambda' = \lambda^{-1}$ on $\mathbb{C}' = \mathbb{C} \cup \infty - 0$. The bundle $\mathcal{O}(n)$ can then be given coordinates (μ, λ) on the fibres over $\mathbb{C}$ and $(\mu', \lambda') = (\mu\lambda^{-n}, \lambda^{-1})$ on the fibres over $\mathbb{C}'$. It can be shown that an element of the space of holomorphic sections of the bundle is of the form

$$\mu_n = \sum_{i=0}^{n} t_i \lambda^i, \quad (t_i) \in \mathbb{C}^{n+1}.$$

For the KPII case, we consider the coordinates (t_i) to be real.

2 The Dirac Operator

The standard Ward construction involves solving the equation $\bar{\partial}_E f = 0$ on a vector bundle E. The idea of Mason, in [2], was to replace the $\bar{\partial}_E$-operator with a Dirac operator $\not{D}_\alpha$, where

$$\not{D}_\alpha \underline{\phi} := \begin{pmatrix} \partial_{\bar{\lambda}} & \alpha \\ \bar{\alpha} & \partial_\lambda \end{pmatrix} \begin{pmatrix} \phi \\ \tilde{\phi} \end{pmatrix} = 0. \tag{1}$$

where α is a smooth function on $\mathcal{O}(n)$. Mason and Woodhouse, in [3], show that if we impose certain symmetry conditions on α and $\underline{\phi}$, the we can derive the equations of the KP hierarchy. These conditions are;

$$\begin{array}{ll} i) & \tilde{\phi} = \bar{\phi}, \\ ii) & \phi(\lambda, \bar{\lambda}) = \overline{\phi(\bar{\lambda}, \lambda)}, \\ iii) & \alpha = \exp(\bar{\mu} - \mu)\alpha_0(\lambda, \bar{\lambda}). \end{array}$$

In the next section, we proceed in the opposite direction, and summarise a method of deriving the Dirac operator with the given symmetries from the operators of the KP hierarchy.

3 Derivation of the Dirac operator

Consider the equation $L\psi = 0$ where L is the first operator in the KP hierarchy, $L = \partial_2 - \partial_1^2 + u$, where $u(t) \in L^1 \cap L^2(\mathbb{R}^2)$ and $t = (t_1, t_2)$. If we assume that $u(t)$ is zero in a neighbourhood of $|t| = \infty$, then $\psi \sim \exp \mu_2$ as $|t| \to \infty$. If we write $\psi = e^{\mu_2}\phi$, then ϕ satisfies

$$\left[(\partial_1 + \lambda)^2 - (\partial_2 + \lambda^2)\right] \phi(t;\lambda) = u(t)\phi(t;\lambda). \tag{2}$$

As detailed in Wickerhauser, [5], there is a unique solution to this equation, satisfying $\phi \to 1$ as $|t| \to \infty$.

We shall denote by $\hat{\phi}$ the Fourier transform of ϕ with respect to the variable $\xi = (\xi_1, \xi_2) \in \mathbb{R}^2$, defined by

$$\hat{\phi}(\xi;\lambda)) = \int_{\mathbb{R}^2} e^{-i\xi \cdot t} \phi(t;\lambda) dt, \quad dt = dt_1 dt_2.$$

Define the polynomials $P(\xi) = \xi_1^2 - \xi_2$ and $P_\lambda(\xi) = P(i\xi_1 + \lambda, i\xi_2 + \lambda^2)$. By taking Fourier transforms of equation (2), one obtains

$$\phi(t;\lambda) = 1 + G\phi = 1 + \frac{1}{4\pi^2} \int_{\mathbb{R}^2} e^{i\xi \cdot t} \frac{[u\phi]^\wedge(\xi;\lambda)}{P_\lambda(\xi)} d\xi.$$

We can make $||G||$ less than unity by decreasing $||u||_{L^1} + ||u||_{L^2}$. We can then write $\phi = (I - G)^{-1}1$, and hence

$$\partial_{\bar{\lambda}}\phi = (I - G)^{-1}(\partial_{\bar{\lambda}}G)\phi.$$

After a little work, we obtain the expression

$$(\partial_{\bar{\lambda}}G)\phi = \alpha_0(\lambda)\exp(\bar{\mu_2} - \mu_2),$$

where

$$\alpha_0(\lambda) = \frac{-1}{2\pi i}\int_{\mathbb{R}^2}\exp(\mu_2 - \bar{\mu_2})u(t)\phi(t;\lambda)dt.$$

We now consider the behaviour of $\nu(t;\lambda) = (I - G)^{-1}\exp(\bar{\mu_2} - \mu_2)$. It is bounded and satisfies equation (2) with the boundary condition $\nu(t;\lambda) \sim \exp(\bar{\mu_2} - \mu_2)$ as $|t| \to \infty$. Define $\tilde{\nu}(t;\lambda) = \exp(\mu_2 - \bar{\mu_2})\nu(t;\lambda)$. This has the boundary condition $\tilde{\nu}(t;\lambda) \to 1$ as $|t| \to \infty$, and satisfies the equation

$$\left[(\partial_1 + \bar{\lambda})^2 - (\partial_2 + \bar{\lambda}^2)\right]\tilde{\nu}(t;\lambda) = u(t)\tilde{\nu}(t;\lambda). \qquad (3)$$

