On group theoretical aspects of the non-linear twistor transform

Sergey A. Merkulov (Glasgow)

0. Introduction. Let $X \hookrightarrow Y$ be a rational curve $X = \mathbb{CP}^1$ embedded into a complex 3-fold Y with normal bundle $N = \mathcal{O}(1) \oplus \mathcal{O}(1)$. According to Kodaira [7], the moduli space of all rational curves obtained by holomorphic deformations of X inside Y is a complex manifold M of dimension $h^0(X, N) = 4$. More strikingly, according to Penrose [9] this M comes also equipped with a canonically induced self-dual conformal structure, and, moreover, *any* local conformal self-dual structure arises in this way. Later several other manifestations of this strange phenomenon have been observed when a complex analytic data of the form $(X \hookrightarrow Y, N)$ gives rise to the full category of local geometric structures C_{geo} (more precisely, it is the successful choice of a pair (X, N) consisting of a complex homogeneous manifold X and a homogeneous vector bundle N on X which uniquely specifies C_{geo} — the choice of a particular ambient manifold Y corresponds to the choice of a particular object in C_{geo}).

The questions then arise: How is it possible that a complex analytic data of the form (X, N) can be used as the building block for such basic differential-geometric quantities as metrics, affine connections, curvatures? What is the guiding principle for making a *successful* choice of (X, N)? How general is this phenomenon?

In this note we attempt to illucidate these questions by unveiling very strong links between the (curved, or non-linear) twistor approach to differential geometry and Borel-Weil approach to representation theory (in this context, see also the works of Baston and Eastwood [5, 1] on applications of the Bott-Borel-Weil technique to the linear twistor transform between "flat" models).

1. Complex G-structures. Let M be an n-dimensional complex manifold and V a fixed reference n-dimensional vector space (typically, $V = \mathbb{C}^n$). Let $\pi : \mathcal{L}^*M \to M$ be the holomorphic bundle of V-valued coframes, whose fibres $\pi^{-1}(t)$ consist, by definition, of all $\mathbb{C}$-linear isomorphisms $e : T_tM \to V$, where T_tM is the cotangent space at $t \in M$. The space $\mathcal{L}^*M$ is a principle

right $\mathrm{GL}(V)$-bundle with the right action given by $R_g(e) = g^{-1} \circ e$. If G is a closed complex subgroup of $\mathrm{GL}(V)$, then a complex G-structure on M is a principal holomorphic subbundle $\mathcal{G}$ of $\mathcal{L}^*M$ with the group G. It is clear that there is a one-to-one correspondence between the set of G-structures on M and the set of holomorphic sections σ of the quotient bundle $\tilde{\pi} : \mathcal{L}^*M/G \to M$ whose typical fibre is isomorphic to $\mathrm{GL}(V)/G$. A G-structure on M is called *locally flat* if there exits a coordinate patch in the neighbourhood of each point $t \in M$ such that in the associated canonical trivialization of $\mathcal{L}^*M/G$ over this patch the section σ is represented by a constant $GL(m, \mathbb{C})/G$-valued function. A G-structure is called *k-flat* if, for each $t \in M$, the k-jet of the associated section σ of $\mathcal{L}^*M/G$ at t is isomorphic to the k-jet of some locally flat section of $\mathcal{L}^*M/G$. It is not difficult to show that a G-structure admits a torsion-free affine connection if and only if it is 1-flat (cf. [4]). A G-structure on M is called *irreducible* if the action of G on V leaves no non-zero invariant subspaces.

