In search of a twistor correspondence for the KP equations

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1 Introduction

This article summarises work done in an attempt to develop a twistor correspondence for the Kadomtsev-Petviashvili equations,

$$(4u_3 + 6uu_1 - u_{111})_1 - 3\sigma^2 u_{22} = 0,$$

where $\sigma^2 = -1$ for the KPI equation and $\sigma^2 = +1$ for the KPII equation, and other related equations as yet unattainable as reductions of the anti-self-dual Yang-Mills equations. The motivation comes from [2] and the work done on developing the Dirac operator and the nonlocal Riemann-Hilbert problem is adapted from work in [1] and [5].

We shall take as our twistor space $\mathcal{O}(n)$, the twisted line bundle of Chern class $n$, fibred over $\mathbb{CP}^1$, which is viewed stereographically as $\mathbb{C} \cup \infty$. The Riemann sphere will be given coordinates $\lambda$ on $\mathbb{C}$ and $\lambda' = \lambda^{-1}$ on $\mathbb{C}' = \mathbb{C} \cup \infty - 0$. The bundle $\mathcal{O}(n)$ can then be given coordinates $(\mu, \lambda)$ on the fibres over $\mathbb{C}$ and $(\mu', \lambda') = (\mu \lambda^{-n}, \lambda^{-1})$ on the fibres over $\mathbb{C}'$. It can be shown that an element of the space of holomorphic sections of the bundle is of the form

$$\mu_n = \sum_{i=0}^n t_i \lambda^i, \quad (t_i) \in \mathbb{C}^{n+1}.$$

For the KPII case, we consider the coordinates $(t_i)$ to be real.

2 The Dirac Operator

The standard Ward construction involves solving the equation $\bar{\partial}_E f = 0$ on a vector bundle $E$. The idea of Mason, in [2], was to replace the $\bar{\partial}_E$-operator
with a Dirac operator $\mathcal{D}_{\alpha}$, where

$$\psi \mathcal{D}_{\alpha} \phi := \begin{pmatrix} \partial_{\lambda} & \alpha \\ \bar{\alpha} & -\partial_{\lambda} \end{pmatrix} \begin{pmatrix} \phi \\ \bar{\phi} \end{pmatrix} = 0.$$ \hspace{1cm} (1)

where $\alpha$ is a smooth function on $\mathcal{O}(n)$. Mason and Woodhouse, in [3], show that if we impose certain symmetry conditions on $\alpha$ and $\bar{\phi}$, then we can derive the equations of the KP hierarchy. These conditions are;

$$i) \quad \tilde{\phi} = \bar{\phi},$$

$$ii) \quad \phi(\lambda, \bar{\lambda}) = \bar{\phi}(\bar{\lambda}, \lambda),$$

$$iii) \quad \alpha = \exp(\bar{\mu} - \mu)\alpha_0(\lambda, \bar{\lambda}).$$

In the next section, we proceed in the opposite direction, and summarise a method of deriving the Dirac operator with the given symmetries from the operators of the KP hierarchy.

3 Derivation of the Dirac operator

Consider the equation $L \psi = 0$ where $L$ is the first operator in the KP hierarchy, $L = \partial_{\tau} - \partial_{\tau}^2 + u$, where $u(t) \in L^1 \cap L^2(\mathbb{R}^2)$ and $t = (t_1, t_2)$. If we assume that $u(t)$ is zero in a neighbourhood of $|t| = \infty$, then $\psi \sim \exp \mu_2$ as $|t| \to \infty$. If we write $\psi = e^{\mu_2 \phi}$, then $\phi$ satisfies

$$\left[ (\partial_\tau + \lambda) \left( \partial_\tau + \lambda^2 \right) \right] \phi(t; \lambda) = u(t)\phi(t; \lambda). \hspace{1cm} (2)$$

As detailed in Wickerhauser, [5], there is a unique solution to this equation, satisfying $\phi \to 1$ as $|t| \to \infty$.

We shall denote by $\hat{\phi}$ the Fourier transform of $\phi$ with respect to the variable $\xi = (\xi_1, \xi_2) \in \mathbb{R}^2$, defined by

$$\hat{\phi}(\xi; \lambda)) = \int_{\mathbb{R}^2} e^{-i\xi \cdot t} \phi(t; \lambda) dt, \quad dt = dt_1 dt_2.$$

Define the polynomials $P(\xi) = \xi_1^2 - \xi_2$ and $P_\lambda(\xi) = P(i\xi_1 + \lambda, i\xi_2 + \lambda^2)$. By taking Fourier transforms of equation (2), one obtains

$$\phi(t; \lambda) = 1 + G\phi = 1 + \frac{1}{4\pi^2} \int_{\mathbb{R}^2} e^{i\xi \cdot t} \frac{[u\phi]^\wedge(\xi; \lambda)}{P_\lambda(\xi)} d\xi.$$
We can make \( \|G\| \) less than unity by decreasing \( \|u\|_{L^1} + \|u\|_{L^2} \). We can then write \( \phi = (I - G)^{-1} \), and hence

\[
\partial_{\lambda} \phi = (I - G)^{-1}(\partial_{t} G)\phi.
\]

After a little work, we obtain the expression

\[
(\partial_{t} G)\phi = \alpha_0(\lambda) \exp(\bar{\mu}_2 - \mu_2),
\]

where

\[
\alpha_0(\lambda) = -\frac{1}{2\pi i} \int_{\mathbb{R}^2} \exp(\mu_2 - \bar{\mu}_2)u(t)\phi(t; \lambda)dt.
\]