But $\phi(t;\bar{\lambda})$ is the unique solution to this problem, therefore

$$\partial_{\bar{\lambda}}\phi(t;\lambda) - \exp(\bar{\mu_2} - \mu_2)\alpha_0(\lambda)\phi(t;\bar{\lambda}) = 0. \qquad (4)$$

As u is real-valued, it follows that $\overline{\phi(\lambda)} = \phi(\bar{\lambda})$ and $\overline{\alpha_0(\lambda)} = \alpha_0(\bar{\lambda})$ and hence we obtain the conjugate equation

$$\overline{\exp(\bar{\mu_2} - \mu_2)\alpha_0(\lambda)}\phi(\lambda) + \partial_\lambda\phi(\bar{\lambda}) = 0.$$

Thus far we have only considered the variables t_1 and t_2. We now wish to include the t_3 dependence of ϕ, arising from the second operator in the KP hierarchy,

$$M = \partial_3 - \partial_1^3 + \frac{3}{2}u\partial_1 + \frac{3}{2}u_1 - v,$$

where v is defined in terms of u by the compatibility conditions. Writing equation (4) with $\mu_2 = \mu_3\exp(-\lambda^3 t_3)$ and multiplying both sides by $\exp\mu_3$, we then apply L_3 to both sides, giving that $\alpha_0(\lambda;t_3) = \alpha_0(\lambda)\exp-(\lambda^3 - \bar{\lambda}^3)t_3$ and hence

$$\partial_{\bar{\lambda}}\phi(t;\lambda) + \exp(\bar{\mu_3} - \mu_3)\alpha_0(\lambda)\phi(t;\bar{\lambda}) = 0, \qquad (5)$$

where t is now an element of $\mathbb{R}^3$. It follows that if $\phi\exp\mu_n$ satisfies the first $(n-1)$ operators in the KP hierarchy, equation (5) holds, with n replacing the subscript 3.

4 The KPI equation

Whereas the inverse scattering for the KPII equation produces a d-bar problem relating $\phi(\lambda)$ and $\phi(\bar{\lambda})$, where ϕ is nowhere analytic in λ, the KPI equation gives a substantially different result. The first operator in the KPI hierarchy is given by $L = \partial_2 - \partial_1^2 + u$, so proceeding as in the KPII case, taking $\mu_3(\lambda) = i(\lambda t_1 + \lambda^2 t_2 - \lambda^3 t_3)$. The problem arising in the fact that for KPII, $P_\lambda(\xi)^{-1}$ has discrete singularities for all λ, but in the KPI case, $P_\lambda(\xi)^{-1}$ has continuous singularities for real λ. This causes the Green's function to be undefined on the real λ-axis. The resulting calculation leads to ϕ being a sectionally meromorphic function, satisfying a nonlocal Riemann-Hilbert problem on the real λ-axis,

$$(\phi_+ - \phi_-)(t;\lambda) = \int_{\mathbb{R}} F(\lambda,\kappa)\exp\{\mu(\kappa) - \mu(\lambda)\}\phi_-(\kappa)d\kappa,$$

$\phi_+(\phi_-)$ being holomorphic in the upper (lower) half-plane.

At this time it is unclear as to the connection between the two results. One feels that they may be parts of a larger construction (for complex t_i?) which reduces to the d-bar relation when $t_i \in \mathbb{R}$ and to the nonlocal Riemann-Hilbert problem when some t_i are imaginary.

5 The twistor correspondence

The main problem with this theory is its global nature. The Ward correspondence for the anti-self-dual Yang-Mills equations is a local construction, therefore it would be sensible to attempt to develop a localised version of the preceding theory. One possible direction is to use ideas from Segal and Wilson, [4], concerning the theory of Grassmannians and the KP hierarchy. A link exists between the theory, if we insist in equation (1) that α vanishes in a neighbourhood of infinity. Then in that neighbourhood, $\partial_{\bar{\lambda}}\phi = 0$ and ψ has the general form of the Segal-Wilson Baker function,

$$\psi(t;\lambda) = \exp\mu\left(1 + \sum_{i=1}^{\infty} a_i\lambda^{-i}\right).$$

It may well be possible to extend the Baker function to obtain the global ψ of the Dirac operator.

An alternative avenue of approach is to consider what happens if the coordinates t_i are complex. The Davey-Stewartson equations generate similar constructions to the KP equations, but with the t_1-coordinate being complex. In this sense, the DS-equations are more fundamental than the KP equations. By complexifying the other coordinates, it may be possible to generate higher-dimensional integrable systems.

References

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