The notion of G-structure is a unifying idea for a variety of popular themes in differential geometry. For example, (i) if $G \subset \mathrm{GL}(n, \mathbb{C})$ is the standardly represented special orthogonal group $\mathrm{SO}(n, \mathbb{C})$, then the associated G-structure is nothing but a complex Riemannian structure; (ii) if $G = \mathrm{CO}(n, \mathbb{C}) \subset \mathrm{GL}(n, \mathbb{C})$, then G-structure coincides with the complex conformal structure; (iii) if $G = \mathrm{GL}(2,\mathbb{C})\mathrm{GL}(n,\mathbb{C}) \subset \mathrm{GL}(2n,\mathbb{C})$, $n \geq 3$, then G-structure is precisely almost quaternionic structure. In the first two examples G-structures are always 1-flat, in the third example this is not true — 1-flat $\mathrm{GL}(2,\mathbb{C})\mathrm{GL}(n,\mathbb{C})$-structures are called complex quaternionic structures.

The notion of G-structure is proved to very useful in the study of affine connections, especially in the context of classifying the irreducibly acting holonomies of torsion-free affine

connections [4]. Given an affine connection ∇ on a connected simply connected complex manifold M with the holonomy group G, the associated G-structure $\mathcal{G}_\nabla \subset \mathcal{L}^*M$ can be constructed as follows. Define two points u and v of $\mathcal{L}^*M$ to be equivalent, $u \sim v$, if there is a holomorphic path γ in M from $\pi(u)$ to $\pi(v)$ such that $u = P_\gamma(v)$, where $P_\gamma : \Omega^1_{\pi(v)}M \rightarrow \Omega^1_{\pi(u)}M$ is the parallel transport along γ. Then $\mathcal{G}_\nabla$ can be defined, up to an isomorphism, as $\{u \in \mathcal{L}^*M \mid u \sim v\}$ for some coframe v. The G-structure $\mathcal{G}_\nabla$ is the smallest subbundle of $\mathcal{L}^*M$ which is invariant under ∇-parallel translations.

The basic question about the local geometry of G-structures — what is the obstruction for a k-flat G-structure $\mathcal{G} \rightarrow M$ to be $(k+1)$-flat? — has been answered by Guillemin [6] and Singer and Sternberg [10]. The obstruction is given locally by a function on M with values in the Spencer cohomology $\mathrm{H}^{k,2}(\mathfrak{g})$ associated with the given representation of G in V (more accurately, with the associated representation of the Lie algebra $\mathfrak{g}$ in V). In the next paragraph we recall the definition of $\mathrm{H}^{k,l}(\mathfrak{g})$.

2. Spencer cohomology. Let V be a vector space and $\mathfrak{g}$ a Lie subalgebra of $gl(V) \simeq V \otimes V^*$. Define recursively the $\mathfrak{g}$-modules

$$\begin{aligned} \mathfrak{g}^{(-1)} &= V \\ \mathfrak{g}^{(0)} &= \mathfrak{g} \\ \mathfrak{g}^{(k)} &= [\mathfrak{g}^{(k-1)} \otimes V^*] \cap [V \otimes \odot^{k+1} V^*]\,,\ k = 1, 2, \ldots, \end{aligned}$$

and define the map

$$\mathfrak{g}^{(k)} \otimes \Lambda^{l-1} V^* \longrightarrow \mathfrak{g}^{(k-1)} \otimes \Lambda^l V^*$$

as the antisymmetrisation over the last l indices.

Since $\partial^2 = 0$, there is a complex

$$\mathfrak{g}^{(k)} \otimes \Lambda^{l-1} V^* \xrightarrow{\partial} \mathfrak{g}^{(k-1)} \otimes \Lambda^l V^* \xrightarrow{\partial} \mathfrak{g}^{(k-2)} \otimes \Lambda^{l+1} V^*$$

whose cohomology at the center term is denoted by $\mathrm{H}^{k,l}(\mathfrak{g})$ and is called the (k,l) Spencer cohomology group. In particular,

$$\begin{aligned} \mathrm{H}^{k,1}(\mathfrak{g}) &= 0 \\ \mathrm{H}^{k,2}(\mathfrak{g}) &= \frac{\mathrm{Ker} : \mathfrak{g}^{(k-1)} \otimes \Lambda^2 V^* \xrightarrow{\partial} \mathfrak{g}^{(k-2)} \otimes \Lambda^3 V^*}{\mathrm{Image} : \mathfrak{g}^{(k)} \otimes V^* \xrightarrow{\partial} \mathfrak{g}^{(k-1)} \otimes \Lambda^2 V^*}. \end{aligned} \tag{1}$$