We now consider the behaviour of \( \nu(t; \lambda) = (I - G)^{-1} \exp(\bar{\mu}_2 - \mu_2) \). It is bounded and satisfies equation (2) with the boundary condition \( \nu(t; \lambda) \sim \exp(\bar{\mu}_2 - \mu_2) \) as \( |t| \to \infty \). Define \( \bar{\nu}(t; \lambda) = \exp(\mu_2 - \bar{\mu}_2)\nu(t; \lambda) \). This has the boundary condition \( \bar{\nu}(t; \lambda) \to 1 \) as \( |t| \to \infty \), and satisfies the equation

\[
\left[ (\partial_1 + \bar{\lambda}^2) (\partial_2 + \bar{\lambda}^2) \right] \bar{\nu}(t; \lambda) = u(t)\bar{\nu}(t; \lambda).
\]

But \( \phi(t; \bar{\lambda}) \) is the unique solution to this problem, therefore

\[
\partial_{\lambda} \phi(t; \lambda) - \exp(\bar{\mu}_2 - \mu_2)\alpha_0(\lambda)\phi(t; \bar{\lambda}) = 0.
\]

As \( u \) is real-valued, it follows that \( \overline{\phi(\lambda)} = \phi(\bar{\lambda}) \) and \( \overline{\alpha_0(\lambda)} = \alpha_0(\bar{\lambda}) \) and hence we obtain the conjugate equation

\[
\exp(\bar{\mu}_2 - \mu_2)\alpha_0(\bar{\lambda})\phi(\bar{\lambda}) + \partial_{\lambda} \phi(\bar{\lambda}) = 0.
\]

Thus far we have only considered the variables \( t_1 \) and \( t_2 \). We now wish to include the \( t_3 \) dependence of \( \phi \), arising from the second operator in the KP hierarchy,

\[
M = \partial_3 - \partial_1^3 + \frac{3}{2}u_2 - \frac{3}{2}u_1 - v,
\]

where \( v \) is defined in terms of \( u \) by the compatibility conditions. Writing equation (4) with \( \mu_2 = \mu_3 \exp(-\lambda^3 t_3) \) and multiplying both sides by \( \exp \mu_3 \), we then apply \( L_3 \) to both sides, giving that \( \alpha_0(\lambda; t_3) = \alpha_0(\lambda) \exp(-\lambda^3 \bar{\lambda}^3) \) and hence

\[
\partial_{\lambda} \phi(t; \lambda) + \exp(\bar{\mu}_3 - \mu_3)\alpha_0(\lambda)\phi(t; \bar{\lambda}) = 0,
\]

where \( t \) is now an element of \( \mathbb{R}^3 \). It follows that if \( \phi \exp \mu_n \) satisfies the first \( (n - 1) \) operators in the KP hierarchy, equation (5) holds, with \( n \) replacing the subscript 3.
4 The KPI equation

Whereas the inverse scattering for the KPI equation produces a d-bar problem relating $\phi(\lambda)$ and $\phi(\bar{\lambda})$, where $\phi$ is nowhere analytic in $\lambda$, the KPI equation gives a substantially different result. The first operator in the KPI hierarchy is given by $L = \partial_{\lambda} - \partial_{\bar{\lambda}} + u$, so proceeding as in the KPII case, taking $\mu_{3}(\lambda) = i(\lambda t_{1} + \lambda^{2}t_{2} - \lambda^{3}t_{3})$. The problem arising in the fact that for KPII, $P_{\lambda}(\xi)^{-1}$ has discrete singularities for all $\lambda$, but in the KPI case, $P_{\lambda}(\xi)^{-1}$ has continuous singularities for real $\lambda$. This causes the Green's function to be undefined on the real $\lambda$-axis. The resulting calculation leads to $\phi$ being a sectionally meromorphic function, satisfying a nonlocal Riemann-Hilbert problem on the real $\lambda$-axis,

$$(\phi_{+} - \phi_{-})(t; \lambda) = \int_{\mathbb{R}} F(\lambda, \kappa) \exp\{\mu(\kappa) - \mu(\lambda)\} \phi_{-}(\kappa) d\kappa,$$

$\phi_{+}(\phi_{-})$ being holomorphic in the upper (lower) half-plane.

At this time it is unclear as to the connection between the two results. One feels that they may be parts of a larger construction (for complex $t_{i}$) which reduces to the d-bar relation when $t_{i} \in \mathbb{R}$ and to the nonlocal Riemann-Hilbert problem when some $t_{i}$ are imaginary.

5 The twistor correspondence

The main problem with this theory is its global nature. The Ward correspondence for the anti-self-dual Yang-Mills equations is a local construction, therefore it would be sensible to attempt to develop a localised version of the preceding theory. One possible direction is to use ideas from Segal and Wilson, [4], concerning the theory of Grassmannians and the KP hierarchy. A link exists between the theory, if we insist in equation (1) that $\alpha$ vanishes in a neighbourhood of infinity. Then in that neighbourhood, $\partial_{\lambda} \phi = 0$ and $\psi$ has the general form of the Segal-Wilson Baker function,

$$\psi(t; \lambda) = \exp \mu \left( 1 + \sum_{i=1}^{\infty} a_{i} \lambda^{-i} \right).$$

It may well be possible to extend the Baker function to obtain the global $\psi$ of the Dirac operator.
An alternative avenue of approach is to consider what happens if the co-
ordinates \( t_i \) are complex. The Davey-Stewartson equations generate similar
constructions to the KP equations, but with the \( t_1 \)-coordinate being com-
plex. In this sense, the DS-equations are more fundamental than the KP
equations. By complexifying the other coordinates, it may be possible to
generate higher-dimensional integrable systems.

References

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