In addition to $\mathrm{H}^{k,2}(\mathfrak{g})$, the $\mathfrak{g}$-module $\mathfrak{g}^{(1)}$ also has a clear geometric meaning. If a G-structure $\mathcal{G} \rightarrow M$ is 1-flat then the set of all local torsion-free affine connections in $\mathcal{G}$ is the affine space modelled on the vector space of local functions on M with values in $\mathfrak{g}^{(1)}$. In particular, if $G \subseteq GL(V)$ is such that $\mathfrak{g}^{(1)} = 0$, then any G-structure admits at most one torsion-free affine connection. If $K(\mathfrak{g})$ is the $\mathfrak{g}$-module of formal curvature tensors of torsion-free affine connections with holonomy in $\mathfrak{g}$, i.e.

$$K(\mathfrak{g}) = [\mathfrak{g} \otimes \Lambda^2 V^*] \cap [\mathrm{Ker} : V \otimes V^* \otimes \Lambda^2 V^* \to V \otimes \Lambda^3 V^*],$$

then

$$\mathrm{H}^{2,2}(\mathfrak{g}) = \frac{K(\mathfrak{g})}{\partial(g^{(1)} \otimes V^*)}$$

i.e. the cohomology group $\mathrm{H}^{2,2}(\mathfrak{g})$ represents the part of $K(\mathfrak{g})$ which is invariant under $\mathfrak{g}^{(1)}$-valued shifts in a formal torsion-free affine connection with holonomy in $\mathfrak{g}$. For example, if $(G, V) = (\mathrm{CO}(n, \mathbb{C}), \mathbb{C}^n)$, then $\mathfrak{g}^{(1)} = V^*$ and $\mathrm{H}^{2,2}(\mathfrak{g})$ is the vector space of formal Weyl tensors.

If $\mathfrak{g}^{(1)} = 0$, then $\mathrm{H}^{2,2}(\mathfrak{g})$ is exactly $K(\mathfrak{g})$, the $\mathfrak{g}$-module which plays a key role in the theory of torsion-free affine connections with holonomy in $\mathfrak{g}$, especially in the Berger classification context [2, 3, 4]. The case $\mathfrak{g}^{(1)} = 0$ is generic — there are very few irreducibly acting Lie subgroups $\mathfrak{g} \subset gl(V)$ which have $\mathfrak{g}^{(1)} \neq 0$ and they are all known by now.

3. A simple group-theoretical explanation of the non-linear twistor transform. Let V be a finite dimensional complex vector space and $G \subsetneq \mathrm{GL}(V)$ an irreducible representation of a reductive complex Lie group in V. Then G also acts irreducibly in V^* via the dual representation. Let $\tilde{X}$ be the G-orbit of a highest weight vector in $V^* \backslash 0$. Then the quotient $X := \tilde{X}/\mathbb{C}^*$ is a generalised flag variety (i.e. a compact complex homogeneous-rational manifold) canonically embedded into $\mathbb{P}(V^*)$ and there is a commutative diagram

$$\begin{array}{ccc} \tilde{X} & \hookrightarrow & V^* \backslash 0 \\ \downarrow & & \downarrow \\ X & \hookrightarrow & \mathbb{P}(V^*) \end{array}$$

In fact, $X = G_s/P$, where G_s is the semisimple quotient of G and P is the parabolic subgroup of G_s leaving a highest weight vector in V^* invariant up to a scale. Let L be the restriction of the hypèrplane section bundle $\mathcal{O}(1)$ on $\mathbb{P}(V^*)$ to the submanifold X. Clearly, L is an ample homogeneous line bundle on X.

In summary, there is a natural map

$$(G, V) \longrightarrow (X, L)$$

associating with any irreducibly acting reductive Lie group $G \subsetneq \mathrm{GL}(V)$ a pair (X, L) consisting of a generelised flag variety X and an ample line bundle L on X. Can this map be reversed?

According to Borel-Weil (see, e.g., [1]), the representation space V can be reconstructed very easily:

$$V = \mathrm{H}^0(X, L).$$

What about G?

Theorem 1 (Onishchik 1962) *Let G be a simply connected simple complex Lie group, P a parabolic subgroup of G, and $X = G/P$ the associated generalised flag variety. Then*

$$\mathfrak{g} \subseteq \mathrm{H}^0(X, TX),$$

where $\mathfrak{g}$ *is the Lie algebra of* G. *Moreover,*

$$\mathfrak{g} = \mathrm{H}^0(X, TX)$$

unless one of the following holds

(i) *G is the representation of* $\mathrm{Sp}(n,\mathbb{C})$ *in* $\mathbb{C}^{2n}$ *(in this case* $\mathrm{H}^0(X,TX) \simeq sl(n,\mathbb{C})$*);*

(ii) *G is the representation of G_2 in* $\mathbb{C}^7$ *(*$\mathrm{H}^0(X,TX) \simeq so(7,\mathbb{C})$*);*

(iii) *G is the fundamental spinor representation of* $\mathrm{SO}(2n+1,\mathbb{C})$ *(*$\mathrm{H}^0(X,TX) \simeq so(2n+2, \mathbb{C})$*);*

Therefore, if $G \subset GL(V)$ is semisimple then, with a few exceptions, G can be reconstructed from (X, L). However, it is often undesirable to restrict oneself to semisimple groups only (especcially in the context of the Berger holonomy classification problem). There is a natural central extension of the Lie algebra $\mathrm{H}^0(X,TX)$ which is canonically associated with the pair (X, L).

Fact 2 *For any (X, L)* $\mathfrak{g} := \mathrm{H}^0(X, L \otimes (J^1L)^*)$ *is a reductive Lie algebra canonically represented in* $\mathrm{H}^0(X, L)$.

This fact is easy to explain — $\mathrm{H}^0(X, L\otimes(J^1L)^*)$ is exactly the Lie algebra of the Lie group G of all global biholomorphisms of the line bundle L which commute with the projection $L \to X$ [8].

In conclusion, with a given irreducible representation $G \subseteq \mathrm{GL}(V)$ there is canonically associated a pair (X, L) consisting of a generalised flag variety X and a very ample line bundle on X such that much of the original information about G can be restored from (X, L). In the twistor theory context, the crucial observation is that the $\mathfrak{g}$-modules $\mathfrak{g}^{(k)}$ and $\mathrm{H}^{k,2}(\mathfrak{g})$ also admit a nice description in terms of (X, L).

Theorem 3 *For any compact complex manifold X and any very ample line bundle L on X, there is an isomorphism*

$$\mathfrak{g}^{(k)} = \mathrm{H}^0(X, L \otimes \odot^{k+1} N^*), \quad k = 0, 1, 2, \ldots$$

and an exact sequence of $\mathfrak{g}$-modules,

$$0 \longrightarrow \mathrm{H}^{k,2}(\mathfrak{g}) \longrightarrow \mathrm{H}^1(X, L \otimes \odot^{k+2} N^*) \longrightarrow \mathrm{H}^1(X, L \otimes \odot^{k+1} N^*) \otimes V^*, \quad k = 1, 2, \ldots$$

where $\mathfrak{g} := \mathrm{H}^0(X, L \otimes (J^1L)^*)$, $N = J^1L$, *and* $\mathrm{H}^{k,2}(\mathfrak{g})$ *are the Spencer cohomology groups associated with the canonical representation of $\mathfrak{g}$ in the vector space* $V := H^0(X, L)$.

Proof. Since L is very ample, there is a natural "evaluation" epimorhism

$$V \otimes \mathcal{O}_X \longrightarrow J^1L \longrightarrow 0$$

whose dualization gives rise to the canonical monomorphism

$$0 \longrightarrow N^* \longrightarrow V^* \otimes \mathcal{O}_X.$$

Then one may construct the following sequences of locally free sheaves

$$0 \longrightarrow L \otimes \odot^{k+1} N^* \longrightarrow L \otimes \odot^k N^* \otimes V^* \longrightarrow L \otimes \odot^{k-1} N^* \otimes \Lambda^2(V^*) \qquad (2)$$

and

$$0 \longrightarrow L \otimes \odot^{k+2} N^* \longrightarrow L \otimes \odot^{k+1} N^* \otimes V^* \longrightarrow L \otimes \odot^k N^* \otimes \Lambda^2(V^*) \longrightarrow L \otimes \odot^{k-1} N^* \otimes \Lambda^3(N^*). \quad (3)$$

One may notice that both these sequences are exact. [Hint: for any vector space W one has $W \otimes \Lambda^2 W \bmod \Lambda^3 W \simeq W \otimes \odot^2 W \bmod \odot^3 W$.]

Then computing $\mathrm{H}^0(X, \ldots)$ of (2) and using the inductive definition of $\mathfrak{g}^{(k)}$ as $[\mathfrak{g}^{(k-1)} \otimes V^*] \cap [V \otimes \odot^{k+1)} V^*]$ (with $\mathfrak{g}^{(0)} := \mathfrak{g}$ and $\mathfrak{g}^{(-1)} := V$) immediately implies the first statement of the Theorem.

The second statement of the theorem follows from (3) and the definition (1) of $\mathrm{H}^{k,2}(\mathfrak{g})$. Indeed, define E_k by the exact sequence

$$0 \longrightarrow L \otimes \odot^{k+2} N^* \longrightarrow L \otimes \odot^{k+1} N^* \otimes V^* \longrightarrow E_k \longrightarrow 0$$

The associated long exact sequence implies the following *exact* sequence of vector spaces

$$0 \longrightarrow \mathrm{H}^0(X, E_k)/\partial[\mathfrak{g}^{(k)} \otimes V^*] \longrightarrow \mathrm{H}^1(X, L \otimes \odot^{k+2} N^*) \longrightarrow \mathrm{H}^1(X, L \otimes \odot^{k+1} N^*) \otimes V^*.$$

On the other hand, the exact sequence

$$0 \longrightarrow E_k \longrightarrow L \otimes \odot^k N^* \otimes \Lambda^2(V^*) \longrightarrow L \otimes \odot^{k-1} N^* \otimes \Lambda^3(N^*)$$

implies

$$\mathrm{H}^0(X, E_k) = \ker : \mathfrak{g}^{(k-1)} \otimes \Lambda^2(V^*) \xrightarrow{\partial} \mathfrak{g}^{(k-2)} \otimes \Lambda^3(V^*).$$

which in turn implies

$$\mathrm{H}^{k,2}(\mathfrak{g}) = \mathrm{H}^0(X, E_k)/\partial[\mathfrak{g}^{(k)} \otimes V^*].$$

This completes the proof of the second part of the Theorem. □

Therefore, the pair consisting from a homogeneous-rational manifold X and a holomorphic vector bundle bundle E on X is more than a natural building block for basic differential-geometric objects. If $\mathrm{rank} E \geq 2$, then, following a common practice in complex analysis, one should replace the pair (X, E) by an equivalent one $(\hat{X} = \mathbb{P}(E^*), L = \mathcal{O}(1))$ and apply Theorem 3 to find out which geometric category C_{geo} may correspond to the twistorial data (X, E). Applying this procedure, e.g., to the pair $(\mathbb{CP}^1, \mathbb{C}^k \otimes \mathcal{O}(1))$, $k \geq 3$, one immediately concludes that C_{geo} is the category of complexified quaternionic manifolds. Also, this purely group theoretical result suggests that there should exist a universal twistor construction for *all* torsion-free geometries. For details of this construction we refer to [8].

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